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'FALLING CAT' CONNECTIONS CONSTRUCTED FROM THE MOMENTUM MAP

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We consider a standard symplectic dynamics on TM generated by a natural Lagrangian L. The Lagrangian is assumed to be invariant with respect to the action TR_g of a Lie group G lifted from the free and proper action R_g of G on M. It is shown that under these conditions a connection on principal bundle $\pi : M \to M/G$ can be constructed based on the momentum map corresponding to the action TR_g. A simple explicit formula for the connection form is given. For the special case of the standard action of $G = \mathrm{SO}(3)$ on $M = \mathbb{R}^3 \times \cdots \times \mathbb{R}^3$ corresponding to a rigid rotation of a N-particle system the formula obtained earlier by Guichardet and Shapere and Wilczek is reproduced.

1. Introduction

A.Guichardet [1] and A.Shapere and F.Wilczek [2] pointed out that the phenomenon of reorientation of deformable bodies (molecules in [1] or cats, astronauts etc. in [2]) in space, based physically on the angular momentum conservation, lends itself to a simple and powerful description within the framework of the theory of connections. They showed, in particular, that in the center-of-mass system ($\vec{P} = \vec{0}$) the condition of vanishing of the total angular momentum ($\vec{L} = \vec{0}$) can be rephrased in terms of the SO(3)-connection in the principle bundle $\pi : M \to M/SO(3)$, where M is the configuration space of the deformable body, where SO(3) acts by rigid rotations (without deformation). In more detail the trajectories fulfilling $\vec{L} = \vec{0}$ represent the *horizontal* curves in the sense of the connection ("vibrational curves" in [1] as opposed to purely rotational ones given by rigid rotations).
In what follows we try to understand the origin of the connection within the standard framework [3] of lagrangian mechanics on TM.

The central object providing the link between the symmetries and conserved quantities in symplectic dynamics is the *momentum map* [4,5]. Now both $\vec{P}$ and

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$\vec{L}$ result (being linear in velocities) from the symmetries of rather special type, namely those *lifted* to TM from M. That is why the situation under consideration is the following : we study a lagrangian system (TM, L) with appropriate action of a Lie group G lifted from the configuration space M. Then we show how one can construct (under some restrictions on the Lagrangian L) a connection in the principal bundle $\pi : M \to M/G$. This connection happens to coincide with the one in [1,2] in the particular case treated there, i.e. for $G = \mathrm{SO}(3)$, M being the configuration space of N-particle system.

2. Construction of the connection form

Let

$$R_g : M \to M$$

be a right action of a Lie group G on a manifold M. Then the lifted action of G on TM is

$$TR_g : TM \to TM$$

Let $L : TM \to \mathbb{R}$ be a G-invariant Lagrangian, i.e.

$$L \circ TR_g \equiv (TR_g)^* L = L$$

for all $g \in G$. The (exact) symplectic form [3] on TM is given by

$$\omega_L = d\theta_L = dS(dL)$$

where (1,1)-type tensor field S on TM (almost tangent structure $\equiv$ vertical endomorphism) is a lift of the identity tensor on M ($S = I^{\uparrow}$; in canonical local coordinates x^i, v^i on TM, $S = dx^i \otimes \frac{\partial}{\partial v^i}$ or $S = dx^i \otimes \frac{\partial}{\partial \dot{x}^i}$ if the notation $v^i \equiv \dot{x}^i$ is used). Since ω_L is to be maximum rank 2-form, the condition

$$det(\frac{\partial^2 L}{\partial v^i \partial v^j}) \neq 0$$

must be fulfilled (nondegenerate Lagrangian).
The *momentum map* associated with the (exact symplectic) action TR_g on TM

$$P : TM \to \mathcal{G}^*$$

is given by

$$< P(v), a > := P_a(v) \quad v \in TM$$

where [7]

$$P_a = X_a^{\uparrow} L$$

$X_a^{\uparrow}$ being the *vertical lift* of the fundamental field X_a of the action R_g corresponding to the element $a \in \mathcal{G}$, where $\mathcal{G}$ is the Lie algebra of G. Fixing a basis $E_\alpha, \alpha = 1, \ldots, dim\mathcal{G}$ in $\mathcal{G}$ and the dual one E^α in $\mathcal{G}^*$ one can write

$$P = P_\alpha E^\alpha$$

where

$$P_\alpha \equiv P_{E_\alpha} : TM \to \mathbb{R}$$

are the components of P with respect to E^α.

It holds

$$(TR_g)^* P = Ad_g^* P$$

where $Ad_g^* : \mathcal{G}^* \to \mathcal{G}^*$ is the coadjoint action of G on $\mathcal{G}^*$. If $\Omega^k(\mathcal{M}, \rho)$ denotes the space of k-forms of type ρ on G-space $\mathcal{M}$ (i.e. V-valued k-forms on $\mathcal{M}$ obeying $R_g^*\sigma = \rho(g^{-1})\sigma$, ρ being a representation of G in V), we see that

$$P \in \Omega^0(TM, Ad^*)$$

- it is $\mathcal{G}^*$ - valued 0-form of type Ad^* on TM.
Thus $P \in \Omega^0(TM, Ad^*)$ is available. What we need is a connection on

$$\pi : M \to M/G$$

i.e. a connection form

$$A \in \Omega^1(M, Ad)$$

such that

$$< A, X_a >= a$$

There is a way to construct some A from P. It is done in 3 steps :
1. if P_α are linear in velocities, then

$$P_\alpha(x, v) = P_{\alpha i}(x) v^i \mapsto \tilde{P}_\alpha \equiv P_{\alpha i}(x) dx^i \in \Omega^1(M, Ad^*)$$

This restrics L to "natural" lagrangian

$$L(x, v) = \frac{1}{2} g_{ij}(x) v^i v^j - U(x)$$

2. Ad^* changes to Ad when some $\hat{h} : \mathcal{G} \to \mathcal{G}^*$ is used :

$$\hat{A} := \hat{h}^{-1} \circ \tilde{P} \in \Omega^1(M, Ad)$$

3. It holds

$$< \hat{A}, X_a >= C(x)(a) \qquad C(x) : \mathcal{G} \to \mathcal{G}$$

$$C_\alpha^\gamma := g_{\alpha\beta}(x) h^{\beta\gamma} \qquad g_{\alpha\beta}(x) := g(X_\alpha, X_\beta)$$

Then

$$A := C^{-1} \circ \hat{A} = C^{-1} \circ \hat{h}^{-1} \circ \tilde{P}$$

is already a connection form . Explicitly [7]

$$A = A^\alpha E_\alpha = (g^{\alpha\beta} \tilde{P}_\beta) E_\alpha = g^{\alpha\beta} (\flat_g X_\beta) E_\alpha$$

($\hat{h}$ dropped from the resulting formula - it is not necessary).

3. Properties of A

1. The physical meaning of horizontality : Let $\gamma : \mathbb{R} \to M$ be a curve on M representing some motion of the system under consideration. Then

$$< A, \dot{\gamma} >= 0 \ \Rightarrow \ P_\alpha(\hat{\gamma}(t)) = 0$$

($\hat{\gamma}$ is the lift of γ to TM). Thus a horizontal curve is such motion of the system that all conserved quantities P_α have all the time zero value (remember $\vec{P} = \vec{0}$, $\vec{L} = \vec{0}$ in the Introduction).
2. if W horizontal, then

$$0 =< A, W >= g^{\alpha\beta} g(X_\beta, W) E_\alpha$$

or

$$Ver_x M \perp Hor_x M$$

(can serve [1] as a definition of connection).

4. Special cases

1. A man walking on a gramophone disc.
Let us have a gramophone disc (its moment of inertia with respect of the axis being I) and a "man" (point mass m) on it. If the angle α measures the orientation of the disc with respect to the outer space and r, φ are the polar coordinates of the man with respect to the disc, the Lagrangian of the system is

$$L(r, \varphi, \alpha, \dot{r}, \dot{\varphi}, \dot{\alpha}) = \frac{1}{2} I \dot{\alpha}^2 + \frac{1}{2} m (\dot{r}^2 + r^2 (\dot{\alpha} + \dot{\varphi})^2) - U(r, \varphi)$$

(interaction of the point mass m with the disc depends only on their relative position). Then the computation gives ($G = SO(2)$ here)

$$A = A^1 E_1 = g^{11} (\flat_g X_1) E_1 = \frac{1}{I + mr^2} ((I + mr^2) d\alpha + mr^2 d\varphi) = d\alpha + \frac{mr^2}{I + mr^2} d\varphi$$

(one can take $E_1 = 1$ since $\mathcal{G} = \mathbb{R}$). A curve $\gamma \leftrightarrow (r(t), \varphi(t), \alpha(t))$ is horizontal if

$$I\dot{\alpha}(t) + mr^2(t)(\dot{\alpha}(t) + \dot{\varphi}(t)) = 0$$

- just vanishing of the total angular momentum of the system.

2. N-particle system (deformable body)
Let $\vec{r}_a$, $a = 1, \ldots N$ denote the radius vector of a-th particle , x^i_a its i-th component (i = 1,2,3), m_a its mass. There is a natural action of the Euclidean group $G = E(3)$ on the configuration space of the N-particle system, consisting in rigid rotations and translations

$$\vec{r}_a \mapsto \vec{r}_a B + \vec{b} \qquad B \in SO(3)$$

Connection form computed in this case is

$$A = A_{tr} + A_{rot} = \frac{m_a dx_a^i}{m}\mathcal{E}_i +$$

$$+(-I^{ij}(\vec{r}_1,\ldots,\vec{r}_N)(\sum_a m_a \vec{r}_a \times d\vec{r}_a)_j)E_i \equiv$$

$$\equiv \frac{\tilde{p}^i}{m}\mathcal{E}_i - I^{ij}\tilde{L}_j E_i$$

where $I_{ij}(\vec{r}_1,\ldots\vec{r}_N)$ is the inertia tensor of the configuration (the part A_{rot} coincides with the results obtained in [1,2]). Horizontal motion :

$$m_a \dot{\vec{r}}_a(t) \equiv \vec{P}(t) = \vec{0}$$

$$\sum_a m_a \vec{r}_a \times \dot{\vec{r}}_a \equiv \vec{L} = \vec{0}$$

i.e. such that the total (linear) momentum $\vec{P}$ as well as the total angular momentum $\vec{L}$ of the system vanish.

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