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TECHNIQUES OF HIGHER ORDER OSCULATOR BUNDLE IN GENERALIZED GAUGE THEORY

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Abstract. Given a certain state of a mechanical system, the object of many physical theories is to determine the way in which that state changes in time. The evolution of the system is usually gouverned by Euler-Lagrange equations and this evolution must look the same in every coordinate system. In gauge theories, in addition to the invariance under changes of coordinates, is involved another invariance, under the so called gauge transformations. In classical gauge theory the physical system is represented by sections in an associate bundle, and is required to be invariant with respect to this gauge change.

In this paper we shall generalize the concept of gauge transformation of one system developed in a k-osculator bundle ([7],[8]), the associate bundle of the bundle of k-frames. The geometrical notion of gauge derivatives is studied and the k-order Einstein Yang-Mills equations are obtained. A general class of gauge invariant Lagrangians is given.

1. Gauge transformations on $\mathrm{Osc}^{(k)}(M)$-bundle

Let M be a differentiable manifold, $\dim M = n$, $(U,\phi) a$ local map at $x \in U$, $x = (x^i)$ the local coordinates. If (U',ϕ') is another local map at x, then the expression of $\phi'\phi^{-1} : \mathbf{R}^n \to \mathbf{R}^n$ is $x'^i = x'^i(x)$.

The k-osculator bundle, E_k=$\mathrm{Osc}^{(k)}M$ is a fiber bundle locally isomorphical with the k-jet bundle of M. In a local map $((\Pi_0^k)(U), h^k)$ in $E_{k,z}$ the coordinates are $z = (x^i, y^{(1)i}, \dots y^{(k)i}), \Pi_0^k(z) = x$, with:

$$x'^i = x'^i(x), \qquad y'^i = \frac{\partial x'^i}{\partial x^j} y^{(1)j} \tag{1.1}$$

$$2y'^{(2)i} = \frac{\partial y'^{(1)i}}{\partial x^j} y^{(1)j} + 2\frac{\partial y'^{(1)i}}{\partial y^{(1)j}} y^{(2)j}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

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$$ky'^{(k)i} = \frac{\partial y'^{(k-1)i}}{\partial x^j} y^{(1)j} + 2\frac{\partial y'^{(k-1)i}}{\partial y^{(1)j}} y^{(2)j} + \cdots + k\frac{\partial y'^{(k-1)i}}{\partial y^{(k-1)j}} y^{(k)j}$$

Let us consider $a = j_0^k(\phi'\phi^{-1})$ the k-jet on $\mathbf{R}^n$, represented by

$$a = (\frac{\partial x'^i}{\partial x^j}, \frac{\partial^2 x'^i}{\partial x^{j_1} \partial x^{j_2}}, \ldots, \frac{\partial^k x'^i}{\partial x^{j_1} \partial x^{j_2} \ldots \partial x^{j_k}}) \tag{1.2}$$

and $F = \mathbf{R}^{nk}, G_n^k$ the prolongation of the linear group that acts usually on $F, G_n^k \times F \to$ F.

The associate bundle of the k-principal frame bundle $P^{(k)}(M) \times F/G_n^k$ is the bundle of classes $u = [z, p] = \{(za, a^{-1}p), a \in G_n^k, z \in P^{(k)}(M)\}$ and locally is applied on $M \times F$, hence u always can be written in the form $u = (x^i, y^{(1)i}, 2y^{(2)i}, \ldots, ky^{(k)i})$. Then the action on F, for a given by (1.2) induces the same relations (1.1). It results that the k-osculator bundle E_k has a canonic symmetric structure of associate k-frames bundle over M.

In classical gauge theory are studied the automorphisms of the associate bundle ($k = 1$) that induce the identity on G_n, usually called gauge transformations.

G.S.Asanov ([1]) considers a more general type of 'gauge transformations' and this concept will be developed here for the bundle $\pi_0^k : E_k \to$ M. First, let us remark that E_k has a bundle structure over $E_h, 0 \leq h < k, \pi_h^k : E_k \to E_h$([7]), with E_0=M.

Definition 1.1. A gauge transformation on E_k is a sequence $(f_0, f_1, f_2, \ldots, f_k)$ of diffeomorphisms, $f_\alpha : E_\alpha \to E_\alpha$, such that

$$\pi_{\alpha-1}^\alpha \circ f_{\alpha-1} = f_\alpha \circ \pi_{\alpha-1}^\alpha, \alpha = \overline{0,k}. \tag{1.3}$$

Obviously, from the definition follows that $\pi_\beta^\alpha \circ f_\beta = f_\beta \circ \pi_\beta^\alpha, \alpha > \beta \geq 0$.

In a local map at $x \in M, \phi f_0 \phi^{-1} : \mathbf{R}^n \to \mathbf{R}^n$ is represented by $\tilde{x}^i = X^i(x)$ and representing in the same manner the diffeomorphisms $h_{\alpha-1}^\alpha f_\alpha h_{\alpha-1}^\alpha$, it results that a gauge transformation on E_k is locally given by:

$$\begin{aligned} \tilde{x}^i &= X^i(x) \\ \tilde{y}^{(1)i} &= Y^{(1)i}(x, y^{(1)}), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \tilde{y}^{(k)i} &= Y^{(k)i}(x, y^{(1)}, \ldots, y^{(k)}). \end{aligned} \tag{1.4}$$

Of course, (1.4) is a sequence of diffeomorphisms if the Jacobian matrices are nondegenerate, that is:

$$\det(\tilde{z}) = \det(\frac{\partial X^i}{\partial x^j}) \cdot \det(\frac{\partial Y^{(1)i}}{\partial y^{(1)j}}) \cdot \ldots \cdot \det(\frac{\partial Y^{(k)i}}{\partial y^{(k)j}}) \neq 0. \tag{1.4'}$$

Particulary gauge transformations are:

1) $\quad \tilde{x}^i = x^i, \tilde{y}^{(\alpha)i} = Y^{(\alpha)i}(x, y^{(1)}, \ldots, y^{(k)}), \alpha = \overline{1,k},$

2) $\tilde{x}^i = X^i(x), \tilde{y}^{(\alpha)i} = Y_j^{(\alpha)i}(x)y^{(\alpha)j}$, or more generally :

$$\tilde{x}^i = X^i(x), \qquad \tilde{y}^{(\alpha)i} = A^i_{j_1 j_2 \dots j_\alpha}(x)y^{(1)j_1}\dots y^{(1)j_\alpha} + \tag{1.5}$$

$$+A^i_{j_1 j_2 \dots j_{\alpha-1}}(x)y^{(1)j_1}\dots y^{(1)j_{(\alpha-2)}}y^{(2)j_{(\alpha-1)}} + \dots + A^i_{j_1}(x)y^{(\alpha)j_1}, \alpha = \overline{1,k},$$

$$\text{where } (A^i_{j_1}(x), A^i_{j_1 j_2}(x), \dots, A^i_{j_1 j_2 \dots j_\alpha}(x)) \in G^k_n.$$

An interesting situation is when we consider a diffeomorphism $f_0 : M \to M$ and $f_\alpha, \alpha = \overline{1,k}$ its k-jets from $P^{(k)}(M)$. Then in the associate bundle E_k is induced a sequence of diffeomorphisms denoted by $\overline{f} = (\overline{f}_0, \overline{f}_1, \dots, \overline{f}_k)$, called the f_0-jet gauge transformation of f_0. The local representation satisfies in this case the conditions:

$$\frac{\partial \tilde{x}^i}{\partial x^j} = \frac{\partial \tilde{y}^{(1)i}}{\partial y^{(1)j}} = \dots = \frac{\partial \tilde{y}^{(k)i}}{\partial y^{(k)j}}.$$

Now, let us return to the gauge transformations.

Proposition 1.1. The set of all gauge transformations on E_k=Osc$^{(k)}M$ bundle is a subgroup of DiffM $\times$ Diff $E_1 \times \dots \times$ Diff E_k, denoted by Aut$(M \times E_1 \times \dots \times E_k)$.

Let's consider the tangent space $T_z(E_k)$ at $z \in E_k$; its local basis

$$\left(\frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^{(0)i}}, \frac{\partial}{\partial y^{(1)i}}, \dots, \frac{\partial}{\partial y^{(k)i}}\right)$$

changes by the following rules ([7]):

$$\frac{\partial}{\partial y^{(\alpha)i}} = \frac{\partial y'^{(\alpha)j}}{\partial y^{(\alpha)i}}\frac{\partial}{\partial y'^{(\alpha)j}} + \frac{\partial y'^{(\alpha+1)j}}{\partial y^{(\alpha)i}}\frac{\partial}{\partial y'^{(\alpha+1)j}} + \dots + \frac{\partial y'^{(k)j}}{\partial y^{(k)i}}\frac{\partial}{\partial y'^{(k)j}}, \tag{1.6}$$

$\alpha = \overline{0,k}$ and the corresponding change imposed by (1.4):

$$\frac{\partial}{\partial y^{(\alpha)i}} = Y_i^{(\alpha)j}\frac{\partial}{\partial \tilde{y}^{(\alpha)j}} + \frac{\partial Y^{(\alpha+1)j}}{\partial y^{(\alpha)i}}\frac{\partial}{\partial \tilde{y}^{(\alpha+1)j}} + \dots + \frac{\partial Y^{(k)j}}{\partial y^{(\alpha)i}}\frac{\partial}{\partial \tilde{y}^{(k)j}}, \tag{1.7}$$

$$\alpha = \overline{0,k}, \text{ where } Y_i^{(\alpha)j} = \frac{\partial Y^{(\alpha)j}}{\partial y^{(\alpha)i}}, Y_i^{(0)j} = X_i^j.$$

One fundamental class of geometric objects on $E_k$is that of d-tensors, which can formally be defined as a system of functions $W^{i_1\dots i_r}_{j_1\dots j_s}$ with the rules of change from ([7]).

Let us denote by $\overline{X}^i_j$ and $\overline{Y}^{(\alpha)i}_j$ the elements of the inverse matrices of X^i_j and $Y^{(\alpha)i}_j$ respectively.

Definition 1.2. A $v_\alpha - d$-gauge tensor (shortly $v_\alpha - d$-g.t.) is a d-tensor $W^{i_1\dots i_r}_{j_1\dots j_s}$ on E_k which satisfies:

$$W^{i_1\dots i_r}_{j_1\dots j_s}(\tilde{x}, \tilde{y}^{(1)}, \dots, \tilde{y}^{(k)}) = Y^{(\alpha)i_1}_{h_1}\dots Y^{(\alpha)i_r}_{h_r}Y^{(\alpha)l_1}_{j_1}\dots Y^{(\alpha)l_s}_{j_s}\cdot \tag{1.8}$$

$$\cdot W^{h_1\dots h_r}_{l_1\dots l_s}(x, y^{(1)}, \dots, y^{(k)}), \alpha = \overline{0,k}.$$

$A(v_0 - d$-g.t) will be called horizontal d-g.t. The set of all $(v_\alpha - d$-g.t$), \alpha = \overline{0,k}$, and their combinations will be called d-gauge tensors. For instance, $\frac{\partial}{\partial y^{(k)i}}$ is a $(v_k - d$-g.t$)$.

2. Gauge covariant derivative

Let's consider a nonlinear connection N on the bundle E_k=Osc$^{(k)}M$, i.e. $T(E_k) = N(E_k) \oplus V(E_k)$, where $V(E_k)$=Ker$(\pi_0^k)^T$ is the vertical subbundle ([7]). The horizontal lift of $\frac{\partial}{\partial x^i}$ is $\frac{\delta}{\delta x^i}$ and the set $\{\frac{\delta}{\delta x^i}\}$ determines a local basis of the horizontal distribution $N_z(E_k)$,

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_{(1)i}\frac{\partial}{\partial y^{(1)j}} - \ldots - N^j_{(k)i}\frac{\partial}{\partial y^{(k)j}}. \tag{2.1}$$

The functions $N^j_{(\alpha)i}$ are called the coefficients of the nonlinear connection. If denoting by J the natural k-tangent structure on $E_k, (J^k = 0)$, then in $z \in E_k$ we have the sequence of distributions $N_0 = N_z(E_k), N_1 = J(N_0), \ldots, N_{k-1} = J(N_{k-2})$ and V_k=Ker$(\pi^k_{k-1})^T, T_z(E_k) = N_0 \oplus N_1 \oplus \ldots \oplus N_{k-1} \oplus V_k$, in which the corresponding adapted bases are

$$\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}} = J(\frac{\partial}{\partial x^i}), \ldots, \frac{\delta}{\delta y^{(k-1)i}} = J(\frac{\delta}{\delta y^{(k-2)i}}), \frac{\delta}{\delta y^{(k)i}} = \frac{\partial}{\partial y^{(k)i}} = J(\frac{\partial}{\partial y^{(k-1)i}}).$$

If the coefficients of the nonlinear connection have the transformation laws:

$$N'^i_{(\beta)m}\frac{\partial x'^m}{\partial x^j} = \frac{\partial x'^i}{\partial x^m}N^m_{(\beta)j} + \frac{\partial y'^{(1)i}}{\partial x^m}N^m_{(\beta-1)j} + \ldots + \frac{\partial y'^{(\beta-1)i}}{\partial x^m}N^m_{(1)j} - \frac{\partial y'^{(\beta)i}}{\partial x^j}, \tag{2.2}$$

$\beta = \overline{1,k}$, then the adapted basis has simple transformation rules

$$\frac{\delta}{\delta x^i} = \frac{\partial x'^j}{\partial x^i}\frac{\delta}{\delta x'^j}, \frac{\delta}{\delta y^{(\beta)i}} = \frac{\partial x'^j}{\partial x^i}\frac{\delta}{\delta y'^{(\beta)j}}, \beta = \overline{1,k}.$$

Let us consider the dual adapted basis:

$$\{\delta x^i; \delta y^{(1)i} = dy^{(1)i} + M^i_{(1)j}x^j; \ldots; \delta y^{(k)i} = dy^{(k)i} + M^i_{(1)j}dy^{(k-1)j} + \ldots + M^i_{(k)j}dy^{(k)j}\}$$

that changes to (1.1) by the rules:

$$M^m_{(\beta)j}\frac{\partial x'^i}{\partial x^m} = M'^i_{(\beta)m}\frac{\partial x'^m}{\partial x^j} + M'^i_{(\beta-1)m}\frac{\partial y'^{(1)m}}{\partial x^j} + \ldots + M'^i_{(1)m}\frac{\partial y'^{(\beta-1)m}}{\partial x^j} + \frac{\partial y'^{(\beta)i}}{\partial x^j}. \tag{2.2'}$$

Note that $\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(\beta)i}}, \beta = \overline{1,k}$, are d-tensors but except $\frac{\delta}{\delta y^{(k)i}}$ none of them is a gauge d-tensor. Next, we shall constraint $\frac{\delta}{\delta x^i}$ and $\frac{\delta}{\delta y^{(\beta)i}}$ to be d-gauge tensors. It follows that:

$$\tilde{N}^i_{(1)m} \cdot Y^{(\alpha)m}_j = Y^{(\alpha+1)i}_m \cdot N^m_{(1)j} - \frac{\partial {Y^{(\alpha+1)}}^i}{\partial x^j}, \alpha = \overline{0, k-1} \tag{2.3}$$

$$\tilde{N}^i_{(2)m} \cdot Y^{(\alpha)m}_j = Y^{(\alpha+2)i}_m \cdot N^m_{(2)j} + \frac{\partial Y^{(\alpha+2)i}}{\partial y^{(\alpha+1)m}} \cdot N^m_{(1)j} - \frac{\partial {Y^{(\alpha+2)}}^i}{\partial y^{(\alpha)j}},$$

....

$$\tilde{N}^i_{(k-\alpha)m} \cdot Y^{(\alpha)m}_j = Y^{(k)i}_m \cdot N^m_{(k-\alpha)j} + \frac{\partial Y^{(k)i}}{\partial y^{(k-1)m}} \cdot N^m_{(k-\alpha-1)j} + \dots + \frac{\partial Y^{(k)i}}{\partial y^{(\alpha+1)m}} \cdot N^m_{(1)j} - \frac{\partial Y^{(k)i}}{\partial y^{(\alpha)j}}$$

where $\tilde{N}^i_{(\beta)j} = N^i_{(\beta)j}(\tilde{x}, \tilde{y}^{(1)}, \dots, \tilde{y}^{(k)}), \beta = \overline{1,k}$.

The coefficients of dual basis have to satisfy

$$Y^{(\beta)i}_m \cdot M^m_{(1)j} = \tilde{M}^i_{(1)m} \cdot Y^{(\beta-1)m}_j + \frac{\partial Y^{(\beta)i}}{\partial y^{(\beta-1)j}}, \tag{2.3'}$$

$$Y^{(\beta)i}_m \cdot M^m_{(2)j} = \tilde{M}^i_{(2)m} \cdot Y^{(\beta-1)m}_j + \tilde{M}^i_{(1)m} \cdot \frac{\partial Y^{(\beta-1)m}}{\partial y^{(\beta-1)j}} + \frac{\partial Y^{(\beta)i}}{\partial y^{(\beta-1)j}},$$

....

$$Y^{(\beta)i}_m \cdot M^m_{(\beta)j} = \tilde{M}^i_{(\beta)m} \cdot X^m_j + \tilde{M}^i_{(\beta-1)m} \cdot \frac{\partial Y^{(1)m}}{\partial x^j} + \dots + \tilde{M}^i_{(1)m} \frac{\partial {Y^{(\alpha-1)}}^m}{\partial x^j} + \frac{\partial {Y^{(\alpha)}}^i}{\partial x^j},$$

Definition 2.1. A nonlinear gauge connection on the bundle $E = \mathrm{Osc}^{(k)}M$ is a system of functions $N = (N_{(1)}, N_{(2)}, \dots, N_{(k)})$, (or equivalently $M = (M_{(1)}, M_{(2)}, \dots, M_{(k)})$) which satisfy the conditions (2.2) and (2.3) (or equivalently (2.2') and (2.3')).

Remark 1. If $k = 1$, the first conditions in (2.2) and (2.3) provide the gauge nonlinear connection considered in ([2],[6]).

Remark 2. In particular cases of gauge transformations the number of conditions in (2.3) can be diminished. For example, for the case of f_0-jet gauge transformations, the k systems of (2.3) reduce to a single one, and moreover, the Liouville fields on E_k ([7]):

$$\Gamma^{(\alpha)} = y^{(1)i} \frac{\partial}{\partial y^{(1)i}} + 2y^{(2)i} \frac{\partial}{\partial y^{(2)i}} + \dots + \alpha y^{(\alpha)i} \frac{\partial}{\partial y^{(\alpha)i}}, \alpha = \overline{1,k}$$

satisfiy with respect to the tangent structure the conditions

$$J(\tilde{\Gamma}^{(k)}) = \tilde{\Gamma}^{(k-1)}; J(\tilde{\Gamma}^{(k-1)}) = \tilde{\Gamma}^{(k-2)}; \dots J(\tilde{\Gamma}^{(2)}) = \tilde{\Gamma}^{(1)}; J(\tilde{\Gamma}^{(1)}) = 0.$$

A f_0-gauge spray on E_k is defined to be a field S, such that $J(S) = \Gamma^{(k)}$ and $J(\tilde{S}) = \tilde{\Gamma}^{(k)}$. From the first condition results that:

$$S = y^{(1)i} \frac{\partial}{\partial x^i} + 2y^{(2)i} \frac{\partial}{\partial y^{(1)i}} + \dots + ky^{(k)i} \frac{\partial}{\partial y^{(k-1)i}} - (k+1)G^i(z) \frac{\partial}{\partial y^{(k)i}} \tag{2.4}$$

and verifies the second condition.

In corroboration with the results from [7], it follows:

Proposition 2.1. A k-gauge spray determines a gauge nonlinear connection for a given f_0-jet gauge transformation with the dual coefficients:

$$M^i_{(1)j} = \frac{\partial G^i}{\partial y^{(k)j}}; M^i_{(2)j} = \frac{1}{2}(SM^i_{(1)j} + M^i_{(1)m} \cdot M^m_{(1)j}); \dots ; \tag{2.5}$$
$$M^i_{(k)j} = \frac{1}{k}(SM^i_{(k-1)j} + M^i_{(1)m} M^m_{(k-1)j})$$

Now, let's consider in general, the distributions $N_0, N_1, \dots, N_{k-1}, V_k$ determined by a given nonlinear connection N on E_k, $E_{k,z} = N_0 \oplus N_1 \oplus \dots \oplus N_{k-1} \oplus V_k$, and denote by $h = v_0, v_1, \dots, v_{k-1}, v_k$ the corresponding projectors and by $hX = X^h, v_\alpha X = X^{v_\alpha}$, the horizontal or vertical part of $X \in \mathcal{X}(E_k)$.

Then we have the covariant derivative operators ([7]):

$D^{(\alpha)h}_{X^h} Y = D_{X^h} Y^{v_\alpha}$ and $D^{(\alpha)v_\beta}_X Y = D_{X^{v_\beta}} Y^{v_\alpha}, \alpha = \overline{0,k}, \beta = \overline{1,k}$

are locally defined by the coefficients $L^{(\alpha)m}_{ij}$ and $C^{(\alpha)m}_{(\beta)ij}$:

$$D_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta y^{(\alpha)i}} = L^{(\alpha)m}_{ij} \frac{\delta}{\delta y^{(\alpha)m}}, D_{\frac{\delta}{\delta y^{(\beta)j}}} \frac{\delta}{\delta y^{(\alpha)i}} = C^{(\alpha)m}_{(\beta)ij} \frac{\delta}{\delta y^{(\alpha)m}} \tag{2.6}$$

$(\alpha = \overline{0,k}, \beta = \overline{1,k}, y^{(0)i} = x^i)$, with

$$L^{(\alpha)'i}_{pq} \frac{\partial x'^p}{\partial x^h} \frac{\partial x'^q}{\partial x^m} = L^{(\alpha)p}_{hm} \frac{\partial x'^i}{\partial x^p} - \frac{\partial^2 x'^i}{\partial x^h \partial x^m} \tag{2.7}$$

$$C^{(\alpha)'i}_{(\beta)pq} \frac{\partial x'^p}{\partial x^h} \frac{\partial x'^q}{\partial x^m} = \frac{\partial x'^i}{\partial x^j} C^{(\alpha)i}_{(\beta)hm}.$$

Let's remark that $L^{(\alpha)i}_{pq}$ can be all identical, and $C^{(\alpha)i}_{(\beta)pq}$ are d-tensor fields which can coincide for any $\alpha = \overline{0,k}$. In this particular situation the covariant derivative D will be called a M-linear connection.

The actions of D^h and D^{v_β} on one d-gauge field $W^{i_1\dots i_r}_{j_1\dots j_s}$ will be denoted by $W^{(\alpha)i_1\dots i_r}_{j_1\dots j_s|m}$, respectively by $W^{(\alpha)i_1\dots i_r}_{j_1\dots j_s}|^{(\beta)}_m, \alpha = \overline{0,k}, \beta = \overline{1,k}$, and are d-tensor fields, but not d-gauge tensors. For that we impose in addition the conditions

$$\tilde{L}^{(\alpha)r}_{pq} = \overline{Y}^{(\alpha)i}_p \cdot \overline{X}^j_q \cdot Y^{(\alpha)r}_m \cdot L^{(\alpha)m}_{ij} - \overline{X}^j_q \cdot \overline{Y}^{(\alpha)i}_p \cdot \frac{\delta Y^{(\alpha)r}_i}{\delta x^j} \tag{2.8}$$

$$C^{(\alpha)r}_{(\beta)pq} = \overline{Y}^{(\alpha)i}_p \cdot \overline{Y}^{(\beta)j}_q \cdot Y^{(\alpha)r}_m \cdot C^{(\alpha)m}_{(\beta)ij} - \overline{Y}^{(\beta)j}_q \cdot \overline{Y}^{(\alpha)i}_p \cdot \frac{\delta Y^{(\alpha)r}_i}{\delta y^{(\beta)j}}, \alpha = \overline{0,k}, \beta = \overline{1,k}.$$

Definition 2.2. A d-linear connection D is a d-gauge linear connection if its coefficients satisfy the transformation rules (2.7) and (2.8). Particularly, if the coefficients do not depend on $\alpha = \overline{0,k}$, then we call it a M-gauge linear connection.

The gauge derivatives will be denoted by $d^{(\alpha)h}_m W^{i_1\dots i_r}_{j_1\dots j_s}$ or $d^{(\alpha)^{v}\beta}_m W^{i_1\dots i_r}_{j_1\dots j_s}$.

3. Metric d-gauge connections of order k

Let be $g_{ij}^{(\alpha)}(x, y^{(1)}, y^{(2)}, \ldots, y^{(k)})a$ system of $(k+1)d$-tensors, symmetric and positively defined, with rank $(g_{ij}^{(\alpha)}) = n, g_{ij}^{(0)}$ being $(h - d$-g.t) and $g_{ij}^{(\beta)}$ are $(v_\beta - d$-g.t), $\beta = \overline{1,k}$, that we shall call h- and v_β-gauge metrics respectively.

If N is a d-gauge linear connection, then:

$$G = g_{ij}^{(0)} dx^i \otimes dx^j + g_{ij}^{(1)} \delta y^{(1)i} \otimes \delta y^{(1)j} + \ldots + g_{ij}^{(k)} \delta y^{(k)i} \otimes \delta y^{(k)j}$$

is globally defined on E_k and the pair (E_k, G) is called metric gauge structure.

Definition 3.1. A d-gauge linear connection D on E_k is a h- (resp. $v_\beta -$) metric connection iff

$$g_{ij|m}^{(\alpha)} = 0 \text{ and } d_m^{(\alpha)h} g_{ij}^{(\alpha)} = 0 \tag{3.1}$$

(resp. $g_{ij|m}^{(\alpha)(\beta)} = 0$ and $d_m^{(\alpha)v_\alpha} g_{ij}^{(\alpha)} = 0), \forall \alpha = \overline{0,k}$.

If D is both h- and v_β-metric then we say that D is a d-gauge metric connection.

Theorem 3.1. The connection

$$L_{ij}^{(\alpha)m} = \frac{1}{2} g^{(\alpha)ms} \{ \frac{\delta g_{sj}^{(\alpha)}}{\delta x^i} + \frac{\delta g_{\text{is}}^{(\alpha)}}{\delta x^j} - \frac{\delta g_{ij}^{(\alpha)}}{\delta x^s} \} \tag{3.2}$$

$$C_{(\beta)ij}^{(\alpha)m} = \frac{1}{2} g^{(\alpha)ms} \{ \frac{\delta g_{sj}^{(\alpha)}}{\delta y^{(\beta)i}} + \frac{\delta g_{\text{is}}^{(\alpha)}}{\delta y^{(\beta)j}} - \frac{\delta g_{ij}^{(\alpha)}}{\delta y^{(\beta)s}} \}, \alpha = \overline{0,k}, \beta = \overline{1,k}$$

is a symmetric d-gauge connection.

An interesting situation is that of f_0-jet gauge transformation. Then $g_{ij}^{(\alpha)}$ can have identical values for $\alpha = \overline{0,k}$, and the conditions (2.7), (2.8) do not depend on $\alpha = \overline{0,k}$. We have:

Proposition 3.1. If $N = (N_{(1)}, \ldots, N_{(k)})$ is a gauge nonlinear gauge connection and f is a f_0-jet gauge transformation, then the following d-connection

$$L_{ij}^m = \frac{\delta N_{(1)j}^m}{\delta y^{(1)i}}, \qquad C_{(\beta)ij}^m = 0, \beta = \overline{1,k} \tag{3.3}$$

is a d-gauge metric connection, and the following connection obtained from (3.3):

$$L_{ij}^m = \frac{\delta N_{(1)j}^m}{\delta y^{(1)i}} + \frac{1}{2} g^{ms} \{ \frac{\delta g_{\text{is}}}{\delta x^j} + \frac{\partial N_{(1)j}^r}{\delta y^{(1)i}} g_{rs} - \frac{\partial N_{(1)j}^r}{\delta y^{(1)s}} g_{ri} \} \tag{3.3'}$$
$$C_{(\beta)ij}^m = \frac{1}{2} g^{ms} \{ \frac{\delta g_{sj}}{\delta y^{(\beta)i}} + \frac{\delta g_{\text{is}}}{\delta y^{(\beta)j}} - \frac{\delta g_{ij}}{\delta y^{(\beta)s}} \},$$

with $g_{ij}^{(\alpha)} = g_{ij}$ is a d-gauge metric connection depending only on N and g_{ij}.

The proof looks like that from [6].

For a f_0-jet gauge transformation we can consider the following k-tangent structure associated to a given nonlinear connection:

$$F(\frac{\delta}{\delta y^{(\alpha)i}}) = \frac{\delta}{\delta y^{(\alpha+1)i}}, F(\frac{\delta}{\delta y^k}) = 0, \alpha = \overline{0, k-1} \tag{3.4}$$

$$F^*(\frac{\delta}{\delta y^{(\alpha+1)i}}) = \frac{\delta}{\delta y^{(\alpha)i}}, F^*(\frac{\delta}{\delta x^i}) = 0$$

globally defined on E_k and determine d-gauge tensors.

The pair (F, G) and (F^*, G) define k-metric almost tangent structures on E_k (for $k = 2$ see [9]). Extending the results from [9] we can prove by analogy that a linear connection D on E_k is a M-linear connection iff $DF = DF^* = 0$.

Theorem 3.2. Let $\bar{\nabla}$ be the Levi-Civita connection associated to the metric G, with $g_{ij}^{(\alpha)} = g_{ij}, \alpha = \overline{0, k}$, for a f_0-jet gauge transformation. Then:

$$D_X Y = v_1 \bar{\nabla}_X (v_1 Y) + v_2 \bar{\nabla}_X (v_2 Y) + \ldots + v_k \bar{\nabla}_X (v_k Y) + \tag{3.5}$$

$$+ F^* v_k \bar{\nabla}_X (v_k F Y) + F^{*2} v_k \bar{\nabla}_X (v_k F^2 Y) + \ldots + F^{*k-1} v_k \bar{\nabla}_X (v_k F^{k-1} Y)$$

is a metric M-linear connection.

The local expression of (3.5) is just (3.2), with $g_{ij}^{(\alpha)} = g_{ij}$, so that it is a symmetric gauge metric connection, called canonic connection.

4. Einstein-Yang-Mills equations of k order

Let be $L_0(x, y^{(1)}, y^{(2)}, \ldots, y^{(k)})$ a Lagrangian defined on a compact set $\Omega \subset \mathbf{R}^{n(k+1)}$, a nonlinear gauge connection $N = (N_{(1)}, \ldots, N_{(k)})$, and a metric gauge structure G on E_k defined by the metric d-gauge fields $g_{ij}^{(\alpha)}(x, y^{(1)}, y^{(2)}, \ldots, y^{(k)})$, $\alpha = \overline{0, k}$.

Let's consider a gauge field Φ, that in applications will be mentioned in the bundle of linear connections. Usually, in calculus, L_0 depends on $z = (x, y^{(1)}, y^{(2)}, \ldots, y^{(k)})$ through Φ and its derivations $\frac{\delta\Phi}{\delta x^i}, \frac{\delta\Phi}{\delta y^{(\beta)i}}$, i.e.

$$L_0(x, y^{(1)}, y^{(2)}, \ldots, y^{(k)}) = L_0(\Phi, \frac{\delta\Phi}{\delta x^i}, \frac{\delta\Phi}{\delta y^{(1)i}}, \ldots, \frac{\delta\Phi}{\delta y^{(k)i}}). \tag{4.1}$$

The action $\int_\Omega L_0(x, y^{(1)}, y^{(2)}, \ldots, y^{(k)}) d\omega$, where $d\omega = dx^1 \wedge \ldots \wedge dy^{(k)n}$ depends on the local coordinates. In order to remove this inconvenience, we consider the Lagrangian density:

$$\mathcal{L}(x, y^{(1)}, \ldots, y^{(k)}) = L_0(x, y^{(1)}, \ldots, y^{(k)}) \cdot \sqrt{g^{(0)} g^{(1)} \ldots g^{(k)}}, \tag{4.2}$$

where $g^{(\alpha)} = \det(g_{ij}^{(\alpha)})$.

Then we have:

$$\mathcal{L}(x, y^{(1)}, \ldots, y^{(k)}) = \mathcal{L}(x, y'^{(1)}, \ldots, y'^{(k)}) \cdot \mathcal{J}, \text{ where } \mathcal{J} = \det(\frac{\partial x'^i}{\partial x^j}). \tag{4.3}$$

Now, the action $I(\Phi) = \int_\Omega \mathcal{L}(x, y^{(1)}, \dots, y^{(k)})d\omega$ is independent of the coordinates on E_k=Osc$^{(k)}$M.

We further assume that the equations of motion for the field Φ follow from the variational principle, $\Delta I(\Phi) = 0$. Applying the rule of product derivation after usual calculation in which we assume the commutation of ∂ with Δ and $\Delta\Phi_{|\Omega} = 0$, it follows that $\Delta I(\Phi) = 0$ iff the Einstein-Yang-Mills equation of k order is satisfied:

$$\frac{\partial\mathcal{L}}{\partial\Phi} - \frac{\partial}{\partial x^i}\frac{\partial\mathcal{L}}{\partial(\frac{\partial\Phi}{\partial x^i})} - \frac{\partial}{\partial y^{(1)i}}\frac{\partial\mathcal{L}}{\partial(\frac{\partial\Phi}{\partial y^{(1)i}})} - \dots - \frac{\partial}{\partial y^{(k)i}}\frac{\partial\mathcal{L}}{\partial(\frac{\partial\Phi}{\partial y^{(k)i}})} = 0. \tag{4.4}$$

Taking into consideration the gauge transformation, is more convenient to express (4.4) in the adapted basis $(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \dots, \frac{\delta}{\delta y^{(k)i}})$. The relation obtained from (4.4) in the adapted basis looks more complicated , but it's easy to verify that (4.4) is invariant with respect to the change of coordinates (1.1) on E_k , but in general, not invariant relative to the gauge transformations (1.4). The main problem is to determine the Lagrangian densities which are gauge-invariant. A solution is provided by [5] for the first order Lagrangian density in the gauge natural bundle. The Utiyama theorem [12] is generalized, and is proven that if $\mathcal{L}$ is a first order gauge invariant Lagrangian of the connection bundle, then $\mathcal{L}$ depends only on the curvature of a given connection ($\mathcal{L} = \mathcal{L}_0 \circ \Omega$, where Ω is the curvature form and $\mathcal{L}_0$ is a fixed Lagrangian density).

In the last section in [5], the idea of gauge invariant Lagrangian of order k is extended to the jet bundle of a G-module. If we consider the frames bundle of order k, the structure equations on this bundle $P^{(k)}(M)$ induce the structure equations on the associated bundle E_k=Osc$^{(k)}M$([10]).

It follows that an important class of gauge invariant Lagrangians is given by the curvatures and the torsions of the d-gauge connection.

The set of all torsions and curvatures for a given d-connection is given in [8] for $k = 2$. In the general case the torsion of a d-connection on E_k has 10 components expressed by the coefficients of the nonlinear connection, the coefficients of the d-connection and their derivatives with respect to the adapted basis $(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \dots, \frac{\delta}{\delta y^{(k)i}})$. For a M-linear connection there exist four components of curvature.

If N is a nonlinear gauge connection, the curvature and torsion coefficients are d-gauge fields. Raising or lowering the indices of these coefficients with the d-gauge metric $g_{ij}^{(\alpha)}$, new gauge tensors are obtained.

In conclusion, a general gauge invariant Lagrangian which depends only on the curvatures and torsions of a metric gauge connection is $\mathcal{L} = L\sqrt{g}$, where L is a linear combination of $\tau_{ij}^m \cdot \tau_m^{ij}$ and $R_{ijk}^m \cdot R_m^{ijk}$, and τ_{ij}^m and R_{ijk}^m are torsions and curvatures of the metric gauge connection.

Particular Lagrangians determined by convenient gauge metrics will be considered in a forecoming paper.

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