DIFFERENTIAL GEOMETRY AND APPLICATIONS
Proc. Conf., Aug. 28 – Sept. 1, 1995, Brno, Czech Republic
Masaryk University, Brno 1996, 285–292

DIFFERENTIAL STRUCTURES FOR FUNCTION SPACES

PIOTR MULTARZYŃSKI

ABSTRACT. In this paper there are presented and analyzed different differential structures on the space of smooth mappings. It turns out that the *proper* structures are especially interesting since they are well organized by the partial order relation. Then there are *admissible* structures defined and compared with the *proper* structures. There are also presented differential structures induced from the spaces of jets.

1. Introduction

Generalization of the differentiable manifold concept is motivated by many aspects among which there are applications to physical (technical) problems and those which are purely mathematical. In physics, for example, the classical problem of spacetime singularities shows clearly that the notion of a differentiable manifold is not sufficient since the manifold structure breaks down in singularities. Other examples where the manifold differential structure cannot be defined are related to function spaces the points of which are the solutions to fundamental physical equations, i.e. spaces of states of dynamical systems. An important example of a space which does not allow for a differentiable manifold structure is the so called *superspace* which is a very convenient tool in the global cosmological analysis [4]. Geometrization of function spaces is also a problem in pure mathematics. A suitable structure on the set of smooth mappings may deliver some efficient methods for the global analysis. Similar situation one can find in topology where the suitable topology defined on the family of continuous mappings allows for very interesting and elegant formulations of many problems. In analogy to the topological case there is a great variety of differential structures which organize a set of smooth maps into a differential space. So there arises a question about their classification and the criteria of choice. Some suggestions may come from the specific applications but independently it is a purely theoretical problem in geometry.

1991 *Mathematics Subject Classification.* 58A40; 58D99; 53C99; 16W99; 83E05.
Key words and phrases. Differentiable manifold; differential space; differential structure; space of smooth mappings; proper and admissible differential structures; structured spaces; Whitney topology.
This paper is in final form and no version of it will be submitted for publication elsewhere.

One also may ask a question about the reasonable directions of different generalizations of the classical differentiable manifold concept. Suggestions of the answer may come from applications and also from the analysis of geometric properties of function spaces. It seems that those directions of the generalization are the most sensible which offer interesting and non trivial geometry for function spaces within a proposed category. Such generalizations in the frame of which there are some natural proposals (or possibly the unique one !) for differential structures on the totality of all smooth mappings should interest us more than the others. As we shall observe below, the category of Sikorski differential spaces does not uniquely distinguish any specific differential structure on the family of all smooth maps but some of them seem to be remarkable.

2. Preliminaries

The differentiable manifold concept has been generalized independently in works of several authors and the need of the generalization has become evident in many mathematical problems. Physical applications of the generalization (within the Sikorski framework) are presented in [7, 8, 9, 14, 18]. R. Sikorski introduced his concept of differential spaces in [20], [21]. The literature of the subject is presented in [3]. Let M be any set and C a set of real functions defined on M. The weakest topology on M such that all functions of C are continuous will be denoted by τ_C. For any subset $A \subset M$, let C_A denote the set of all real functions β on A such that, for any $p \in A$, there exist an open neighbourhood $U \in \tau_C$ of p and a function $\alpha \in C$ satisfying the condition $\beta_{A\cap U} = \alpha_{A\cap U}$. In turn, let scC denote the family of all real functions on M which are of the form $\omega \circ (\alpha_1, \dots, \alpha_n)$ for some $\omega \in C^\infty(R^n)$, $\alpha_1, \dots, \alpha_n \in C$, and $n \in N$. A family C is called the *differential structure* (in the sense of Sikorski) on M if

$$C = C_M = scC. \tag{1}$$

The elements of a differential structure are called the *smooth functions* on M [20]. The pair (M, C) is said to be a *differential space* (*d-space*, for short). For an arbitrary set C_0 of real functions on M, the set $(scC_0)_M$ is the smallest differential structure containing C_0. A differential structure C on M is said to be *generated* by C_0 if $C = (scC_0)_M$. We shall write also $C = gen(C_0)$, if M is known from the context. Let M be an arbitrary set and let $SIK(M)$ denote the set of all Sikorski differential structures on M. There is a natural partial ordering relation (dictated by the inclusion relation) in $SIK(M)$. Namely, for any two differential structures $C, D \in SIK(M)$ we define the relation $\prec$ as follows

$$C \prec D \Longleftrightarrow C \subset D. \tag{2}$$

For any set M, the family $SIK(M)$ is a lattice, i.e. for any two differential structures C an D on M there is the least upper bound $C \vee D$ and the greatest lower bound $C \wedge D$. Moreover, every subset $T \subset SIK(M)$ has the least upper bound (briefly: *join*) $\bigvee T$ and the greatest lower bound (briefly: *meet*) $\bigwedge T$, which are defined by the formulas,

$$\bigvee T = gen(\bigcup T), \tag{3}$$

$$\bigwedge T = \bigcap T, \tag{4}$$

where $\bigcup T$ and $\bigcap T$ denote the union and the intersection of all differential structures which are elements of the family T. Thus $SIK(M)$ is a complete lattice with the greatest and the least members. The totality of all real functions on M and the collection of all constant functions on M are the greatest and the least members of $SIK(M)$, respectively. Let $(M,\mathcal{M})$ and $(N,\mathcal{N})$ be two differential spaces. A mapping $f : M \longrightarrow N$ is said to be *smooth* if $\beta \circ f \in \mathcal{M}$, for any function $\beta \in \mathcal{N}$. The totality of all smooth mappings from a d-space $(M,\mathcal{M})$ into a d-space $(N,\mathcal{N})$ will be denoted by $C^\infty(M,N)$ or by N^M . The class of all differential spaces in the sense of Sikorski forms a category whose morphisms are all smooth mappings from one differential space into another. A differential space $(M,\mathcal{M})$ will sometimes be denoted shortly by M if the differential structure is clear from the context. Let M be a set, $(N,\mathcal{N})$ be a d-space and let F be a family of some mappings from M into N. A differential structure $\mathcal{M}$ on the set M is said to be *induced* from $(N,\mathcal{N})$ by F if $\mathcal{M}$ is the smallest differential structure on M such that all mappings $f \in F$ are smooth. For the induced differential structure $\mathcal{M}$ on M we shall use the notation $\mathcal{M} = ind(F)$

$$\mathcal{M} = ind(F) = gen(\beta \circ f : f \in F \wedge \beta \in \mathcal{N}). \tag{5}$$

In turn, let $(M,\mathcal{M})$ be a d-space, N be a set and F be a family of some maps from M into N. A differential structure $\mathcal{N}$ on N is said to be *coinduced* from $(M,\mathcal{M})$ by F if N is the greatest differential structure on N such that all mappings $f \in F$ are smooth. For any two d-spaces $(M,\mathcal{M})$ and $(N,\mathcal{N})$ one defines their *cartesian product* $(M \times N, \mathcal{M} \sqcap \mathcal{N})$ which is a d-space with the *product differential structure* $\mathcal{M} \sqcap \mathcal{N}$ defined as follows,

$$\mathcal{M} \sqcap \mathcal{N} = gen(\{\alpha \circ \pi_M : \alpha \in \mathcal{M}\} \cup \{\beta \circ \pi_N : \beta \in \mathcal{N}\}), \tag{6}$$

where π_M and π_N are the projections on M and N, respectively. Let $(K,\mathcal{K})$, $(M,\mathcal{M})$ and $(N,\mathcal{N})$ be d-spaces. For any smooth map $h \in N^{K\times M}$ we define the map $\hat{h} \in (N^M)^K$ by the formula,

$$\hat{h}(k)(m) = h(k,m), \tag{7}$$

where $k \in K$, $m \in M$. The correspondence $\Lambda : h \longmapsto \hat{h}$ is a bijection and it is called the *exponential mapping*. Two mappings h and $\hat{h}$ which are related by (7) will be called the *associated maps*.

3. Proper and admissible differential structures

The terminology and the roots of the concepts presented in this section have been adapted from Arens and Dugundji [2]. As it was mentioned above a great variety of differential structures can be superimposed on the set N^M of all smooth mappings from M into N, for M and N being arbitrary differential spaces. Therefore one needs a motivation to distinguish some of them as being more satisfactory than the others. In this paper we shall concentrate on a purely mathematical suggestion which comes from homotopy theory. Namely, in the topology of deformations, equivalence of the concepts of "homotopy" and of "function space path" is usually required [5]. We shall consider the following two types of differential structures on the set N^M of all smooth mappings between differential spaces M and N. Let $(K, \mathcal{K})$ be a differential space. We shall say that L is a *proper K-differential structure* on the family N^M if for any smooth homotopy $h \in N^{K\times M}$ also the associated generalized path $\hat{h} \in \left(N^M\right)^K$ is smooth. Analogously, we shall say that U is an *admissible K-differential structure* if for any smooth generalized path $\hat{h} \in \left(N^M\right)^K$ also the associated homotopy $h \in N^{K\times M}$ is smooth. In the above formulation we use the term *generalized path* since the space of parameters is an arbitrary d-space $(K, \mathcal{K})$ instead of the standard unit interval $(I, C^\infty(I))$. By $P_K(M,N)$ we shall denote the set of all proper and by $A_K(M,N)$ the set of all admissible K-differential structures on the family N^M. The families $P_K(M,N)$ and $A_K(M,N)$ are in the natural way the partial orderings with respect to the relation $\prec$ defined by (2). The above proper and admissible K-structures are well motivated provided the differential space K itself is somehow distinguished for our purpose. Otherwise it is natural to formulate a general definition which is free of a particular d-space $(K, \mathcal{K})$. Therefore we shall define a differential structure on N^M to be *proper* if for every differential space $(K, \mathcal{K})$, the smoothness of a map $h : K \times M \longrightarrow N$ implies the smoothness of its associated map $\hat{h} : (K, \mathcal{K}) \longrightarrow N^M$. In turn, a differential structure on N^M is said to be *admissible* if for every differential space $(K, \mathcal{K})$, the smoothness of a map $\hat{h} : K \longrightarrow N^M$ implies the smoothness of its associated map $h : K \times M \longrightarrow N$.

By $P(M,N)$ and $A(M,N)$ we shall denote the family of all proper and all admissible differential structures on N^M, respectively. Directly from definition one obtains an elegant charakterization of proper and admissible structures:

$$\mathcal{L}' \prec \mathcal{L} \Longrightarrow \mathcal{L} \in P(M,N), \tag{8}$$

$$\mathcal{U} \prec \mathcal{U}' \Longrightarrow \mathcal{U}' \in A(M,N), \tag{9}$$

$$\mathcal{L} \prec \mathcal{U}, \tag{10}$$

$$P(M,N) \cap A(M,N) \ = \ \textit{empty set} \quad \textit{or} \quad \textit{one element set}, \tag{11}$$

for any $\mathcal{L}', \mathcal{U}' \in SIK(N^M)$, $\mathcal{L} \in P(M,N)$ and $\mathcal{U} \in A(M,N)$. Closely analogous relations characterize also all proper and all admissible K-structures, for any d-space $(K, \mathcal{K})$. The above statements mean that any differential structure which is smaller than some proper differential structure is also proper, and any structure which is greater than some admissible structure automatically is admissible. In turn, there is always a comparison possible between proper and admissible structures: every proper structure is smaller than every admissible one. Consequently we get the statement that there exists at most one proper-admissible differential structure on the family N^M of all smooth maps, and such a structure is simultanously the greatest proper and the least admissible differential structure. The differential structure $\bigvee P(M,N)$ may also be thought of as the differential structure coinduced on the family M^N by all those "generalized paths" in N^M , for any d-space $(K, \mathcal{K})$ of generalized parameters, whose associated homotopies are smooth. The definition and the property (8) suggest that the proper differential structures are rather weak ones. Indeed, there are very many of them which are really weak (there is even the trivial structure amongst them). Since, on the other hand, the greatest element $\bigvee L(M,N)$ of the (sub)lattice $L(M,N)$ belongs to the family of coinduced differential structures, it has a chance to be quite a strong differential structure. Therefore it is better to think of proper structures as of the "relatively" weak differential structures. The totality $P(M,N)$ of all proper differential structures constitutes a sublattice of the lattice $SIK(N^M)$ of all differential structures on N^M. Moreover, since, for any M and N, there is always the greatest proper differential structure,

$$\bigvee P(M,N) = gen\left(\bigcup P(M,N)\right), \tag{12}$$

$P(M,N)$ is the principal ideal in the lattice $SIK(N^M)$. It means that any differential structure $\mathcal{L} \in SIK(N^M)$ is proper if and only if $\mathcal{L} \prec \bigvee P(M,N)$. Unfortunately, the totality $U(M,N)$ of all admissible differential structures is much more disorganized. Sometimes there exists a structure $\mathcal{U} \in U(M,N)$ such that the condition $\mathcal{U} \prec \mathcal{U}'$ is fulfilled precisely by admissible structures $\mathcal{U}'$, but it is often so that not even $\mathcal{U} \wedge \mathcal{U}'$ is admissible for some admissible $\mathcal{U}$ and $\mathcal{U}'$. So far it remains an open question whether the meet $\bigwedge U_0$ of an arbitrary chain $U_0 \subset U(M,N)$ is, in general, an admissible structure or not. Of course, the meet of any finite chain $U_0 \subset U(M,N)$ is admissible. The above comment motivates the terminology assumed: certainly "*proper*" and merely "*admissible*". The class of admissible differential structures can be equivalently characterized as follows. Let $(M, \mathcal{M})$ and $(N, \mathcal{N})$ be arbitrary differential spaces. We shall use the so called *smooth evaluation mapping* $\Omega : M^N \times N \longrightarrow N$, defined by

$$\Omega(f,p) = f(p). \tag{13}$$

Then a differential structure $\mathcal{U} \in SIK(N^M)$ is admissible if and only if the evaluation mapping Ω is smooth. To prove this statement let us assume first that $\mathcal{U} \in SIK(N^M)$ is an admissible structure. Since the identity map $id_{N^M} : N^M \longrightarrow N^M$

is always smooth, independently of the choice of a differential structure assumed on N^M, the associated mapping

$$\Lambda^{-1}(id_{N^M}) : N^M \times M \longrightarrow N, \tag{14}$$

must be also smooth. But $\Lambda^{-1}(id_{N^M}) = \Omega$, which means that Ω is smooth. On the other hand, if the evaluation Ω is smooth then, for any differential space $(K, \mathcal{K})$, any smooth mapping $g \in \left(N^M\right)^K$ and $(k, m) \in K \times M$,

$$\{[\Lambda(\Omega \circ (g \times id_M))]\,(k)\}\,(m) = [\Omega \circ (g \times id_M)]\,(k, m) = \Omega(g(k), m) = [g(k)]\,(m),$$

i.e. $\Lambda(\Omega \circ (g \times id_M)) = g$. This means that $\Lambda^{-1}(g) = \Omega \circ (g \times id_M)$ is a smooth mapping from $K \times M$ into N (since it is a composition of smooth maps), which ends the proof.

Example: Every Euclidean space $(R^n, C^\infty(R^n))$ may be interpreted as the space $R^{\{1,2,\dots,n\}}$ of all smooth mappings from the discrete space $\{1, 2, \dots, n\}$ into the Euclidean line $(R, C^\infty(R))$, and one can prove that the Euclidean differential structure $C^\infty(R^n)$ is the (unique) proper-admissible differential structure. It means that the standard differential structure $C^\infty(R^n)$ is distinguished among all the other structures on R^n. In particular no *exotic* manifold structure on R^n can be simultaneously proper and admissible.

Among all proper differential structures the particular place takes the class of differential structures on N^M which are induced from the differential spaces $(J^k(M, N), \mathcal{F}(J^k(M, N))$ of k-jets, for $k = 0, 1, 2, \dots, \infty$ (for definition see [13]):

$$\mathcal{F}_k(N^M) = ind(\{j^k_m : N^M \ni f \longmapsto j^k_m f \in J^k(M, N), m \in M\}). \tag{15}$$

For the case $k = 0$, we have the differential structure $\mathcal{F}_0(N^M)$ induced on N^M from $(N, \mathcal{N})$ by the following evaluations (i.e. 0-jet maps j^0_m)

$$ev_m : N^M \ni f \longmapsto ev_m(f) \equiv f(m) \in N, \tag{16}$$

which means that

$$\mathcal{F}_0(N^M) = ind(Ev) \equiv gen(\{\beta \circ ev_m : \beta \in \mathcal{N}, m \in M\}). \tag{17}$$

The structure $\mathcal{F}_0(N^M)$ given by definition (17) is a generalized version of the finite or countable product differential structure. One can show that this structure is proper. Indeed, if $h : K \times M \longrightarrow N$, for some $(K, \mathcal{K})$, then $\Lambda(h) : (K, \mathcal{K}) \longrightarrow (N^M, ind(Ev))$ is smooth since, for any $m \in M$, $\beta \in N$ the function

$$\beta \circ ev_m \circ \Lambda(h) = \beta \circ h(\cdot, m) \in \mathcal{K} \tag{18}$$

is a smooth real function on K. Notice that the differential structure induced on N^M by the evaluations ev_m is proper while all structures on N^M in which the evaluation Ω (defined by (13)) is smooth turn out to be admissible.

4. Final remarks

The differential structure $\mathcal{F}_k(N^M)$, for any $k = 0, 1, 2, \dots, \infty$, determines a topology in N^M which is not trivial and not discrete. It is always a Hausdorff topology. In general case, the topology determined on N^M by the differential structure $\mathcal{F}_k(N^M)$ is not comparable with the so called *strong Whitney topology* [13]. But if the space $\{m\} \subset M$ are closed sets) then the two topologies are comparable, and the topology of the differential structure $\mathcal{F}_k(N^M)$ turns out to be weaker than the corresponding Whitney topology. But the "distance" between the two topologies is not too big. One obtains Whitney topology on N^M by adding to the topology of the differential structure $\mathcal{F}_k(N^M)$ all intersections of the open sets. A natural geometric framework for spaces whose topology is stronger than the topology determined by the differential structure is the category of the so called *structured spaces*. The notion of a structured space is a generalization of the differential space concept introduced by Sikorski [20]. This generalization was originally considered by Mostov [12]. Some foundations of structured spaces with applications to relativistic physics are presented in [10, 11]. Thus the differential space of all smooth mappings N^M with the structural sheaf determined by the Sikorski differential structure $\mathcal{F}_\infty(N^M)$ induced from the space of all ∞-jets and with the Whitney topology is an object in the category of structured spaces.

References

1. R. Arens, *A topology for spaces of transformations*, Ann. of Math. **47** (1946) 480-495.
2. R. Arens, J. Dugundji, *Topologies for function spaces*, Pacific Journ. of Math. **1** (1951) 5-31.
3. K. Buchner, M. Heller, P. Multarzyński, W.Sasin, *Literature on differential spaces*, Acta Cosmol. (Cracow) , **119** (1993) 111-129.
4. A.E. Fischer, *The Theory of Superspace*, Plenum Press, New York - London, 1970.
5. R. H. Fox, *On topologies for function spaces*, Bull. Amer. Math. Soc. **51** (1945) 429-432.
6. A. Frölicher, A. Kriegl, *Linear spaces and differentiation theory*, J. Wiley and Sons, Chichester, (1988).
7. J. Gruszczak, M.Heller, P. Multarzyński, *A generalization of manifolds as spacetime models*, J. Math. Phys. **29** (1988) 2576-2580.
8. J. Gruszczak, M. Heller, P. Multarzyński, *Physics with and without the equivalence principle*, Found. Phys. **19** (1989) 607-618.
9. M. Heller, P. Multarzyński, W. Sasin, *The algebraic approach to spacetime geometry*, Acta Cosmol. **16** (1989) 53-85.
10. M. Heller, W. Sasin, *The Structure of the b-Completion of Space-Time*, General Rel. and Gravitation 26 (1994).
11. M. Heller, W. Sasin, *Structured spaces and their application to relativistic physics*, J. Math. Phys. **36** (1995) 3644-3662.
12. M. A. Mostov, *The differentiable space structures of Milnor classifying spaces, simplicial complex and geometric relations*, J. Diff. Geom. **14** (1979) 255-293.
13. P. Multarzyński, *Whitney topology and structural stability of smooth mappings in differential spaces*, Demonstratio Math. **24** (1991) 495-514.
14. P. Multarzyński, M. Heller, *The differential and cone structures of spacetime*, Found. Phys. **20** (1990) 1005-1015.
15. P. Multarzyński, W. Sasin, *On the dimension of differential spaces*, Demonstratio Math. **23** (1990) 405-415.
16. P. Multarzyński, W. Sasin, *Algebraic characterization of the dimension of differential spaces*, Supplemento ai Rendiconti del Circolo Matematico di Palermo, Ser. II No **22** (1989) 193-199.

17. P. Multarzyński, W. Sasin, Z. Żekanowski, *Vectors and vector fields of kth order*, Demonstratio Math. **24** (1991) 557-572.
18. P. Multarzyński, Z. Żekanowski, *On general hamiltonian dynamical systems in differential spaces*, Demonstratio Math. **24** (1991) 539-555.
19. I. Satake, *On generalization of the notion of manifold*, Proc. Natl. Acad. Sci. **42** (1956) 359-363.
20. R. Sikorski, *Abstract covariant derivative*, Colloq. Math. **18** (1967) 251-271.
21. R. Sikorski, *Differential modules*, Colloq. Math. **24** (1971) 45-70.
22. W. Sasin, P. Multarzyński, *The Legendre transformation in differential spaces*, Supplemento ai Rendiconti del Circolo Matematico di Palermo, Ser. II No **30** (1993) 35-46.
23. K. Spallek, *Differenzierbare Raume*, Math. Ann. **180** (1969) 269-296.
24. W. Waliszewski, *Regular and coregular mappings of differential spaces*, Ann. Polon. Math. **30** (1975) 263-281.

Institute of Mathematics, Warsaw University of Technology
00-661 Warsaw, Pl. Politechniki 1, Poland