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ANOTHER LOOK ON RECURSION OPERATORS

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ABSTRACT. Recursion operators of partial differential equations are identified with Bäcklund auto-transformations of linearized diffieties. Relations to the classical concept and its recent Guthrie's generalization are discussed.

Traditionally, a recursion operator of a PDE is a linear operator L acting on symmetries: if f is a symmetry then so is Lf. This provides a convenient way to generate infinite families of symmetries (Olver [13]). Standard reference is [14].

The presence of a recursion operator has been soon recognized as one of the attributes of the complete integrability. An exact and rich theory has been developed within the class of evolution equations (see Fokas [2] and references therein).

Recursion operators are, as a rule, nonlocal and so may turn out to be the symmetries they generate (giving a powerful source of nonlocal symmetries [10, 5]). A remarkable problem of inverting a recursion operator motivated Guthrie [3] to a generalization, with consequences for generation of nonlocal symmetries [4]. His nonlocalities are no longer limited to mere inverses D^{-1} of total derivatives.

It seems to be an interesting problem to give an interpretation of recursion operators in terms of the Vinogradov [17] category of diffieties. Recently, Krasilshchik and Kersten [6, 8] established a fundamental relation between recursion operators and deformations of diffiety structures.

In this paper (Section 3) we propose another answer, according to which recursion operators are Bäcklund auto-transformations of so called linearized diffiety (Section 2). Classical recursion operators are discussed in Section 4. The Guthrie operators turn out to be equivalent to what we call finitary linear operators, in the case of two independent variables (Section 5). To cover known examples in multidimension (e.g., the KP equation in [2]) finitary constructions are manifestly insufficient, but a theory of infinitary coverings still awaits a necessary development. In

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Section 6 we discuss the inversion of a recursion operator, within what we call a local setting. Finally, in Section 7 we conclude that a direct computational procedure for finding recursion operators exists, based on the mere definition.

1. Preliminaries

We work in the category of diffieties introduced by A.M. Vinogradov [17, 9]. The objects, *diffieties*, are infinite-dimensional smooth manifolds equipped with an involutive distribution D of dimension $m < \infty$. For simplicity we assume here that our diffieties are fibred manifolds $E \to M$ over a fixed base manifold M of the same dimension m, and that the projection of the distribution on M is of maximal rank, so that D is a horizontal distribution of a flat connection in the fibration $E \to M$ (see [7]). The lift of a vector field X from M to E will be denoted by D_X.

A map $h : E \to E'$ is a morphism of diffieties if h commutes with the projections on M, and $D_X(f \circ h) = (D'_X f) \circ h$ for any $f \in C^\infty E'$ and any vector field X on M. So arising category is denoted by DE_M here. For a different description see [11].

The passage from a system of n partial differential equations

$$F^l = 0, \quad l = 1, \dots, n \tag{1}$$

to a diffiety is as follows. Consider a fibred manifold $Y \to M$. Its fibred jet manifold $j^\infty Y \to M$ becomes a diffiety when endowed with a connection D such that D_X on $j^\infty Y$ satisfies $D_X f \circ j^\infty \gamma = X(f \circ j^\infty \gamma)$ for every section $\gamma : M \to Y$. Within any jet coordinate system x^i on M, and u^k_I in the fibres of $j^\infty Y \to M$, where I runs over symmetric multiindices over the i's, we have vector fields $D_i := D_{\partial/\partial x^i} = \partial/\partial x^i + \sum_I u^k_{Ii}\, \partial/\partial u^k_I$. Operators D_i, called *total derivatives*, commute. Suppose now that functions F^l from (1) are defined on $j^\infty Y$. If conditions $F^l = 0$ together with all their "differential consequences" $D_{i_1} \dots D_{i_k} F^l = 0$ determine a submanifold in $j^\infty Y$ then it admits a restriction of the total derivatives D_i (which we denote by the same symbol D_i) and, consequently, of the whole Cartan connection D. This is the diffiety naturally representing the system (1).

A *covering* (Krasilshchik and Vinogradov [10]) of a diffiety E is defined as another diffiety $\tilde E$ together with a projection (surjective submersion and DE_M-morphism) $p : \tilde E \to E$. This concept is crucial for nonlocal aspects of the PDE's.

Any covering $p : \tilde E \to E$ is locally determined by a set of p-vertical vector fields $X_{(i)}$ on $\tilde E$, $i = 1, \dots, \dim M$, such that vector fields $\tilde D_i = D_i + X_{(i)}$ commute, i.e.,

$$D_i X_{(j)} - D_j X_{(i)} + [X_{(i)}, X_{(j)}] = 0. \tag{2}$$

If the covering is finite-dimensional (fibres of p have a finite dimension) and we denote by w^k its coordinates along the fibres then we can write $X_{(i)} = X^k_{(i)}\, \partial/\partial w^k$, with $X^k_{(i)} \in C^\infty \tilde E$. The diffiety $\tilde E$ then corresponds to a system of equations

$$w^k_{x^i} = X^k_{(i)} \tag{3}$$

(additionally to (1)) of which (2) is the integrability condition.

A finitary covering is said to be *linear* if equations (3) are linear. Linear isomorphism classes of linear one-dimensional coverings correspond to one-dimensional *horizontal cohomology* classes, cf. [5]. Let us recall the essentials from [18].

Let Λ^*E denote the noncommutative $C^\infty E$-algebra of antisymmetric forms on E, with respect to the wedge product $\wedge$, let $\mathcal{C}\Lambda^*E$ denote the ideal of contact forms, i.e., forms vanishing on the Cartan distribution. Then $\bar\Lambda^*E = \Lambda^*E/\mathcal{C}\Lambda^*E$ is a $C^\infty E$-algebra, naturally identified with the algebra of horizontal forms (forms vanishing whenever at least one of the arguments is a vertical vector field). With respect to the differential $d : \Lambda^k E \to \Lambda^{k+1}E$ we have $d(\mathcal{C}\Lambda^k E) \subseteq \mathcal{C}\Lambda^{k+1}E$, so that a well-defined differential $\bar d : \bar\Lambda^k E \to \bar\Lambda^{k+1}E$ is induced, giving rise to a cohomology theory. This cohomology, called *horizontal*, is computable by means of the $\mathcal{C}$-*spectral sequence* [18], based on the filtration of $\Lambda^k E$ by powers of the ideal $\mathcal{C}\Lambda^k E$.

By an *abelian* covering [5] we mean a covering which is a composition (tower) of linear one-dimensional coverings. As a consequence of [18, Two-line theorem] no nontrivial abelian coverings exist for a non-overdetermined system if $\dim M > 2$ (more than two independent variables). When, on the other side, $\dim M = 2$, the one-dimensional horizontal cohomology group coincides with the group of conservation laws, and nontrivial abelian coverings may occur.

Bäcklund transformations (see [15]) admit a very transparent description in the above terms: A Bäcklund transformation from a diffiety E_1 to a diffiety E_2 is simply a pair of coverings $\tilde E \to E_1$, $\tilde E \to E_2$ with a common total space $\tilde E$ (see [10]).

2. Linearization

Denote by VE the vector bundle of vertical tangent vectors to a diffiety E over M and by τ_E the projection $VE \to E$. Then VE is a diffiety in a natural way, τ_E is a covering, and V is a functor $DE_M \to DE_M$ (see [11] and also [10, 7, 16]). We give a self-contained exposition below.

We start with two canonical ways of lifting functions from E to VE. The first is $f \mapsto f \circ \tau_E$, the second is $f \mapsto \dot f$, where $\dot f \in C^\infty VE$ satisfies

$$\dot f \circ \xi = \xi f \tag{4}$$

(composition of maps on the left side, action of fields on functions on the right side) for any vertical vector field $\xi : E \to VE$. We call $\dot f$ a *linearization* of the function f.

If q^κ are local coordinates along the fibres of $E \to M$ then functions $\dot q^\kappa$ may serve as coordinates along the fibres of $VE \to E$. Such coordinate systems on VE will be called *standard*. For any function $f \in C^\infty E$ we then have

$$\dot f = \sum_\kappa \dot q^\kappa \frac{\partial f}{\partial q^\kappa}. \tag{5}$$

Similarly, there are two canonical ways of lifting k-forms from E to VE. The first is the plain pull-back along τ_E, $\omega \mapsto \tau_E^*\omega$, and the second is as follows.

2.1. Proposition. There exists a unique map $\Lambda^k E \to \Lambda^k VE$, $\omega \mapsto \dot\omega$ such that

1°. $\xi^*\dot\omega = \mathcal{L}_\xi\omega$ (the Lie derivative) for any vertical vector field $\xi : E \to VE$.

2°. $(\omega + \sigma)^{\cdot} = \dot\omega + \dot\sigma$.

3°. $(\omega \wedge \sigma)^{\cdot} = \dot\omega \wedge \tau_E^*\sigma + \tau_E^*\omega \wedge \dot\sigma$.

4°. $(d\omega)^{\cdot} = d\dot\omega$.

Proof. For $k = 0$ we have the above linearization of functions. Conditions 2°–4° then determine $\dot\omega$ uniquely for any k (force a coordinate formula). Finally, 1° holds for a sum, a wedge product and a differential if it holds for the arguments.

The form $\dot\omega \in \Lambda^k VE$ is said to be a *linearization* of the k-form ω.

For a vector field Ξ on E we define a *V-lift* Ξ^V as a vector field on VE satisfying

$$\Xi^V(q \circ \tau_E) = \Xi q \circ \tau_E, \qquad \Xi^V(\dot q) = (\Xi q)^{\cdot} \tag{6}$$

for any function $q \in C^\infty E$. Obviously, these formulas determine the field Ξ^V uniquely within one coordinate chart, hence its global existence follows.

Now, let E be a diffiety. For any vector field X on M the field D_X^V, the V-lift of the field D_X, depends on the fields X $C^\infty M$-linearly, i.e., induces a connection. It is easily verified that the connection is flat, giving rise to a diffiety (VE, D^V). We call it a *linearization* of the diffiety E.

Let us show that this diffiety coincides with that introduced in [11]. Indeed, the Cartan connection on E can be equivalently described as a section e_D of the projection $j^1E \to E$. This section induces a section $e_D^V : VE \xrightarrow{Ve_D} Vj^1E \cong j^1VE$ of the projection $j^1VE \to VE$, i.e., a connection on VE. In standard coordinates e_D is described by $q_i^k = D_i q^k$, so that e_D^V is given by $q_i^k = D_i q^k$, $(\dot q^k)_i = (q_i^k)^{\cdot} = (D_i q^k)^{\cdot}$. The verification is finished by comparison with formulas (6).

Obviously, if the diffiety E is determined by equations (1) then the linearization VE is determined by equations (1) together with their linearizations (5):

$$\sum_{k,I} \dot u_I^k \frac{\partial F^l}{\partial u_I^k} = 0. \tag{7}$$

In accordance with [9, Section 7.4] we define a *symmetry* of a diffiety $E \in DE_M$ as a vertical vector field ξ satisfying $[\xi, D_X] = 0$ for any vector field X on M.

2.2. Proposition. *A section $\xi : E \to VE$ of the projection τ_E is a symmetry of E if and only if it is a DE_M-morphism $E \to VE$.*

Proof. A map $\xi : E \to VE$ is a DE_M-morphism if and only if it preserves the Cartan connection, i.e., if $D_X(f \circ \xi) = (D_X^V f) \circ \xi$ for any X and any function f on VE. It is sufficient to check this condition for functions of the form $q \circ \tau_E$ and $\dot q$, for $q \in C^\infty E$. Obviously, $D_X(q \circ \tau_E \circ \xi) = D_X q = D_X q \circ \tau_E \circ \xi = D_X^V(q \circ \tau_E) \circ \xi$ always holds, while $D_X(\dot q \circ \xi) - (D_X^V \dot q) \circ \xi = D_X(\xi q) - (D_X q)^{\cdot} \circ \xi = D_X(\xi q) - \xi(D_X q) = [D_X, \xi]q$ is zero for any $q \in C^\infty E$ if and only if $[D_X, \xi] = 0$, which finishes the proof.

2.3. Proposition. *For any form $\omega \in \Lambda^k E$ and any vector fields $\Xi_1, \dots, \Xi_k$ on E it holds $\dot\omega(\Xi_1^V, \dots, \Xi_k^V) = \omega(\Xi_1, \dots, \Xi_k)^{\cdot}$.*

Proof. By induction on k. For $k = 1$ and a generator $\omega = f\,dg$, where $f, g \in C^\infty E$, we have $\dot\omega(\Xi^V) = \dot f\,dg(\Xi^V) + f\,d\dot g(\Xi^V) = \dot f \Xi^V g + f \Xi^V \dot g = \dot f \Xi g + f(\Xi g)^{\cdot} = (f \Xi g)^{\cdot} = \omega(\Xi)^{\cdot}$. The induction step follows by formulas 2°, 3° of Proposition 2.1.

2.4. Corollary. *For any contact form $\omega \in \mathcal{C}\Lambda^k E$ the form $\dot\omega$ is contact again: $\dot\omega \in \mathcal{C}\Lambda^k VE$. Consequently, the map $\omega \mapsto \dot\omega$ maps horizontal forms on E to horizontal forms on VE and commutes with the horizontal differential.*

This corollary opens the way for lifting the whole $\mathcal{C}$-spectral sequence from E to VE, different from simple pulling-back by τ_E. However, we shall not need this here.

3. Recursion operators

3.1. Definition. A *recursion operator* for a diffiety E is a pair of coverings $K, L : R \to VE$ over the linearization VE of E such that the diagram

$$
\begin{tikzcd}
 & R \arrow[dl, "K"'] \arrow[dr, "L"] & \\
VE \arrow[dr, "\tau_E"'] & & VE \arrow[dl, "\tau_E"] \\
 & E, &
\end{tikzcd}
\tag{8}
$$

is commutative. A recursion operator is said to be *linear* whenever

1° The covering R is linear over E.

2° The morphisms K, L are linear over E.

A recursion operator is said to be *finitary* whenever

3° Both coverings K, L are finite-dimensional over VE.

Thus, a recursion operator is a special case of a Bäcklund auto-transformation of the linearized diffiety. Note that coverings K, L may have different dimensions.

3.2. Example (KdV). The KdV equation, which we write as $u_t = u_{xxx} + uu_x$, is known to admit a recursion operator (Lenard operator) $L : v \mapsto D_{xx}v + \frac{2}{3}uv + \frac{1}{3}u_x D_x^{-1}v$. The corresponding commutative diagram (8) is easily seen to be

$$
\begin{tikzcd}
 & \boxed{\begin{array}{c} w_x = v \\ w_t = v_{xx} + uv \\ v_t = v_{xxx} + uv_x + u_x v \\ u_t = u_{xxx} + uu_x \end{array}} \arrow[dl, "K : v = w_x"'] \arrow[dr, "L : v = v_{xx} + \frac{2}{3}uv + \frac{1}{3}u_x w"] & \\
\boxed{\begin{array}{c} v_t = v_{xxx} + uv_x + u_x v \\ u_t = u_{xxx} + uu_x \end{array}} \arrow[dr] & & \boxed{\begin{array}{c} v_t = v_{xxx} + uv_x + u_x v \\ u_t = u_{xxx} + uu_x \end{array}} \arrow[dl] \\
 & \boxed{u_t = u_{xxx} + uu_x .} &
\end{tikzcd}
$$

Here the linearized equation $v_t = v_{xxx} + uv_x + u_x v$ comes out by formula (7) with $\dot{u} = v$. The variables v, v_x, v_{xx}, etc. are coordinates in the fibres of the projection $\tau_{\mathrm{KdV}} : V(\mathrm{KdV}) \to \mathrm{KdV}$. The covering K is one-dimensional. Its unique nonlocal coordinate w satisfies $w_x = v$, $w_t = v_{xx} + uv$, i.e., corresponds to the traditional $D_x^{-1}v$. It can be shown that the covering L is of dimension three.

3.3. Action on symmetries. If f is a symmetry of an equation E possessing a recursion operator L then Lf may turn out to be a symmetry, or merely a *shadow* of a symmetry (see Krasilshchik and Vinogradov [10]). Let us look at this point here.

Let s be a symmetry of the diffiety E, i.e., a morphism $s : E \to VE$ which is a section of the projection $\tau_E : VE \to E$. Consider a diagram

$$\begin{tikzcd} \tilde E \arrow[r, "P'"] \arrow[d, "P"'] & R \arrow[r, "L"] \arrow[d, "K"] & VE \arrow[d, "\tau_E"] \\ E \arrow[r, "s"'] \arrow[rr, bend right, "\mathrm{id}"'] & VE \arrow[r, "\tau_E"'] & E, \end{tikzcd}$$

where the left-hand square is a pullback, i.e., $P : \tilde E \to E$ is a covering over E induced by s. The whole diagram is commutative and hence the morphism $\tilde s := L \circ P' : \tilde E \to VE$ satisfies $\tau_E \circ \tilde s = P$, i.e., is a morphism of coverings over E.

When the covering P is trivial ($\tilde E \cong E \times \mathbb{R}^n$) then $\tilde s$ is simply an n-parametric family of symmetries of E, otherwise $\tilde s$ is the *shadow* of a nonlocal symmetry [10] (shadows may be defined as morphisms $\tilde E \to VE$ over E). The process of reconstruction of the true nonlocal symmetry out of its shadow (detailed in loc. cit.) is left out in this paper.

3.4. Example (KdV). We start with the KdV symmetry $s : v = 1 + tu_x$. From the defining equations of R we easily see that w, now as the nonlocal variable of the induced covering $\tilde E$, satisfies $w_x = 1 + tu_x$, $w_t = u + tu_t$. These equations are readily integrated as $w = tu + x + \mathrm{const}$ (w represents a trivial conservation law) so that the induced covering is trivial via a map $E \times \mathbb{R} \to \tilde E$, $c \mapsto (w = tu + x + c)$. Hence, $v' = v_{xx} + \frac{2}{3}uv + \frac{1}{3}u_x w = tu_t + \frac{1}{3}xu_x + \frac{2}{3}u + \frac{1}{3}cu_x$ is a one-parameter family of symmetries (cf. a similar discussion of mKdV in [3]).

Continuing with $s = tu_t + \frac{1}{3}xu_x + \frac{2}{3}u$, we see that the defining equations of $\tilde E$, $w_x = tu_{xxx} + tuu_x + \frac{1}{3}xu_x + \frac{2}{3}u$, $w_t = tu_{txx} + tuu_t + \frac{1}{3}xu_t + \frac{4}{3}u_{xx} + \frac{2}{3}u^2$ represent a nontrivial conservation law (reducible to $z_x = u$, $z_t = u_{xx} + \frac{1}{2}u^2$ by $w = tu_{xx} + \frac{1}{3}xu + \frac{1}{2}tu^2 + \frac{1}{3}z$). Thus, the covering $\tilde E$ is nontrivial and we obtain a shadow of a true nonlocal symmetry in this case.

4. Classical recursion operators

4.1. Definition. A linear finitary recursion operator (8) is said to be *classical* if

4° The covering $K : R \to VE$ is abelian

(see Section 2 for the definition of abelian coverings).

For finding classical recursion operators it is important to know the horizontal cohomology of linearized diffieties VE.

4.2. Definition. A diffiety E is said to be V-*simple* if there exist no nontrivial horizontal one-dimensional cohomology classes of the linearized diffiety VE other than $[\tau_E^* \sigma]$ and $[\dot\sigma]$ for horizontal forms $\sigma \in \bar\Lambda^1 E$ (cf. Corollary 2.4).

Concrete but unpublished computations support a hypothesis that equations of mathematical physics are, as a rule, V-simple. Below we sketch two examples to support another hypothesis, that the abelian coverings for classical recursion operators are due to cohomology classes of the form $[\dot\sigma]$, and not $[\tau_E^* \sigma]$.

4.3. Example (KdV). In Example 3.2 the underlying exact 1-form of the covering R over VE was just $v\,dx + (v_{xx} + uv)\,dt$, a linearization of the KdV conservation law $u\,dx + (u_{xx} + \frac{1}{2}u^2)\,dt$.

4.4. Example (sin-Gordon equation). We write the sin-Gordon (Bonnet) equation B as $u_{xy} = \sin u$. Olver [13] found its recursion operator to be $v \mapsto v' = D_{xx}v + u_x^2 v - u_x D_x^{-1}(u_{xx}v)$. The linearization VB is given by a pair $u_{xy} = \sin u$, $v_{xy} = v\cos u$, while the covering R over VB is determined by equations $w_x = u_{xx}v$, $w_y = u_x v_y$. This covering is abelian, due to an exact horizontal 1-form $\sigma = u_{xx}v\,dx + u_x v_y\,dy$, which is not a linearization of any exact 1-form on B.

However, σ may be adjusted by adding an appropriate horizontal differential $\bar{d}f$, $f \in C^\infty VB$, since $[\sigma + \bar{d}f] = [\sigma]$. Choosing $f = -u_x v$, we get $\sigma + \bar{d}f = -u_x v_x\,dx + v\sin u\,dy$, a linearization of the conservation law $-\frac{1}{2}u_x^2\,dx + \cos u\,dy$. The recursion operator then becomes $v' = v_{xx} - u_x\tilde{w}$ for $\tilde{w}_x = -u_x v_x$, $\tilde{w}_t = v\sin u$.

5. Guthrie's recursion operators

We assume in this section a linear finitary recursion operator $K, L : R \to VE$ to be given. We denote by v^ι, $\iota \in I$, the nonlocal variables of the covering $VE \to E$, and by w^k, $k = 1, \dots, \dim_K R$, the nonlocal variables of the covering K.

As it has been explained in Section 2, R has to be given by a set of equations (3). These equations are linear over E if and only if $X^k_{(i)} = A_{(i)}{}^k_l w^l + B^k_{(i)}$, where each $B^k_{(i)}$ is linear in v^ι, i.e., $B^k_{(i)} = B_{(i)}{}^k_\iota v^\iota$. Symbols $A_{(i)}{}^k_l$, $B_{(i)}{}^k_\iota$ denote functions on E. The integrability conditions (2) are (in matrix notation)

$$D_j A_{(i)} - D_i A_{(j)} + A_{(i)}A_{(j)} - A_{(j)}A_{(i)} = 0, \tag{9}$$

$$D_j B_{(i)} - D_i B_{(j)} + A_{(i)}B_{(j)} - A_{(j)}B_{(i)} = 0. \tag{10}$$

The linear operator L is then of the form (F^κ_l, G^κ_ι are functions on E)

$$v'^\kappa = F^\kappa_l w^l + G^\kappa_\iota v^\iota.$$

It is easy to see that these are exactly the formulas of Guthrie [3, (3.2–3)].

5.1. Remarks. (1) Guthrie assumes additionally some covering $\tilde{E}$ over E to be given [3, (3.1)]. The difference is that $A_{(i)}{}^k_l$, $B_{(i)}{}^k_\iota$, F^κ_l, and G^κ_ι are then functions on $\tilde{E}$. However, this clearly amounts to what one might call a *nonlocal setting* (having a recursion operator on $\tilde{E}$ instead of E). In Section 6 we demonstrate that for the purpose of mere inversion the local approach is sufficient.

(2) Obviously, (9) determines a zero-curvature representation of the equation E (is a Maurer–Cartan condition). Moreover, upon introducing extended matrices

$$\mathcal{A}_{(i)} := \left(\begin{array}{c|c} A_{(i)} & B_{(i)} \\ \hline 0 & 0 \end{array}\right) \tag{11}$$

(one column $(B^k_{(i)})$ from the right and one row of zeroes from below) conditions (9, 10) reduce to a single Maurer–Cartan condition $D_j\mathcal{A}_{(i)} - D_i\mathcal{A}_{(j)} + \mathcal{A}_{(i)}\mathcal{A}_{(j)} - \mathcal{A}_{(j)}\mathcal{A}_{(i)} = 0$, and represent a zero-curvature representation for VE.

(3) While the linearity property of recursion operators seems to be universal (symmetries form a linear space), it is clear from the following proposition that general recursion operators in multidimension cannot be finitary.

5.2. Proposition. *For any finitary linear recursion operator* (8) *the covering* $K : R \to E$ *is trivial whenever* $\dim M > 2$ *and the equation* E *is not overdetermined.*

Proof. From [12, Corollary 4.4] it follows that the matrices $A_{(i)}$ can be reduced to zero matrices by an appropriate change of coordinates in the fibres of K. Then (10) simplifies to a condition that columns $B_{(i)}$ consist of components of exact horizontal 1-forms, which are all trivial by [18, Two-line theorem], and hence they determine a trivial covering.

6. Inverting a recursion operator

An inversion of a general Bäcklund transformation amounts to mere interchanging of the two coverings, and the same applies to recursion operators. Hence, for a recursion operator given by diagram (4) the inverted operator is simply

$$\begin{tikzcd} & R \arrow[dl, "K'=L"'] \arrow[dr, "L'=K"] & \\ VE \arrow[dr, "\tau_E"'] & & VE \arrow[dl, "\tau_E"] \\ & E, & \end{tikzcd}$$

Obviously, the inversion of a finitary linear recursion operator is a finitary linear recursion operator. Moreover, as we shall see, the process of inversion is basically algorithmic.

6.1. Example (mKdV). Let us recompute the example of mKdV [3] in the local setting. We write the mKdV equation E as $u_t = u_{xxx} - u^2 u_x$. Then the linearization VE is a pair $u_t = u_{xxx} - u^2 u_x$, $v_t = v_{xxx} - u^2 v_x - 2uu_x v$. The mKdV recursion operator $L : z = v_{xx} - \frac{2}{3}u^2 v - \frac{2}{3}u_x w$ (where z stands for $\dot{u}$ in the second copy of VE) comes out of a covering R, given by $w_x = uv$, $w_t = uv_{xx} - u_x v_x + u_{xx}v - u^3 v$ (due to a linearization of an mKdV conservation law $\frac{1}{2}u^2\,dx + (uu_{xx} - \frac{1}{2}u_x^2 - \frac{1}{4}u^4)\,dt$).

To invert this operator all we need is to reconstruct the covering over the second copy of VE out of the above data. The formula for L gives a relation $v_{xx} = z + \frac{2}{3}u^2 v + \frac{2}{3}u_x w$, which leads us to the choice of the nonlocal variables over VE as w, v and $s = v_x$, subject to the relations

$$\begin{aligned} w_x &= uv, \\ v_x &= s, \\ s_x &= \tfrac{2}{3}u_x w + \tfrac{2}{3}u^2 v + z, \end{aligned}$$

and

$$\begin{aligned} w_t &= \tfrac{2}{3}uu_x w + (u_{xx} - \tfrac{1}{3}u^3)v - u_x s + uz, \\ v_t &= \tfrac{2}{3}u_{xx}w - \tfrac{1}{3}u^2 s + z_x, \\ s_t &= (\tfrac{2}{3}u_{xxx} - \tfrac{2}{9}u^2 u_x)w + (\tfrac{2}{3}uu_{xx} - \tfrac{2}{9}u^4)v - \tfrac{2}{3}uu_x s + z_{xx} - \tfrac{1}{3}u^2 z. \end{aligned}$$

It is easy to verify the integrability conditions (9) and (10) and that v satisfies the linearized equation whenever z does. Thus, we have arrived at a covering $L' : R' \to VE$, $(w, v, s) \mapsto v$. Its fibre dimension is one (recall that $s = v_x$ is a local variable with respect to L'), as is that of the covering $K : R \to VE$. Hence we naturally identify $R' = R$, $L' = K$, $K' = L$ with the inverted recursion operator.

The above covering R' can be simplified (rewriting the defining system via extended matrices of Remark 5.2(2) and applying gauge transformations is very helpful). We present only the result: Upon a coordinate transformation $p^+ = w + \sqrt{3/2}$, $p^- = w - \sqrt{3/2}$, $q = -\frac{2}{3}uw + s$ we get

$$\begin{aligned} p^\pm_x &= \pm\sqrt{\tfrac{2}{3}}up^\pm \pm \sqrt{\tfrac{3}{2}}q, \\ q_x &= z, \\ p^\pm_t &= \pm\sqrt{\tfrac{2}{3}}(u_{xx} - \tfrac{1}{3}u^3)p^\pm - (u_x \pm \tfrac{1}{\sqrt{6}}u^2)q \pm \sqrt{\tfrac{3}{2}}z_x + uz, \\ q_t &= z_{xx} - u^2 z, \end{aligned}$$

while the operator K acquires the form $v = p^+ - p^-$.

The covering $R' \to VE$ is non-abelian, but still decomposes as a tower $R' \cong P^+ \oplus P^- \to Q \to VE$, where $Q \to VE$ is due to the linearization of an mKdV conservation law $u\,dx + (u_{xx} - \frac{1}{3}u^3)\,dt$, and $P^+ \oplus P^- \to Q$ is the Whitney sum of two coverings corresponding to the (decoupled) equations on p^+, p^-, respectively. The coverings P^+, P^- are non-isomorphic.

7. Finding recursion operators

Procedures to find recursion operators have been proposed by many authors, see, e.g., [1, 8, 14]. See also, e.g., [2] for deriving recursion operators from Lax pairs. Finding finitary linear recursion operators on the basis of the mere definition is, however, also possible. Steps to be followed are:

1. Compute a (rich enough) covering R over VE (abelian for classical operators).
2. Find all linear morphisms $R \to VE$ over E.

For methods to find abelian coverings over diffieties see [18, 5]. Hints of Section 4 may be helpful. Finding coverings of Section 5 is possible by the procedure of [12].

Step 2 consists of solving a system of equations in total derivatives, which is, in fact, a part of the defining system for symmetries of R. Therefore, virtually any software to compute generating functions of symmetries is capable of finding recursion operators. Selecting surjective morphisms out of those found in Step 2 (and excluding the identity morphism) is then easy. (If we drop the linearity requirement then also all symmetries s of E, as morphisms $R \to E \xrightarrow{s} VE$, will be found.)

Concrete computations confirm feasibility, and will be published elsewhere.

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