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AN EXISTENCE THEOREM FOR SUBMANIFOLDS OF HIGHER CODIMENSIONS

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1. Introduction. This paper concerns submanifolds of the standard affine space $\mathbf{R}^n$ of codimension greater than one, equipped with affine connections. In [4] the notion of the type number of the second fundamental form was adopted to the affine case and an equivalence theorem was proved, which involved that concept. This time it is used to formulate and prove a theorem about existence of an affine immersion of a manifold with a torsion-free affine connection endowed with objects which will be further the second fundamental form, the shape operator and the normal connection for the obtained immersion. The theorem requires less equations than a general existence theorem formulated in [2].

In this paper some facts proved by Allendoerfer in the Riemannian case (see [1]) are generalized to the affine case. It is done in less complicated way, without using local coordinates. For comparison I also refer to a paper on existence theorems in the case of hypersurfaces ([3]).

At first we recall some basic facts. Let $f : M^n \longrightarrow \mathbf{R}^{n+p}$ be an immersion of an n-dimensional manifold M^n into the space $\mathbf{R}^{n+p}$ equipped with the usual flat affine connection $\widetilde{\nabla}$. If N is a vector bundle over M^nsuch that

$$\forall x \in M^n \quad T_{f(x)}\mathbf{R}^{n+p} = f_*(T_xM^n) \oplus N_x, \tag{1.1}$$

then for all $X, Y \in \mathcal{X}(M^n)$ we have a decomposition

$$\widetilde{\nabla}_X f_*(Y) = f_*(\nabla_X Y) + h(X,Y) \tag{1.2}$$

with respect to the above splitting. Here $\nabla_X Y$ is a local section of T_xM^n and $h(X,Y)$ is a local section of N. Throughout the paper we will use the notation $\nabla_X Y \in TM^n$ and $h(X,Y) \in N$ which means that $(\nabla_X Y)(x) \in T_xM^n$ and $h_x(X,Y) \in N_x$ for every point x in the domain of $\widetilde{\nabla}_X f_*(Y)$. It is easy to prove

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that ∇ is an affine connection without torsion on M^n and h, called the second fundamental form on M^n, is for a fixed $x \in M^n$ a bilinear symmetric mapping from $T_xM^n \times T_xM^n$ into N_x. We say that connection $\tilde{\tilde{\nabla}}$ in $\mathbf{R}^{n+p}$ induces connection ∇ in M^n, of course by immersion f. On the other hand, if we have a given N and an affine connection ∇ on M^n, such that the equations (1.1) and (1.2) are satisfied, we call such an immersion an affine immersion and we will denote it by

$$f : (M^n, \nabla) \longrightarrow (\mathbf{R}^{n+p}, \tilde{\nabla}).$$

For a local section ξ of N and for any $X \in \mathcal{X}(M^n)$ we have also a decomposition

$$\tilde{\nabla}_X \xi = -f_*(A_\xi(X)) + D_X\xi, \tag{1.3}$$

where A_ξ and $D_X\xi$ are defined by the conditions $A_\xi X \in TM$ and $D_X\xi \in N$. A_ξ is a (1,1)-tensor field on M^n, called the shape operator, and $D_X\xi$ defines a connection in the vector bundle N, called the normal connection. Let $\mathcal{X}(N)$ denote a module of local sections of a vector bundle N, $\nabla h(X,Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$, $(\nabla_X A)_\xi(Y) = \nabla_X(A_\xi Y) - A_\xi(\nabla_X Y) - A_{D_X\xi}Y$ for $X, Y, Z \in \mathcal{X}(M)$, and $R^\perp$ denotes the curvature tensor of a linear connection D.

We recall a general existence theorem [2]:

Theorem 1.1 (Dillen, Nomizu, Vrancken) *Let M^n be a simply connected manifold with a torsion-free affine connection ∇. Let N be a vector bundle over M^n and h - (0,2)-symmetric N-valued tensor field on M^n. Let D be a linear connection in N and A_ξ - (1,1)-tensor field on M^n for any $\xi \in N$. Assume that the following equations are fulfilled:*

$$R(X,Y)Z = A_{h(Y,Z)}X - A_{h(X,Z)}Y \qquad \textit{(Gauss)},$$

$$\nabla h(X,Y,Z) = \nabla h(Y,X,Z) \qquad \textit{(Codazzi I)},$$

$$(\nabla_X A)_\xi(Y) = (\nabla_Y A)_\xi(X) \qquad \textit{(Codazzi II)},$$

$$R^\perp(X,Y)\xi = h(X, A_\xi Y) - h(A_\xi X, Y) \qquad \textit{(Ricci)},$$

for every $X, Y, Z \in \mathcal{X}(M^n)$ and $\xi \in \mathcal{X}(N)$. Then there exists a global affine immersion $f : (M^n, \nabla) \longrightarrow (\mathbf{R}^{n+p}, \tilde{\nabla})$, for some $p \in \mathbf{N}$ with h and A_ξ as the second fundamental form and shape operator and D as the normal connection.

A basic notion in this paper is that of the type number of a symmetric bilinear form on a given manifold which has its values in a vector bundle. Let V be an n-dimensional vector space and $h^1, .., h^k : V \times V \longrightarrow \mathbf{R}$ be symmetric bilinear forms on V. Throughout the paper we will denote by the same symbols h^i mappings $V \longrightarrow V^*$ given by

$$h^i(v)(w) := h^i(v, w). \tag{1.4}$$

At first we define the type number of a set of symmetric bilinear forms on a vector space.

Definition 1.2. *The type number of the set $\{h^1, .., h^k\}$ is the greatest number r such that there exist r vectors $v_1, .., v_r \in V$ of the property that the forms $h^i(v_j)$ are linearly independent for i=1,..,k, j=1,..,r.*

Then we can define the type number of a symmetric bilinear form on a manifold.

Definition 1.3. *Let M^n be an n-dimensional manifold, N - a vector bundle over M^n and h - a symmetric N-valued 2-form on M^n. The type number of h at a point x is equal to r if for any local basis $\xi_1, .., \xi_k$ of N the type number of the set $\{h^1, .., h^k\}$, with respect to the decomposition $h = \sum h^i \xi_i$, is equal to r. We will denote it by $t_x h$.*

In paper [4]we proved that $t_x h$ is independent of the choice of a basis of N.

In this paper we will use the notation $\sigma_{X_1,..,X_r}$ for the cyclic sum with respect to $X_1, ..., X_r$. We first prove an algebraic lemma, which is in a way a generalization of Lemma 1.5. from [4].

Lemma 1.4. *Let $\{h^1, .., h^k\}$ be a set of bilinear forms on a vector space V with the type number equal to r. Let $\lambda^1, .., \lambda^k : V^{r-1} \longrightarrow \mathbf{R}^m$ be functions. If the cyclic sum*

$$\sigma_{X_1,..,X_r}\big[\sum_{i=1}^{k} h^i(X_r, W)\lambda^i(X_1, .., X_{r-1})\big] = 0$$

for every $X_1, .., X_r, W \in V$, then $\lambda^i = 0$ for i=1,..,k.

Proof: We can assume that $m = 1$. We obviously have

$$\sigma_{X_1,..,X_r}\big[\sum_{i=1}^{k} \lambda^i(X_1, .., X_{r-1})h^i(X_r)\big] = 0 \tag{1.5}$$

for every $X_1, .., X_r \in V$.

Assume that $Y_1, .., Y_r$ are such vectors that $h^i(Y_j)$ are linearly independent for $i = 1, .., k$, $j = 1, .., r$. We prove that λ^i acting on a sequence of vectors that contains $r - 1 - s$ vectors from $Y_1, .., Y_r$ and arbitrary s vectors for $0 \leq s < r$ are equal to 0. We use induction with respect to s. For $s = 0$ substitute $Y_1, ..., Y_r$ to (1.5). Then we obtain immediately: $\lambda^i(Y_1, .., Y_{r-1}) = \lambda^i(Y_2, .., Y_r) = ... = \lambda^i(Y_r, Y_1, .., Y_{r-2}) = 0$ by the assumption about the type number. We can have here the vectors $Y_1, ..., Y_r$ in an arbitrary order. Assume now that the assertion is fulfilled for a certain $s < r$. Substitute to (1.5) a sequence with $s + 1$ arbitrary vectors $Z_1, .., Z_{s+1}$ (here we can choose their order). In this equation we have now components including $h^i(Z_j)$ multiplied by λ^i acting on sets of vectors with s arbitrary Z_j. By the assumption such λ^i are equal to 0. The other λ^i in (1.5) are multiplied by $h^i(Y_j)$ which are linearly independent. Thus all λ^i with $s + 1$ arbitrary vectors are equal to 0.

The proof is completed.

2. The main theorem. Now we prove two lemmas, which will lead to our main theorem.

Lemma 2.1 *If $t_x h > 2$ for every $x \in M^n$ then the Gauss and the first Codazzi equations imply the second equation of Codazzi.*

Proof: Choose a local basis $\xi_1, .., \xi_k$ of a module of sections of N and denote A_{ξ_i} by A_i. Let $h = \sum h^i \xi_i$. For arbitrary $X, Y, Z, W \in \mathcal{X}(M^n)$ we use Bianchi identity $\sigma_{XYZ}(\nabla R)(X, Y, Z, W) = 0$ and apply it to the Gauss equation:

$$\sigma_{XYZ}\big[\nabla_X(A_{h(Z,W)}Y - A_{h(Y,W)}Z) - (A_{h(Z,W)}\nabla_X Y - A_{h(\nabla_X Y,W)}Z)\big] \tag{2.1}$$

$$-\sigma_{XYZ}\big[(A_{h(\nabla_X Z,W)}Y - A_{h(Y,W)}\nabla_X Z) - (A_{h(Z,\nabla_X W)}Y - A_{h(Y,\nabla_X W)}Z)\big] = 0$$

Consider two these components of (2.1):

$$\sigma_{XYZ}\big[\nabla_X(A_{h(Z,W)}Y) - \nabla_X(A_{h(Y,W)}Z)\big] \tag{2.2}$$

$$= \sigma_{XYZ}\big[\nabla_X(A_{h(Z,W)}Y) - \nabla_Z(A_{h(X,W)}Y)\big]$$

$$= \sigma_{XYZ}\big[X(h^i(Z,W)A_iY + h^i(Z,W)\nabla_X(A_iY) - Z(h^i(X,W))A_iY$$

$$-h^i(X,W)\nabla_Z(A_iY)\big].$$

Use the first Codazzi equation: $(\nabla h)(X, Z, W) = (\nabla h)(Z, X, W)$. Hence

$$A_{(\nabla h)(X,Z,W)}Y - A_{(\nabla h)(Z,X,W)}Y = 0. \tag{2.3}$$

Since

$$A_{(\nabla h)(X,Z,W)}Y = X(h^i(Z,W))A_iY - h^i(\nabla_X Z, W)A_iY$$

$$-h^i(Z, \nabla_X W)A_iY + h^i(Z,W)A_{D_X\xi_i}Y,$$

from (2.3) we have:

$$X(h^i(Z,W))A_iY - Z(h^i(Z,W))A_iY$$

$$= -h^i(\nabla_Z X, W)A_iY - h^i(X, \nabla_Z W)A_iY + h^i(X,W)A_{D_Z\xi_i}Y$$

$$-h^i(\nabla_X Z, W)A_iY + h^i(Z, \nabla_X W)A_iY - h^i(Z,W)A_{D_X\xi_i}Y.$$

Substitute it into (2.2) and then into (2.1). After reduction of some components we obtain:

$$\sigma_{XYZ}\big[h^i(Z,W)\nabla_X(A_iY) - h^i(X,W)\nabla_Z(A_iY)\big]$$

$$+\sigma_{XYZ}\big[A_{h(Y,W)}(\nabla_X Z) - A_{h(Z,W)}(\nabla_X Y)\big]$$

$$-\sigma_{XYZ}\big[A_{h(\nabla_Z X,W)}Y + A_{h(X,\nabla_Z W)}Y\big]$$

$$+\sigma_{XYZ}\big[A_{h(\nabla_X Y,W)}Z + A_{h(Y,\nabla_X W)}Z\big]$$

$$+\sigma_{XYZ}\big[h^i(X,W)A_{D_Z\xi_i}Y - h^i(Z,W)A_{D_X\xi_i}Y\big] = 0.$$

We can make a reduction of the third and fourth components using properties of cyclic sums. Hence we obtain:

$$\sigma_{XYZ}\big[h^i(Z,W)\big(\nabla_X(A_iY)-A_i(\nabla_XY)-A_{D_X\xi_i}Y\big)\big]$$

$$-\sigma_{XYZ}\big[h^i(Z,W)\big(\nabla_Y(A_iX)-A_i(\nabla_YX)-A_{D_Y\xi_i}X\big)\big]=0.$$

Hence

$$\sigma_{XYZ}h^i(Z,W)\big[(\nabla_XA)_{\xi_i}(Y)-(\nabla_YA)_{\xi_i}(X)\big]=0.$$

Applying Lemma 1.4. we immediately obtain

$$(\nabla_XA)_{\xi_i}(Y)=(\nabla_YA)_{\xi_i}(X)$$

at every point of the domain of the transversal frame, for each $X,Y\in\mathcal{X}(M^n)$. Since the mapping $\xi\longmapsto A_\xi$ is $\mathcal{C}^\infty(M)$ -linear, we can replace ξ_i by an arbitrary $\xi\in\mathcal{X}(N)$. The proof is completed.

Denote by $Q(X,Y,Z)$ the difference $\nabla h(X,Y,Z)-\nabla h(Y,X,Z)$. By the first Codazzi equation $Q=0$ on the whole M^n.

Lemma 2.2 *If $t_xh>2$ for every $x\in M^n$ then the Gauss equation and the Codazzi equations imply the Ricci equation*

$$R^\perp(X,Y)\xi=h(X,A_\xi Y)-h(A_\xi X,Y).$$

Proof: We will again use here properties of cyclic sums:

$$\sigma_{XYZ}(\nabla Q)(X,Y,Z,W)=\sigma_{XYZ}\big[D_XD_Yh(Z,W)-D_XD_Zh(Y,W)$$

$$-D_Xh(\nabla_YZ,W)+D_Xh(\nabla_ZY,W)-D_Xh(Z,\nabla_YW)+D_Xh(Y,\nabla_Z,W)\big]$$

$$=\sigma_{XYZ}\big[D_XD_Yh(Z,W)-D_YD_Xh(Z,W)\big]$$

$$-\sigma_{XYZ}\big[\nabla h(X,\nabla_YZ,W)+h(\nabla_X\nabla_YZ,W)+h(\nabla_YZ,\nabla_XW)\big]$$

$$+\sigma_{XYZ}\big[\nabla h(X,\nabla_ZY,W)+h(\nabla_X\nabla_ZY,W+h(\nabla_ZY,\nabla_XW)\big]$$

$$-\sigma_{XYZ}\big[\nabla h(X,Z,\nabla_YW)+h(\nabla_XZ,\nabla_YW)+h(Z,\nabla_X\nabla_YW)\big]$$

$$+\sigma_{XYZ}\big[\nabla h(X,Y,\nabla_ZW)+h(\nabla_XY,\nabla_ZW)+h(Y,\nabla_X\nabla_ZW)\big]$$

$$=\sigma_{XYZ}\big[D_XD_Yh(Z,W)-D_YD_Xh(Z,W)\big]$$

$$-\sigma_{XYZ}\big[\nabla h(X,[Y,Z],W)+h(\nabla_X\nabla_YZ,W)+h(\nabla_YZ,\nabla_XW)\big]$$

$$-\sigma_{XYZ}\big[\nabla h(X,Z,\nabla_YW)+h(\nabla_ZY,\nabla_XW)+h(Z,\nabla_X\nabla_YW)\big]$$

$$+\sigma_{XYZ}\big[h(\nabla_Y\nabla_XZ,W)+h(\nabla_ZY,\nabla_XW)\big]$$

$$+\sigma_{XYZ}\big[\nabla h(Z,X,\nabla_YW)+h(\nabla_YZ,\nabla_XW)+h(Z,\nabla_Y\nabla_XW)\big].$$

According to the first Codazzi equation the above expression is equal to

$$\sigma_{XYZ}\big[R^{\perp}(X,Y)h(Z,W)+D_{[X,Y]}h(Z,W)\big]-\sigma_{XYZ}\big[\nabla h([Y,Z],X,W\big]$$

$$-\sigma_{XYZ}\big[h(\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z,W)+h(Z,\nabla_X\nabla_Y W-\nabla_Y\nabla_X W)\big]$$

$$=\sigma_{XYZ}\big[R^{\perp}(X,Y)h(Z,W)\big]+\sigma_{XYZ}\big[\nabla h([X,Y],Z,W)+h(\nabla_{[X,Y]}Z,W)$$

$$+h(Z,\nabla_{[X,Y]}W)\big]-\sigma_{XYZ}\big[\nabla h([X,Y],Z,W)\big]$$

$$-\sigma_{XYZ}\big[h(R(X,Y)Z,W)+h(\nabla_{[X,Y]}Z,W)+h(Z,R(X,Y)W)$$

$$+h(Z,\nabla_{[X,Y]}W)\big]$$

$$=\sigma_{XYZ}\big[R^{\perp}(X,Y)h(Z,W)-h(X,R(Y,Z)W)\big]$$

$$-h(\sigma_{XYZ}R(X,Y)Z,W).$$

By the Gauss equation the last expression is equal to

$$\sigma_{XYZ}\big[R^{\perp}(X,Y)h(Z,W)-h(X,A_{h(Z,W)}Y)+h(X,A_{h(Y,W)}Z)\big]$$

$$=\sigma_{XYZ}\big[R^{\perp}(X,Y)h(Z,W)-h(X,A_{h(Z,W)}Y)+h(Y,A_{h(Z,W)}X)\big].$$

Take a local frame $\xi_1,..,\xi_k$ of $\mathcal{X}(N)$. We have the decomposition $h=\sum h^i\xi_i$. Taking into account that $\sigma_{XYZ}\nabla Q=0$, we get:

$$\sigma_{XYZ}\big[h^i(Z,W)R^{\perp}(X,Y)\xi_i-h^i(Z,W)h(X,A_{\xi_i}Y)$$

$$+h^i(Z,W)h(Y,A_{\xi_i}X)\big]=0.$$

Aplying at last Lemma 1.4 we obtain:

$$R^{\perp}(X,Y)\xi_i=h(X,A_{\xi_i}Y)-h(Y,A_{\xi_i}X)$$

at any point for every $X,Y\in\mathcal{X}(M^n)$.

As in the previous lemma we can replace ξ_i by an arbitrary $\xi\in\mathcal{X}(N)$ because of linearity of the objects in the above equation. The proof is completed.

Lemmas 2.1, 2.2 and Theorem 1.1 as well as Theorem from [4]give the following result:

Theorem 2.3 *Let M^nbe an n-dimensional simply connected manifold with a torsion-free connection ∇. Let N be a vector bundle over M^n, h - a (0,2)-symmetric N-valued tensor field on $M^n$and A_ξ - a (1,1)-tensor field on M^n. Let D be a linear connection in N. Assume that the following conditions are satisfied:*
1) $t_xh>2$ for every $x\in M^n$.
2) $(\nabla h)(X,Y,Z)=(\nabla h)(Y,X,Z)$ for every $X,Y,Z\in\mathcal{X}(M^n)$.
Then there exists an affine immersion $f:M^n\longrightarrow\mathbf{R}^{n+k}$ (where k is the fibre dimension of N) such that h is its second fundamental form, A_ξ - the shape

operator and D - the normal connection. The immersion is unique up to affine transformations.

3. Some remarks about the Gauss equation. For a local basis $\xi_1,..,\xi_k$ of $\mathcal{X}(N)$ the second fundamental form can be expressed by $h = \sum h^i\xi_i$, where $h^1,..,h^k$ are bilinear real-valued forms on M^n. At every point $x \in M^n$ we can write the Gauss equation in the form:

$$\sum_{i=1}^{k}\big[h^i(X,Z)A_{\xi_i}(Y) - h^i(Y,Z)A_{\xi_i}(X)\big] = R(X,Y)Z$$

for $X,Y,Z \in \mathcal{X}(M^n)$. Denote A_{ξ_i} by A^i. Choose a local frame $X_1,..,X_n$ M^n in a neighbourhood U of a fixed point x. Let $[a^i_{jk}]^n_{j,k=1}$ be matrices of the mappings A^i in the basis $X_1,..,X_n$ at x. Let $h^i_{ab} := h^i(X_a,X_b)$ and R^s_{abc} be components of R such that $R(X_a,X_b)X_c = \sum_{s=1}^n R^s_{abc}X_s$.

Then we have:

$$\sum_{i=1}^{k}(h^i_{ac}a^i_{sb} - h^i_{bc}a^i_{sa}) = R^s_{abc} \tag{3.1}$$

for $a,b,c,s = 1,..,n$. For every s this is a system of linear equations with respect to a^i_{sj}. Because of the skew-symmetry of R each system has $\frac{n}{2}(n^2-n)$ equations and kn variables. Notice that the coefficient matrices of all systems are the same. Denote them by H and denote by R^s the vectors $(R^s_{abc}) \in \mathbf{R}^{n(n^2-n)}$, where $a = 1,..,n$; $a < b$; $c = 1,..,n$.

We can formulate a lemma about solution of the Gauss equation:

Lemma 3.1. *Let $f : M^n \longrightarrow \mathbf{R}^{n+k}$ be an affine immersion with a transversal bundle N, $h^1,..,h^k$ be components of the second fundamental form h with respect to a local basis $\xi_1,..,\xi_k$ of sections of N in a neighbourhood U of a point $x \in M^n$. If for each s=1,..,n $rank H = rank[H,R^s]$ in U, then there exists a solution of the Gauss equation in U. Moreover, if $t_x h \geq 2$ in U, a solution of the Gauss equation is unique and smooth.*

Proof: The first part is obvious. To prove the uniqueness consider the Gauss equation with the right side equal to 0:

$$\sum\big[h^i(X,Z)A^i(Y) - h^i(Y,Z)A^i(X)\big] = 0.$$

Lemma 1.4. enforces the unique solution $A^i = 0$ for i=1,..,k. It means that the matrix H has the maximal rank nk. In equation (3.1) all the functions h^i_{ac} and R^s_{abc} are smooth. Since H has the maximal rank, it implies the smoothness of the solutions a^i_{jk} of (3.1).

Notice that the Gauss equation is independent of the choice of a basis on M^n. Therefore if a solution in one basis exists then it exists in every basis.

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