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A FIRST ORDER PROLONGATION OF THE CONVENTIONAL SPACE

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Abstract. A variational equation of the third order in three-dimensional space is proposed which describes autoparallel curves of some connection.

We shall focus on three-dimensional (pseudo-) Euclidean space and consider the problem of finding a third-order variational-type equation which can be put down in the form of the autoparallel tansport equation for some non-linear connection. It is common to introduce the latter in the following form:

$$\dddot{x}^{\rho} = f^{\rho}(x^{\beta}, \dot{x}^{\beta}, \ddot{x}^{\beta})\,. \tag{1}$$

On the other side, an Euler-Poisson third-order equation is always of the affine type,

$$\mathcal{A}_{\rho\mu}\dddot{x}^{\mu} + k_{\rho}(x^{\beta}, \dot{x}^{\beta}, \ddot{x}^{\beta}) = 0\,, \tag{2}$$

with a skew-symmetric matrix $\boldsymbol{\mathcal{A}} = (\mathcal{A}_{\rho\beta})$, and, consequently, in the case when the number of equations equals *three*, can not be solved with respect to the derivatives of the third order. What one can undertake in this situation is at most to look for such a variational equation, which describes the geodesic curve only up to reparametrization.

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1. General setting

One algorithm for building up an attached connection to a third order differential equation of a certain class was presented in [1], and we shall follow it here. Although only (pseudo-) Euclidean space will be considered, to give the Reader a sense of general setting, some constructions will be described as developed over an n-dimensional manifold M. A differential equation of the third order will be understood to be a cross-section of the third-order velocity manifold $T''M = J_3(\mathbb{R}_0;\, M) = \{\boldsymbol{x}; \boldsymbol{u}, \dot{\boldsymbol{u}}, \ddot{\boldsymbol{u}}\}$, fibred over the second order one, $T'M = J_2(\mathbb{R}_0;\, M) = \{\boldsymbol{x}; \boldsymbol{u}, \dot{\boldsymbol{u}},\}$. These fibred manifolds are associated, as fibre bundles, to the principle fibre bundles of the third-order and second-order frames, $H'' = \tilde{J}_3(\mathbb{R}^n_0;\, M) = \{x^\rho; r^\rho_{\beta\gamma}, r^\rho_{\beta\gamma\delta},\}$, and $H' = \tilde{J}_2(\mathbb{R}^n_0;\, M) = \{x^\rho; r^\rho_{\beta\gamma}\}$, where the tilde means that only invertible jets count, and also we shall denote the inverse to the matrix (r^ρ_β) by $(\overset{-1}{r}{}^\gamma_\delta)$. The cotangent space to the manifold H' is spanned by the following set of differential forms (with coefficients from above the manifold H'')

$$
\begin{aligned}
\omega^\rho &\doteqdot \overset{-1}{r}{}^\rho_\mu dx^\mu \\
\omega^\rho_\beta &\doteqdot \overset{-1}{r}{}^\rho_\mu dr^\mu_\beta - \overset{-1}{r}{}^\rho_\mu r^\mu_{\beta\nu} \overset{-1}{r}{}^\nu_\lambda dx^\lambda \\
\omega^\rho_{\beta\gamma} &\doteqdot \overset{-1}{r}{}^\rho_\mu dr^\mu_{\beta\gamma} - \overset{-1}{r}{}^\rho_\mu r^\mu_{\beta\nu} \overset{-1}{r}{}^\nu_\lambda dr^\lambda_\gamma - \overset{-1}{r}{}^\rho_\mu r^\mu_{\gamma\nu} \overset{-1}{r}{}^\nu_\lambda dr^\lambda_\beta \\
&\qquad + \overset{-1}{r}{}^\rho_\mu r^\mu_{\beta\nu} \overset{-1}{r}{}^\nu_\lambda r^\lambda_{\gamma\iota} \overset{-1}{r}{}^\iota_\sigma dx^\sigma + \overset{-1}{r}{}^\rho_\mu r^\mu_{\gamma\nu} \overset{-1}{r}{}^\nu_\lambda r^\lambda_{\beta\iota} \overset{-1}{r}{}^\iota_\sigma dx^\sigma - \overset{-1}{r}{}^\rho_\mu r^\mu_{\nu\beta\gamma} \overset{-1}{r}{}^\nu_\lambda dx^\lambda \,.
\end{aligned}
$$

To span the cotangent space to the manifold H'' one more string of forms drops in (we present their definition through a recurcive relation, which appears more simple for any order as well),

$$
r^\rho_\delta \omega^\delta_{\iota\beta\gamma} = dr^\rho_{\iota\beta\gamma} - r^\rho_{\mu\iota}\omega^\mu_{\beta\gamma} - r^\rho_{\mu\beta}\omega^\mu_{\iota\gamma} - r^\rho_{\mu\gamma}\omega^\mu_{\beta\iota} - r^\rho_{\mu\iota\beta}\omega^\mu_\gamma - r^\rho_{\mu\gamma\beta}\omega^\mu_\iota - r^\rho_{\mu\iota\gamma}\omega^\mu_\beta - r^\rho_{\mu\iota\beta\gamma}\omega^\mu \,.
$$

These differential forms constitute a global object, intrinsically defined in [2].

Rather then to proceed with the cross-section $f\colon T'M \to T''M$, one could wish to develop some calculus on the corresponding principle bundles. By the commutative diagram

$$
\begin{array}{ccc}
H'' \times \mathbb{V}'' & \xrightarrow{\rho''} & T''M \\
{\scriptstyle\Phi}\big\uparrow & & \big\uparrow{\scriptstyle f} \\
H' \times \mathbb{V}' & \xrightarrow[\rho']{} & T'M
\end{array}
\tag{3}
$$

the mapping Φ has to be both an equivariant one and a cross-section. The typical fibre $\mathbb{V}'' = J_3(\mathbb{R}_0;\, \mathbb{R}^n_0) = \{U^\rho, \dot{U}^\rho, \ddot{U}^\rho\}$ undergoes such left action of the group $GL''(n) = \tilde{J}_3(\mathbb{R}^n_0;\, \mathbb{R}^n_0) = \{s^\rho_\beta, s^\rho_{\beta\gamma}, s^\rho_{\beta\gamma\delta}\}$, that the quotient map $\rho''(r.s, s^{-1}.U) = \rho''(r, U)$ is described explicitly by

$$
\begin{aligned}
u^\rho &= r^\rho_\mu U^\mu \\
\dot{u}^\rho &= r^\rho_\mu \dot{U}^\mu + r^\rho_{\mu\nu} U^\mu U^\nu \\
\ddot{u}^\rho &= r^\rho_\mu \ddot{U}^\mu + 3 r^\rho_{\mu\nu} \dot{U}^\mu U^\nu + r^\rho_{\mu\nu\lambda} U^\mu U^\nu U^\lambda \,.
\end{aligned}
$$

A tangent vector to the product manifold $H'' \times \mathbb{V}''$,

$$\boldsymbol{\mathfrak{a}} = a^\rho \frac{\partial}{\partial x^\rho} + a^\rho_\mu \frac{\partial}{\partial r^\rho_\mu} + a^\rho_{\mu\nu} \frac{\partial}{\partial r^\rho_{\mu\nu}} + a^\rho_{\mu\nu\lambda} \frac{\partial}{\partial r^\rho_{\mu\nu\lambda}} + \upsilon^\rho \frac{\partial}{\partial U^\rho} + \dot{\upsilon}^\rho \frac{\partial}{\partial \dot{U}^\rho} + \ddot{\upsilon}^\rho \frac{\partial}{\partial \ddot{U}^\rho} \,,$$

is vertical with respect to the projection ρ'' if and only if

$$\begin{gathered} a^\rho = 0 \\ r^\rho_\mu \upsilon^\mu + a^\rho_\mu U^\mu = 0 \\ r^\rho_\mu \dot{\upsilon}^\mu + a^\rho_\mu \dot{U}^\mu + a^\rho_{\mu\nu} U^\mu U^\nu - 2 r^\rho_{\mu\nu} \overset{-1}{r}{}^\mu_\lambda a^\lambda_\iota U^\nu U^\iota = 0 \end{gathered}$$

$$\begin{aligned} r^\rho_\mu \ddot{\upsilon}^\mu &+ a^\rho_\mu \ddot{U}^\mu + 3 a^\rho_{\mu\nu} \dot{U}^\mu U^\nu + a^\rho_{\mu\nu\lambda} U^\mu U^\nu U^\lambda \\ &- 3 r^\rho_{\mu\nu} \overset{-1}{r}{}^\mu_\lambda U^\nu (a^\lambda_\iota \dot{U}^\iota + a^\lambda_{\iota\sigma} U^\iota U^\sigma - 2 r^\lambda_{\iota\sigma} \overset{-1}{r}{}^\iota_\beta a^\beta_\delta U^\sigma U^\delta) \\ &\qquad - 3 r^\rho_{\mu\nu} \overset{-1}{r}{}^\nu_\lambda a^\lambda_\iota \dot{U}^\mu U^\iota - 3 r^\rho_{\mu\nu\lambda} \overset{-1}{r}{}^\lambda_\iota a^\iota_\sigma U^\mu U^\nu U^\sigma = 0 \,. \end{aligned}$$

If the map Φ is equivariant, then its Lie derivative with respect to an arbitrary pair of vertical vector fields $\boldsymbol{\mathfrak{a}''}$ and $\boldsymbol{\mathfrak{a}'}$ on the manifolds $H'' \times \mathbb{V}''$ and $H' \times \mathbb{V}'$ is zero (i.e. vector fields $\boldsymbol{\mathfrak{a}''}$ and $\boldsymbol{\mathfrak{a}'}$ are Φ-related):

$$T\Phi \circ \boldsymbol{\mathfrak{a}''} = \boldsymbol{\mathfrak{a}'} \circ \Phi \,.$$

The map $\eta^{-1}\Phi \colon H'' \times \mathbb{V}' \to H'' \times \mathbb{V}''$, induced by the projection $\eta \colon H'' \to H'$,

$$\eta^{-1}\Phi(r, U) = \Phi(\eta r, U) \,,$$

is not fibred over the identity in H''. Nevertheless, there can always be found an element $(\delta^\rho_\mu, 0, s^\rho_{\mu\nu\lambda}) \in GL''(n)$ such that

$$r^\rho_\sigma s^\sigma_{\mu\nu\lambda} + r^\rho_{\mu\nu\lambda} = \Phi^\rho_{\mu\nu\lambda}(x^\beta; r^\beta_\iota, r^\beta_{\iota\gamma}, U^\beta, \dot{U}^\beta) \,. \tag{4}$$

We define the fibred morphism F over the identity in H'' as a family of cross-sections $F(r)$ of the fibration $\pi \colon \mathbb{V}'' \to \mathbb{V}'$ by means of

$$F^\rho(x^\beta; r^\beta_\mu, r^\beta_{\mu\nu}, r^\beta_{\mu\nu\lambda}, U^\beta, \dot{U}^\beta) = \ddot{\Phi}^\rho + s^\rho_{\iota\sigma\gamma} U^\iota U^\sigma U^\gamma \,,$$

with $s^\rho_{\iota\sigma\gamma}$ defined from (4). Then, by virtue of Φ equivariant, for every vertical vector field $\boldsymbol{\mathfrak{a}}$ on the manifold $H'' \times \mathbb{V}''$ we have

$$TF \circ (T(id) \times T\pi) \circ \boldsymbol{\mathfrak{a}} = \boldsymbol{\mathfrak{a}} \circ F \circ (id \times \pi) \,, \tag{5}$$

as can be seen from the next diagram by the appropriate chose of paths

$$\begin{tikzcd}
TH'' \times T\mathbb{V}'' \arrow[d, equal] & TH'' \times T\mathbb{V}' \arrow[l, "TF"'] \arrow[d, equal] \\
TH'' \times T\mathbb{V}'' \arrow[r, "T(id)\times T\pi"] & TH'' \times T\mathbb{V}' \\
H'' \times \mathbb{V}'' \arrow[u, "\boldsymbol{\mathfrak{a}}"] \arrow[r, "id\times\pi"] \arrow[d, equal] & H'' \times \mathbb{V}' \arrow[u, "(T(id)\times T\pi)\circ\boldsymbol{\mathfrak{a}}"'] \arrow[d, equal] \\
H'' \times \mathbb{V}'' & H'' \times \mathbb{V}' \arrow[l, "F"']
\end{tikzcd}$$

The kernel of $T\rho''$ annuls the following one-forms:

$$\begin{aligned}
&\omega^\rho\,; \\
&\triangle U^\rho \doteqdot dU^\rho + U^\mu \omega^\rho_\mu\,; \\
&\triangle \dot U^\rho \doteqdot d\dot U^\rho + \dot U^\mu \omega^\rho_\mu + U^\mu U^\nu \omega^\rho_{\mu\nu}\,; \\
&\triangle \ddot U^\rho \doteqdot d\ddot U^\rho + \ddot U^\mu \omega^\rho_\mu + 3U^\mu \dot U^\nu \omega^\rho_{\mu\nu} + U^\mu U^\nu U^\lambda \omega^\rho_{\mu\nu\lambda}\,.
\end{aligned} \tag{6}$$

By calculating the Lie derivative of the differential forms (6) along a vertical vector field it turns out that the exterior differential system, generated by (6) is invariant under the action of the group $GL''(n)$ upon the manifold $H'' \times \mathbb{V}''$.

If the functions F^ρ *satisfy* (5), *then the differential forms*

$$\triangle F^\rho \doteqdot dF^\rho + \ddot U^\mu \omega^\rho_\mu + 3U^\mu \dot U^\nu \omega^\rho_{\mu\nu} + U^\mu U^\nu U^\lambda \omega^\rho_{\mu\nu\lambda}$$

expand into the differential forms (6) *alone:*

$$\triangle F^\rho = F^{\rho}{}^{2}_{\mu} \triangle \dot U^\mu + F^{\rho}{}^{1}_{\mu} \triangle U^\mu + F^{\rho}{}^{0}_{\mu} \omega^\mu\,. \tag{7}$$

The concept of second order connection involves the quotient manifold $\Gamma H' = TH'/GL'(n)$ with respect to the standard action of the group $GL'(n)$:

$$a^\rho \frac{\partial}{\partial x^\rho} + a^\rho_\mu \frac{\partial}{\partial r^\rho_\mu} + a^\rho_{\mu\nu} \frac{\partial}{\partial r^\rho_{\mu\nu}} \mapsto a^\rho \frac{\partial}{\partial x^\rho} + s^\nu_\mu a^\rho_\nu \frac{\partial}{\partial r^\rho_\mu} + (s^\lambda_{\mu\nu} a^\rho_\lambda + s^\lambda_\mu s^\iota_\nu a^\rho_{\lambda\iota}) \frac{\partial}{\partial r^\rho_{\mu\nu}}\,.$$

The manifold $\Gamma H'$ projects onto the manifold $T''M$ by means of the following mapping of TH', compatible with this action:

$$(a^\rho, a^\rho_\mu, a^\rho_{\mu\nu}) \mapsto (a^\rho, a^\rho_\mu \overset{-1}{r}{}^\mu_\nu a^\nu, a^\rho_{\lambda\iota} \overset{-1}{r}{}^\lambda_\mu \overset{-1}{r}{}^\iota_\nu a^\mu a^\nu - a^\rho_\delta \overset{-1}{r}{}^\delta_\sigma r^\sigma_{\lambda\iota} \overset{-1}{r}{}^\lambda_\mu \overset{-1}{r}{}^\iota_\nu a^\mu a^\nu + a^\rho_\iota \overset{-1}{r}{}^\iota_\lambda a^\lambda_\mu \overset{-1}{r}{}^\mu_\nu a^\nu)\,.$$

Definition[3]**.** A second order connection is given by a map $\gamma\colon T'M \to \Gamma H'$, which is identity in TM

By means of the commutative diagram

$$\begin{tikzcd}
\Gamma H' \arrow[r] & T''M \\
T'M \arrow[u, "\gamma"] \arrow[r, equal] & T'M \arrow[u, "\widehat\gamma"']
\end{tikzcd}$$

every connection γ defines a morphism of manifolds $\widehat\gamma$. The very similar way to (7) this map γ may be described through the structure equations,

$$\begin{aligned}
\triangle \Gamma^\rho_\beta &= \Gamma^{\rho}_{\beta}{}^{2}_{\mu} \triangle \dot U^\mu + \Gamma^{\rho}_{\beta}{}^{1}_{\mu} \triangle U^\mu + \Gamma^{\rho}_{\beta}{}^{0}_{\mu} \omega^\mu\,, \\
\triangle \Gamma^\rho_{\beta\gamma} &= \Gamma^{\rho}_{\beta\gamma}{}^{2}_{\mu} \triangle \dot U^\mu + \Gamma^{\rho}_{\beta\gamma}{}^{1}_{\mu} \triangle U^\mu + \Gamma^{\rho}_{\beta\gamma}{}^{0}_{\mu} \omega^\mu\,,
\end{aligned}$$

where the differential forms $\triangle\Gamma^\rho_\beta$, $\triangle\Gamma^\rho_{\beta\gamma}$ are build up from the differentials of the functions $(\Gamma^\rho_\beta, \Gamma^\rho_{\beta\gamma})$, which represent the map γ, as follows:

$$\begin{aligned}
\triangle\Gamma^\rho_\beta &\doteqdot d\Gamma^\rho_\beta + \Gamma^\mu_\beta\omega^\rho_\mu - \Gamma^\rho_\mu\omega^\mu_\beta + U^\mu\omega^\rho_{\beta\mu}\,,\\
\triangle\Gamma^\rho_{\beta\gamma} &\doteqdot d\Gamma^\rho_{\beta\gamma} + \Gamma^\mu_{\beta\gamma}\omega^\rho_\mu - \Gamma^\rho_{\beta\mu}\omega^\mu_\gamma - \Gamma^\rho_{\mu\gamma}\omega^\mu_\beta + \Gamma^\mu_\beta\omega^\rho_{\mu\gamma} + \Gamma^\mu_\gamma\omega^\rho_{\mu\beta} - \Gamma^\rho_\mu\omega^\mu_{\beta\gamma} + U^\mu\omega^\rho_{\beta\gamma\mu}\,.
\end{aligned}$$

Connection γ is called *stable* if it projects onto the identity in $T'M$. In this case the morphism $\widehat{\gamma}$ is a cross-section and thus defines a third-order differential equation of the type (1).

To discuss a weaker condition of a quasi-stable connection, we recall that the group $GL'(1)$ acts on the right upon the space $T'M$ by parameter transformations. The generators are:

$$\begin{aligned}
\mathfrak{p}_1 &= u^\rho\frac{\partial}{\partial u^\rho} + 2\dot{u}^\rho\frac{\partial}{\partial \dot{u}^\rho}\,,\\
\mathfrak{p}_2 &= u^\rho\frac{\partial}{\partial \dot{u}^\rho}\,.
\end{aligned}$$

The quotient space with respect to this action is the manifold $C'M$ of contact elements, locally arranged as $\mathbb{R}\times T'\mathbb{R}^{n-1}$. Connection γ is said to be *quasi-stable* if it projects onto the identity in $C'M$.

In case of quasi-stable connection it is possible to introduce[1,3] the notion of parallel transport in such a way, that the autoparallel curves of this connection will be described in the typical fibre $\mathbb{V}''$ of the fibre bundle $T''M$ by means of the equation

$$\ddot{U}^\rho = \Gamma^\rho_\mu\dot{U}^\mu + \Gamma^\rho_{\mu\nu}U^\mu U^\nu + \lambda^{(2)}\dot{U}^\rho + \lambda^{(1)}U^\rho\,. \tag{9}$$

If the quasi-stable connection γ is stable, the functions $\lambda^{(1)}$ and $\lambda^{(2)}$ both vanish.

Not every equation (1) can be rearranged in the form (9). The crucial idea consists in applying a somewhat technical trick of reparametrization. If the map f in (1) or (3) defines in the consistent manner some equation on the manifold $C'M$, and if we think of f as of a vector field $\mathfrak{f}$ on the manifold $T'M$ by the inclusion $T''M \hookrightarrow TT'M$, then the Lie brackets $[\mathfrak{p}_1,\mathfrak{f}]$ and $[\mathfrak{p}_2,\mathfrak{f}]$ differ from a multiply of $\mathfrak{f}$ by some vertical field with respect to the projection

$$\wp\colon T'M \to C'M\,. \tag{10}$$

In fact, a stronger condition holds:

$$\begin{cases}(T\wp)\,[\mathfrak{p}_1,\mathfrak{f}] = (T\wp)\,\mathfrak{f}\\ (T\wp)\,[\mathfrak{p}_2,\mathfrak{f}] = 0\end{cases}$$

In terms of the representation (7) the above condition amounts to the following two equations with Lagrange multiplies μ and λ,

$$3F^\rho - F^{\rho}{}^1_\iota U^\iota - 2F^{\rho}{}^2_\iota\dot{U}^\iota = 3\mu U^\rho \tag{11}$$

$$3\dot{U}^\rho - F^{\rho}{}^2_\iota U^\iota = 3\lambda U^\rho\,. \tag{12}$$

The multipliers μ and λ are functions on the manifold $H' \times \mathbb{V}'$, and in order them to represent some well-defined functions on the manifold $T'M$, they both have to satisfy the condition of $GL'(n)$-invariance of the type $\langle \mathfrak{a}, d\mu \rangle = 0$ for any ρ'-vertical vector $\mathfrak{a}$, which amounts to the following system of partial differential equations:

$$\begin{aligned} \frac{\partial\mu}{\partial\dot U^\rho} U^\beta U^\gamma &= r^\nu_\rho \frac{\partial\mu}{\partial r^\nu_{\beta\gamma}} \\ \frac{\partial\mu}{\partial U^\rho} U^\beta - \frac{\partial\mu}{\partial\dot U^\rho} \dot U^\beta &= 2 r^\delta_{\rho\nu} \frac{\partial\mu}{\partial r^\delta_{\beta\nu}} + r^\delta_\rho \frac{\partial\mu}{\partial r^\delta_\beta} \,. \end{aligned} \tag{13}$$

Definition[3]**.** The equation (1) is *reducible* if (11,12) holds for the representation (7). It will be called *strictly reducible* if both $\mu = 0$ and $\lambda = 0$.

Consider now a (second order nonlinear) connection, the coefficients Γ^ρ_β, $\Gamma^\rho_{\beta\gamma}$ of which are constructed from the coefficients of the first-order prolongation of the differential system (7),

$$dF^{\rho\,2}_{\ \beta} + F^{\mu\,2}_{\ \beta}\omega^\rho_\mu - F^{\rho\,2}_{\ \mu}\omega^\mu_\beta + 3U^\mu\omega^\rho_{\beta\mu} = F^{\rho\,2\,0}_{\ \beta\,\mu}\omega^\mu + F^{\rho\,2\,1}_{\ \beta\,\mu}\triangle U^\mu + F^{\rho\,2\,2}_{\ \beta\,\mu}\triangle \dot U^\mu \,,$$

according to the following prescription:

$$\begin{aligned} \Gamma^\rho_\beta &= \tfrac{1}{3} F^{\rho\,2}_{\ \beta}\,; \\ \Gamma^\rho{}_{\beta\gamma} &= \tfrac{1}{2}(\Pi^\rho_{\beta\gamma} + \Pi^\rho_{\gamma\beta})\,, \quad \text{where} \\ \Pi^\rho_{\beta\gamma} &= \tfrac{1}{3} F^{\rho\,2\,0}_{\ \beta\gamma} + \tfrac{1}{9}(F^{\rho\,2\,2}_{\ \beta\,\mu} F^{\mu\,1}_{\ \gamma} + F^{\rho\,2\,1}_{\ \beta\,\mu} F^{\mu\,2}_{\ \gamma}) + \tfrac{2}{27} F^{\rho\,2\,2}_{\ \beta\,\mu} F^{\mu\,2}_{\ \nu} F^{\nu\,2}_{\ \gamma}\,. \end{aligned} \tag{14}$$

Let us agree to call the connection, constructed accordingly to the formulae (14), be *attached* to the differential equation (1)

Proposition[1]**.** *The connection, attached to a reducible differential equation, is quasi-stable. The equation* (9) *of the autoparallel curves of the connection, attached to a reducible differential equation, coincides with the initial equation* (1)*. If* (1) *is strictly reducible, then the attached connection is stable.*

The functions $\lambda^{(1)}$ and $\lambda^{(2)}$ in (9) are expressed through the functions μ from (11) and λ from (12) in terms of the coefficients of the differential $d\lambda$,

$$d\lambda = \lambda^0_\mu \omega^\mu + \lambda^1_\mu \triangle U^\mu + \lambda^2_\mu \triangle \dot U^\mu \,,$$

according to the formulae below:

$$\begin{gathered} \lambda^{(1)} = \lambda^0_\nu U^\nu + \lambda^1_\nu \dot U^\nu \lambda^2_\nu \ddot U^\nu + \mu(1 - \lambda^2_\nu U^\nu) - \lambda(\lambda^1_\nu U^\nu + \tfrac{2}{3}\lambda^2_\nu F^{\nu\,2}_{\ \iota} U^\iota) - 2\lambda^2 \,, \\ \lambda^{(2)} = 2\lambda \,. \end{gathered}$$

In view of (13),

$$\lambda^1_\rho = \frac{\partial\lambda}{\partial U^\rho}\,, \qquad \lambda^2_\rho = \frac{\partial\lambda}{\partial \dot U^\rho}\,.$$

2. Euclidean space. Variational equation.

As declared, we look for a third order differential equation in (pseudo-) Euclidean space $\mathbf{E}_i^3$, which would be derivable from a Lagrangian. The dimension of the space is *three*. As mentioned at the very beginning of the present contribution, we cannot expect such equation to exist in the form, solved with respect to the highest (i.e of the third order) derivatives. So we shall first settle down on the manifold

$$\mathbb{R} \times T'\mathbf{E}_k^2$$

and afterwards go all the way back to the manifold $T'\mathbf{E}_i^3$ along the projection of (10), which in the canonical coordinates is so expressed:

$$\begin{aligned} \mathsf{t} \circ \wp &= x^0 \\ \mathsf{x}^a \circ \wp &= x^a \\ \mathsf{v}^a \circ \wp &= \frac{u^a}{u^0} \\ \mathsf{v}'^a \circ \wp &= \frac{\dot{u}^a}{{u_0}^2} - \frac{\dot{u}_0}{{u_0}^3} u^a \,. \end{aligned}$$

Let us concentrate on a system of two third-order ordinary differential equations

$$\mathsf{E}_a = 0\,. \tag{15}$$

We introduce a vector valued differential one-form

$$\boldsymbol{\epsilon} = \mathsf{E}_a\, d\mathsf{x}^a \otimes d\mathsf{t}\,, \tag{16}$$

where the expressions E_a are called the Euler-Poisson expressions.[1] Applying the general criterion of [5] for an arbitrary system of differential equations to be a system of Euler-Poisson equations, it was established in [6] that the vector expression $\mathbf{E} = \{\mathsf{E}_a\}$ in (15) must have the shape

$$\mathbf{E} = \mathbb{A} \,.\, \mathbf{v}'' + (\mathbf{v}' .\, \partial_{\mathbf{v}})\, \mathbb{A} \,.\, \mathbf{v}' + \mathbb{B} \,.\, \mathbf{v}' + \mathbf{c}\,, \tag{17}$$

where the skew-symmetric matrix $\mathbb{A}$, the matrix $\mathbb{B}$, and the column vector $\mathbf{c}$ depend on the variables t, $\mathbf{x}$, $\mathbf{v} = d\mathbf{x}/d\mathsf{t}$, and satisfy the following system of partial differential equations in t, x^a, and v^a [6, 7]

$$\begin{gathered} \partial_{\mathsf{v}[a} \mathsf{A}_{bc]} = 0 \\ 2\,\mathsf{B}_{[ab]} - 3\,\mathbf{D}_1 \mathsf{A}_{ab} = 0 \\ 2\,\partial_{\mathsf{v}[a} \mathsf{B}_{b]c} - 4\,\partial_{\mathsf{x}[a} \mathsf{A}_{b]c} + \partial_{\mathsf{x}^c} \mathsf{A}_{ab} + 2\,\mathbf{D}_1 \partial_{\mathsf{v}^c} \mathsf{A}_{ab} = 0 \\ \partial_{\mathsf{v}(a} \mathsf{c}_{b)} - \mathbf{D}_1 \mathsf{B}_{(ab)} = 0 \\ 2\,\partial_{\mathsf{v}^c}\, \partial_{\mathsf{v}[a} \mathsf{c}_{b]} - 4\,\partial_{\mathsf{x}[a} \mathsf{B}_{b]c} + {\mathbf{D}_1}^2\, \partial_{\mathsf{v}^c} \mathsf{A}_{ab} + 6\,\mathbf{D}_1 \partial_{\mathsf{x}[a} \mathsf{A}_{bc]} = 0 \\ 4\,\partial_{\mathsf{x}[a} \mathsf{c}_{b]} - 2\,\mathbf{D}_1 \partial_{\mathsf{v}[a} \mathsf{c}_{b]} - {\mathbf{D}_1}^3\, \mathsf{A}_{ab} = 0\,. \end{gathered} \tag{18}$$

[1] This is an alternative way to interpret the notion of the Euler morphism, the latter having been considered by Kolář in [4].

In (18) $\mathbf{D}_1$ and farther below $\mathbf{D}_2$ denote the generators of the Cartan distribution,

$$\mathbf{D}_2 = \mathbf{v}'.\,\partial_{\mathbf{v}} + \mathbf{D}_1\,,$$
$$\mathbf{D}_1 = \partial_{\mathsf{t}} + \mathbf{v}\,.\,\partial_{\mathbf{x}}\,.$$

Let $\boldsymbol{\theta}_2$, $\boldsymbol{\theta}_3$ denote the canonical contact forms

$$\boldsymbol{\theta}_3 = \frac{\partial}{\partial {\mathsf{v}'}^a} \otimes (d{\mathsf{v}'}^a - {\mathsf{v}''}^a d\mathsf{t}) + \boldsymbol{\theta}_2\,,$$
$$\boldsymbol{\theta}_2 = \frac{\partial}{\partial \mathsf{v}^a} \otimes (d\mathsf{v}^a - {\mathsf{v}'}^a d\mathsf{t}) + \frac{\partial}{\partial \mathsf{x}^a} \otimes (d\mathsf{x}^a - \mathsf{v}^a d\mathsf{t})\,.$$

Along with the differential form $\boldsymbol{\epsilon}$ we introduce another one, $\underline{\boldsymbol{\epsilon}}$,

$$\underline{\boldsymbol{\epsilon}} = \mathsf{A}_{ab}\, d\mathsf{x}^a \otimes d{\mathsf{v}'}^b + \mathsf{k}_a\, d\mathsf{x}^a \otimes d\mathsf{t}\,,$$
$$\mathbf{k} = (\mathbf{v}'.\,\partial_{\mathbf{v}})\,\mathbb{A}\,.\,\mathbf{v}' + \mathbb{B}\,.\,\mathbf{v}' + \mathbf{c}\,.$$

Exterior differential systems, generated by the forms $\boldsymbol{\epsilon}$ and $\underline{\boldsymbol{\epsilon}}$, are equivalent:

$$\underline{\boldsymbol{\epsilon}} - \boldsymbol{\epsilon} = (\mathsf{A}_{ab}\, d\mathsf{x}^a \otimes d{\mathsf{v}'}^b)\,\bar{\wedge}\,\boldsymbol{\theta}_3\,.$$

Now it is time to put in the concept of *symmetry*. Let

$$\boldsymbol{\mathfrak{x}} = \tau\frac{\partial}{\partial \mathsf{t}} + \mathfrak{x}^a\frac{\partial}{\partial \mathsf{x}^a} \tag{19}$$

denote the generator of some local group of transformations of the manifold $\mathbb{R}\times\mathbf{E}^2_k$, its successive prolongations to the space $J_s(\mathbb{R};\mathbf{E}^2_k) \approx \mathbb{R}\times T^s\mathbf{E}^2_k$ denoted by $\boldsymbol{\mathfrak{x}}_{\boldsymbol{s}}$:

$$\boldsymbol{\mathfrak{x}}_2 = \mathfrak{v}^a\frac{\partial}{\partial {\mathsf{v}'}^a} + \boldsymbol{\mathfrak{x}}_1\,.$$

The demand that the exterior differential system, generated by the vector valued differential form $\underline{\boldsymbol{\epsilon}}$, be invariant under the infinitesimal transformation $\boldsymbol{\mathfrak{x}}$ incarnates into the following equation[2]

$$\mathbf{L}(\boldsymbol{\mathfrak{x}}_2)(\underline{\boldsymbol{\epsilon}}) = \boldsymbol{\Xi}\,.\,\underline{\boldsymbol{\epsilon}} + \boldsymbol{\beta}\,\bar{\wedge}\,\boldsymbol{\theta}_2\,, \tag{20}$$

where the elements of the matrix $\boldsymbol{\Xi}$ and the coefficients of the semi-basic $T^*\mathbf{E}^2_k$-valued one-form $\boldsymbol{\beta}$ depend upon the variables t, $\mathbf{x}$, $\mathbf{v}$, and $\mathbf{v}'$. Both $\boldsymbol{\Xi}$ and $\boldsymbol{\beta}$ play the role of Lagrange multipliers. Splitting of equation (20) with respect to independent differentials $d\mathsf{t}$, $d\mathsf{x}^a$, $d\mathsf{v}^a$, and $d{\mathsf{v}'}^a$, results in the following system of partial differential equations

$$\begin{aligned} \mathbf{L}(\boldsymbol{\mathfrak{x}}_1)\mathsf{A}_{ab} &= \Xi_a{}^c\,\mathsf{A}_{cb} - \mathsf{A}_{ac}\frac{\partial}{\partial {\mathsf{v}'}^b}\mathfrak{v}^c \\ \mathbf{L}(\boldsymbol{\mathfrak{x}}_2)\mathsf{k}_a &= \Xi_a{}^b\,\mathsf{k}_b - \mathsf{A}_{ab}\mathbf{D}_2\mathfrak{v}^b - \mathsf{k}_a\mathbf{D}_1\tau\,. \end{aligned} \tag{21}$$

[2] The notion of vector bundle valued exterior differential systems invariance was introduced in [8], see also [9].

Variational problem in parametric form. Consider for a moment an r^{th}-order variational problem in parametric form, set by a Lagrangian

$$\ell(\zeta, x^\rho, u^\rho, \ldots, \overset{r-1}{u}{}^\rho)d\zeta$$

on the space $J_r(\mathbb{R}; M)$. As long as we limit ourselves only to the case of autonomous Euler-Poisson equations,

$$\mathcal{E}_\rho = 0\,, \text{ see } (2), \tag{22}$$

the differential form

$$\varepsilon = \mathcal{E}_\rho\, dx^\rho \otimes d\zeta \tag{23}$$

may *globally* be deprived of the factor $d\zeta$, constituting thus a *globally* defined T^*M-valued density

$$\boldsymbol{e} = \mathcal{E}_\rho dx^\rho\,. \tag{24}$$

Now the projection $\wp : T^rM \to C^rM$ can be employed to generate an autonomous variational problem set over T^rM from every one variational problem over C^rM.

Lemma. *In terms of a local chart, if in* (16) *the local semi-basic differential form* ϵ *corresponds to the Lagrangian*

$$L d\mathsf{t}\,,$$

then the vector valued density

$$\boldsymbol{e} = u^a(\mathsf{E}_a \circ \wp)dx^0 + u^0(\mathsf{E}_a \circ \wp)dx^a \tag{25}$$

corresponds to the Lagrangian

$$\ell(\zeta, x^\rho, u^\rho, \ldots, \overset{r-1}{u}{}^\rho)d\zeta = \mathcal{L}(x^\rho, u^\rho, \ldots, \overset{r-1}{u}{}^\rho)d\zeta$$

with the Lagrange function

$$\mathcal{L} = u^0 L \circ \wp\,. \tag{26}$$

Let us return to the third-order case. The relations between quantities, allocated on the space of contact elements $C'M \overset{\text{def}}{=} C^2M$ and the corresponding quantities on the second-order velocity space $T'M \overset{\text{def}}{=} T^2M$, expressed by (22, 24, and 25), say that in (2) we have

$$\boldsymbol{k} \;=\; (\,\dot{\boldsymbol{u}}\,.\,\boldsymbol{\partial}_{\boldsymbol{u}}\,)\,\boldsymbol{\mathcal{A}}\,.\,\dot{\boldsymbol{u}} \;+\; \boldsymbol{\mathcal{B}}\,.\,\dot{\boldsymbol{u}} \;+\; \boldsymbol{c}$$

with

$$\mathcal{A}_{ab} = (u^0)^{-2}\cdot(\mathsf{A}_{ab}\circ\wp), \quad \mathcal{B}_{ab} = (u^0)^{-1}(\mathsf{B}_{ab}\circ\wp), \quad c_a = u^0(\mathsf{c}_a\circ\wp)\,, \tag{27}$$

and that the Weierstrass constraint holds:

$$\boldsymbol{\mathcal{A}}\,.\,\boldsymbol{u} \equiv 0, \quad \boldsymbol{k}\,.\,\boldsymbol{u} \equiv 0\,.$$

Circles and hyperbolae. Let in (19) generator $\mathfrak{x}$ correspond to the (pseudo-) orthogonal transformations of a three-dimensional (pseudo-) Euclidean plain. Solving (18) together with (21), we establish the expressions (17) for this case:

$$\mathbf{E} = -\frac{*\mathbf{v}''}{(1+\mathbf{v}\cdot\mathbf{v})^{3/2}} + 3\,\frac{*\mathbf{v}'}{(1+\mathbf{v}\cdot\mathbf{v})^{5/2}}(\mathbf{v}'{\cdot}\mathbf{v}) + \frac{m}{(1+\mathbf{v}\cdot\mathbf{v})^{3/2}}\big[(1{+}\mathbf{v}{\cdot}\mathbf{v})\mathbf{v}' - (\mathbf{v}'{\cdot}\mathbf{v})\mathbf{v}\big]\,. \tag{28}$$

(The dual to some vector $\mathbf{w}$ is known to be defined with the help of the skew-symmetric Levi-Civita symbol e_{ab} by means of $(*\mathbf{w})_a = \mathsf{e}_{ba}\mathsf{w}^b$.) To convert (28) into a "homogeneous" three-dimensional form one applies (27) and obtains the final Euler-Poisson equations, which are naturally connected to the second prolongation of the transformation group $[\mathbf{E}(3,i), \mathbf{E}_i^3]$:

$$\boxed{\boldsymbol{\mathcal{E}} = \frac{\ddot{\boldsymbol{u}}\times\boldsymbol{u}}{\|\boldsymbol{u}\|^3} \;-\; 3\,\frac{\dot{\boldsymbol{u}}\times\boldsymbol{u}}{\|\boldsymbol{u}\|^5}(\dot{\boldsymbol{u}}\cdot\boldsymbol{u}) \;+\; m\,\frac{\dot{\boldsymbol{u}}\,(\boldsymbol{u}\cdot\boldsymbol{u}) \;-\; \boldsymbol{u}\,(\dot{\boldsymbol{u}}\cdot\boldsymbol{u})}{\|\boldsymbol{u}\|} = 0} \tag{29}$$

Furthermore, we can indicate a general formula for the family of the Lagrange functions which produce the expression (29):

$$\mathcal{L}_{(\rho)} \;=\; \frac{u^\rho[\dot{\boldsymbol{u}}, \boldsymbol{u}, \boldsymbol{e}_{(\rho)}]}{\|\boldsymbol{u}\|\,\|\boldsymbol{u}\times\boldsymbol{e}_{(\rho)}\|^2} \;-\; m\,\|\boldsymbol{u}\| \;+\; \dot{\boldsymbol{u}}\,.\,\boldsymbol{\partial}_{\boldsymbol{u}}\,\phi \;+\; \boldsymbol{a}\,.\,\boldsymbol{u}\,,$$

where an arbitrary row vector $\boldsymbol{a}$ is constant and a function ϕ depending on the variable $\boldsymbol{u}$ is subject to the constraint $\boldsymbol{u}\,.\,\boldsymbol{\partial}_{\boldsymbol{u}}\,\phi \;=\; 0$. (Recall the notation $[\ ,\ ,\ ]$ for the parallelepipedal product of three vectors.) The vector $\boldsymbol{e}_\rho$ denotes the ρ-th component of the (pseudo-) Euclidean frame. *Each* $\mathcal{L}_{(\rho)}$ *fits in.*

Although *there does not exist an invariant (even in extended sense) Lagrange function*, the equations (29) *are invariant* with respect to the group under consideration. Namely, let

$$\mathfrak{x}'' \;=\; [\boldsymbol{\varpi}, \boldsymbol{x}, \boldsymbol{\partial}_{\boldsymbol{x}}] \;+\; [\boldsymbol{\varpi}, \boldsymbol{u}, \boldsymbol{\partial}_{\boldsymbol{u}}] \;+\; [\boldsymbol{\varpi}, \dot{\boldsymbol{u}}, \boldsymbol{\partial}_{\dot{\boldsymbol{u}}}] \;+\; [\boldsymbol{\varpi}, \ddot{\boldsymbol{u}}, \boldsymbol{\partial}_{\ddot{\boldsymbol{u}}}] \tag{30}$$

stand for the third-order prolongation of the infinitesimal (pseudo-) Euclidean transformations to the manifold $T''M$ with $\boldsymbol{\varpi}$ for the group parameter. Then

$$\mathbf{L}(\mathfrak{x}'')(\boldsymbol{\mathcal{E}}) \;=\; \boldsymbol{\varpi}\times\boldsymbol{\mathcal{E}}\,.$$

Remark. Assuming $m = 0$ in (29), we recover geodesic circles as integral curves, and in the case the index i in $\mathbf{E}(3,i)$ equals 2 this amounts to uniformly accelerated motion in three-dimensional special relativity.

2. Euclidean space. Connection

In a (pseudo-) orthonormal frame of reference, the corresponding third-order frame takes on the shape

$$r^\rho_\beta = \delta^\rho_\beta, \quad r^\rho_{\beta\gamma} = 0, \quad r^\rho_{\beta\gamma\nu} = 0\,, \tag{31}$$

so the structure forms ω^ρ_β, $\omega^\rho_{\beta\gamma}$, and $\omega^\rho_{\beta\gamma\nu}$ vanish and we can identify $\boldsymbol{u}$, $\dot{\boldsymbol{u}}$, and $\ddot{\boldsymbol{u}}$ with the "invariant coordinates" U, $\dot{U}$, and $\ddot{U}$ respectively.

In order to construct a connection, consistent with the equation (29), we first supplement the three independent expressions, entering in (29), with an arbitrary additional one. Without loss of generality we can search for the latter in the form

$$\ddot{\boldsymbol{u}} \,.\, \boldsymbol{u} \;=\; ||\boldsymbol{u}||^2 \cdot \Psi(\boldsymbol{u}, \dot{\boldsymbol{u}})\,. \tag{32}$$

The system of equations (29) and (32) can now be solved with respect to the third-order derivatives to produce:

$$\dddot{\boldsymbol{u}} \;=\; 3\,\frac{\dot{\boldsymbol{u}}\cdot\boldsymbol{u}}{||\boldsymbol{u}||^2}\,\dot{\boldsymbol{u}} \;-\; 3\,\frac{(\dot{\boldsymbol{u}}\cdot\boldsymbol{u})^2}{||\boldsymbol{u}||^4}\boldsymbol{u} \;-\; m\,\boldsymbol{u}\times\dot{\boldsymbol{u}} \;+\; \Psi\cdot\boldsymbol{u}\,. \tag{33}$$

Now we proceed further to define more precisely the arbitrary function Ψ. With (11 and 12) we calculate μ and λ for the equation (33):

$$\mu \;=\; \frac{1}{3}\left(2\Psi \;-\; 2\dot{\boldsymbol{u}}\frac{\partial\Psi}{\partial\dot{\boldsymbol{u}}} \;-\; \boldsymbol{u}\frac{\partial\Psi}{\partial\boldsymbol{u}}\right)\;; \tag{34}$$

$$\lambda \;=\; \frac{\boldsymbol{u}\cdot\dot{\boldsymbol{u}}}{||\boldsymbol{u}||^2} \;-\; \frac{1}{3}\,\boldsymbol{u}\frac{\partial\Psi}{\partial\dot{\boldsymbol{u}}}\,. \tag{35}$$

In the reference frame (31) by virtue of (13) we conclude that μ and λ are constant. Then the compatibility conditions for the system of partial differential equations (34) and (35) show, that λ must be equal to zero,

$$\lambda = 0\,. \tag{36}$$

Set

$$\Psi \;=\; \frac{3}{||\boldsymbol{u}||^2}\left(\psi \;+\; \frac{1}{2}||\dot{\boldsymbol{u}}||^2\right)\,.$$

For the function ψ we now get

$$\left(\boldsymbol{u}\cdot\frac{\partial}{\partial\dot{\boldsymbol{u}}}\right)\,\psi \;=\; 0\,. \tag{37}$$

Introducing the intermediate variable $\boldsymbol{z} = \boldsymbol{u}\times\dot{\boldsymbol{u}}$ we see by (37) that the function ψ depends on $\boldsymbol{u}$ and $\dot{\boldsymbol{u}}$ via the variable $\boldsymbol{z}$ only. Now we express the equation (34) in terms of $\boldsymbol{z}$ to get

$$3\,\boldsymbol{z}\cdot\frac{\partial\psi}{\partial\boldsymbol{z}} \;=\; 4\psi \;-\; ||\boldsymbol{u}||^2\mu\,. \tag{38}$$

Again, the compatibility conditions for (38) turn μ to zero,

$$\mu = 0\,. \tag{39}$$

To settle the matter definitely, we call upon the demand of (pseudo-) Euclidean symmetry (with the generator (30)) for the equation (33), which gives

$$\boldsymbol{z} \times \frac{\partial}{\partial \boldsymbol{z}}\, \psi \;=\; 0\,. \tag{40}$$

Altogether (38, 39, and 40) produce for the determination of the function ψ the equation

$$\frac{\partial \psi}{\partial \boldsymbol{z}} \;==\; \frac{4}{3}\, \frac{\boldsymbol{z}}{\|\boldsymbol{z}\|^2}\, \psi$$

with the solution $\psi = \|\boldsymbol{z}\|^{4/3}$. Thus the function Ψ has been found,

$$\Psi \;=\; \frac{3}{2}\, \frac{\|\dot{\boldsymbol{u}}\|^2}{\|\boldsymbol{u}\|^2} \;+\; 3A\, \frac{\|\dot{\boldsymbol{u}} \times \boldsymbol{u}\|^{4/3}}{\|\boldsymbol{u}\|^2}, \qquad A \text{ is a real number.}$$

Let us introduce the following obvious definition:

Definition. For any smooth transformation group G acting upon a manifold M let a curve $\sigma\colon I \to M,\ I \subset \mathbb{R}$ be called an *autogeodesic path* if

(1) the image $\sigma(I)$ is an extremal submanifold of some parameter-independent variation problem;
(2) the curve σ is autoparallel with respect to some (nonlinear, higher-order) connection on M;
(3) the corresponding autoparallel transport equation is G-invariant.

In view of the preceding considerations we now are capable of calculating the coefficients $(\Gamma^{\rho}_{\beta}, \Gamma^{\rho}_{\beta\gamma})$ of the connection, given by (14). Rather then make this, it appears more economic to accomplish only with the presentation of the explicit expression of the corresponding autogeodesic path equation.

Proposition.

(1) *The third-order autogeodesic paths of the three-dimensional (pseudo-) Euclidean space are the solutions of the next differential equation:*

$$\boxed{\ddot{\boldsymbol{u}} = 3\frac{\dot{\boldsymbol{u}} \cdot \boldsymbol{u}}{\|\boldsymbol{u}\|^2}\dot{\boldsymbol{u}} - 3\left[\frac{(\dot{\boldsymbol{u}} \cdot \boldsymbol{u})^2}{\|\boldsymbol{u}\|^4} - \frac{\|\dot{\boldsymbol{u}}\|^2}{2\|\boldsymbol{u}\|^2} - A\frac{\|\dot{\boldsymbol{u}} \times \boldsymbol{u}\|^{4/3}}{\|\boldsymbol{u}\|^2}\right]\boldsymbol{u} - m\,\boldsymbol{u} \times \dot{\boldsymbol{u}}} \tag{41}$$

(2) *The corresponding connection is stable.*

Proof. It is necessary to calculate Γ^{ρ}_{β} and $\Gamma^{\rho}_{\beta\gamma}$ from (14) and to show that the right-hand side of (9) coincides with the right-hand side of (41). The second statement follows from (36 and 39) ■

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