DIFFERENTIAL GEOMETRY AND APPLICATIONS
Proc. Conf., Aug. 28 – Sept. 1, 1995, Brno, Czech Republic
Masaryk University, Brno 1996, 215–229

BUNCH CONNECTIONS

George Virsik

Abstract. A path connection in a principal bundle $P(M, G)$ is a lifting of paths in M onto paths in P. It is a well known fact that such liftings which are independent of reparametrisation, equivariant and infinitesimal of order one are in a one-to-one correspondence with ordinary connections in P. Even if r-th order jets are used, a path connection can be reasonably associated only with an r-th order connection which is the Ehresmann prolongation of a single first order connection. In order to make a similar relation hold for arbitrary (non-holonomic) r-th order connections, one needs to lift r-bunches rather than paths. Here an r-bunch in M is a map $I^r \to M$, not necessarily continuous, with a certain semi-smoothness condition. In particular, the bunch connection associated with an r-th order connection that is the product of r first order connections is investigated.

1. Path connections in principal bundles. All manifolds are assumed smooth and finite dimensional. If M is such a manifold denote by $\Lambda(M)$ the set of piecewise smooth paths in M, i.e. maps $\lambda : I \to M$, where $I = [0, 1]$. If $P(M, G)$ is a principal bundle with projection $p : P \to M$ and structure group G write $\Lambda(M) \times_M P = \{(\lambda, h) : \lambda \in \Lambda(M), h \in P, ph = \lambda(0)\}$.

A *path connection* in $P(M, G)$ ([4], [5]) is a map

$$\begin{aligned} \Sigma : \Lambda(M) \times_M P &\to \Lambda(P) \\ (\lambda, h) &\mapsto [t \mapsto \Sigma_\lambda(h; t)] \end{aligned} \tag{1.1}$$

satisfying

(1.2) $p\Sigma_\lambda(h; t) = \lambda(t)$, $t \in I$ (lifting)

(1.3) $\Sigma_\lambda(h; 0) = h$ (initial condition)

(1.4) if $\psi : [0, 1] \to [a, b] \subseteq [0, 1]$ is a piecewise smooth diffeomorphism then $\Sigma_\lambda(h; \psi(s)) = \Sigma_{\lambda\circ\psi}(\Sigma_\lambda(h; \psi(0)); s)$, $s \in I$ (reparametrisation)

(1.5) $\Sigma_\lambda(hg; t) = \Sigma_\lambda(h; t)g$, $g \in G$, $t \in I$ (equivariance)

1991 *Mathematics Subject Classification.* 53 C 05, 58 A 20.
Key words and phrases. Path connection, semi-smooth, r-bunch, bunch connection, bloop.
This paper is in final form and no version of it will be submitted for publication elsewhere.

If $h \in P(M, G)$ is fixed and $\Omega(M) \subseteq \Lambda(M)$ denotes the set of loops at $x = ph$, then the map $H_h : \Omega(M) \to G$ defined by $\Sigma_\lambda(h; 1) = hH_h(\lambda)$ is known as the *holonomy map* of the path connection Σ at h.

The condition (1.5) guarantees that a path connection defines an equivariant one-to one correspondence between the fibres $P_{\lambda(0)}$ and $P_{\lambda(t)}$ for any $t \in I$. Thus the lift (1.1) of $\lambda \in \Lambda(M)$ is also uniquely determined by some $a \in I$ and $h_a \in P$, where $h_a = \Sigma_\lambda(h; a)$. We shall write $\Sigma_\lambda(h_a; .)$ as well for such a lift. Thus $\Sigma_\lambda(h_a; s) = \Sigma_\lambda(h_0; s)$, where $h_0 \in P_{\lambda(0)}$ is given by the condition $\Sigma_\lambda(h_0; a) = h_a$. Therefore (1.3) and (1.4) can be generalised to $\Sigma_\lambda(h_a; a) = h_a$, where $ph_a = \lambda(a)$, and to $\Sigma_\lambda(h; \psi(s)) = \Sigma_{\lambda\circ\psi}(h; s)$, $s \in I$, respectively.

An infinitesimal non-holonomic r-th order connection in $P(M, G)$ is a bundle map $\Gamma : P \to \widetilde{J}_r P$ over M which satisfies $\pi^r_0 \circ \Gamma = \mathrm{id}_P$ and $\Gamma(hg) = \Gamma(h) \cdot j^r_0(t \mapsto g)$ for any $g \in G$. Here $\widetilde{J}_r P$ is the bundle of r-th order non-holonomic jets of sections of $P(M, G) \to M$ and $\cdot$ denotes the jet prolongation of the action of G on P (c.f. [1], [2] and [3]).

We shal say that a path connection (1.1) is *infinitesimal of order* $r \geqq 1$ if there exists a smooth map $\Gamma : P \to \widetilde{J}_r P$ such that

$$j^r_0 \Sigma_\lambda(h; .) = \Gamma(h) \circ j^r_0 \lambda \text{ for any } \lambda \in \Lambda(M) \text{ and any } h \in P_{\lambda(0)}. \tag{1.6}$$

Theorem 1.1. *The relation* (1.6) *with* $r = 1$ *defines a one-to-one correspondence between first order connections in* P *and path connections in* P *that are infinitesimal of order* $r = 1$.

This is a well known result. For a general r we first observe that if a path connection satisfies (1.6) then it satisfies also

$$\begin{aligned} &j^r_t \Sigma_\lambda(h; .) = \Gamma(h(t)) \circ j^r_t \lambda, \\ &\text{where } h(t) = \Sigma_\lambda(h; t), \text{ for any } h \in P_{\lambda(0)}, \ , t \in I. \end{aligned} \tag{1.7}$$

In fact, fix $t \in I$ and apply (1.4) to $\psi_t : [0, 1] \to [t, 1]$, defined as $\psi_t(s) = t + (1 - t)s$, to obtain $\Sigma_\lambda(h; \psi_t(s)) = \Sigma_{\lambda\circ\psi_t}(\Sigma_\lambda(h; t); s)$. The r-jet of the left hand side of this gives $j^r_0(\Sigma_\lambda(h; .) \circ \psi_t) = j^r_t(\Sigma_\lambda(h; .) \circ j^r_0(\psi_t)$, whereas the right hand side gives by (1.6) $j^r_0(s \mapsto \Sigma_{\lambda\circ\psi_t}(\Sigma_\lambda(h; t); s) = \Gamma(\Sigma_\lambda(h; t)) \circ j^r_0(\lambda \circ \psi_t) = \Gamma(\Sigma_\lambda(h; t)) \circ j^r_t(\lambda) \circ j^r_0(\psi_t)$, whence we get (1.7) since $j^r_0(\psi_t)$ is invertible.

Theorem 1.2. *If* (1.1) *is a path connection which satisfies* (1.6) *for some* $r \geqq 1$ *and some* r*-th order connection* $\Gamma : P \to \widetilde{J}_r P$ *then it also satisfies*

$$j^{r+1}_0 \Sigma_\lambda(h; .) = (\Gamma \star \xi)(h) \circ j^{r+1}_0 \lambda \textit{ for any } \lambda \in \Lambda(M) \textit{ and any } h \in P_{\lambda(0)},$$

where $\xi = \pi^r_1 \circ \Gamma : P \to J_1 P$ *and* $\Gamma \star \xi = \mathbf{J}(\Gamma) \circ \xi : P \to \widetilde{J}_{r+1} P$ *is the Ehresmann prolongation of* Γ (c.f. [3] and [6] for notation).

Proof. We have $(\Gamma \star \xi)(h) \circ j^{r+1}_0 \lambda = \mathbf{J}(\Gamma)\{\xi(h)\} \circ j^{r+1}_0 \lambda$, where $\xi : P \to J_1 P$ satisfies $j^1_0 \Sigma_\lambda(h; .) = \xi(h) \circ j^1_0 \lambda$. If $\xi(h) = j^1_0 \eta$, $x = \lambda(0) \in M$, $\eta : M \underset{x}{\rightsquigarrow} P$, then this says that

$$j^1_0 \Sigma_\lambda(h; .) = j^1_0(\eta \circ \lambda), \tag{1.8}$$

and $(\Gamma \star \xi)(h) \circ j_0^{r+1}\lambda$ can be written as$j_x^1(u \mapsto \Gamma(\eta(u))) \circ j_0^1(t \mapsto j_t^r\lambda) = j_0^1(t \mapsto \Gamma(\eta(t))) \circ j_t^r\lambda$. This last expression depends only on first order derivatives of $\eta \circ \lambda$ and so by (1.8) it equals $j_0^1(t \mapsto \Gamma(\Sigma_\lambda(h;t)) \circ j_t^r\lambda)$ which by (1.7) is $j_0^1(t \mapsto j_t^r\Sigma_\lambda(h;.)) = j_0^{r+1}\Sigma_\lambda(h;)$ as required.

As an immediate corollary we have a result that says that, as far as path connections are concerned, only infinitesimality of firts order is involved.

Theorem 1.3. *If* (1.1) *is a path connection which satisfies* (1.6) *for some* $r \geqq 1$ *and some* r*-th order connection* $\Gamma : P \to \widetilde{J}_r P$ *then it also satisfies*

$$j_0^r\Sigma_\lambda(h;.) = (\xi \star \xi \star \ldots \star \xi)(h) \circ j_0^r\lambda \text{ for any } \lambda \in \Lambda(M) \text{ and any } h \in P_{\lambda(0)},$$

where $\xi = \pi_1^r \circ \Gamma : P \to J_1 P$.

In order to have a formula like (1.6) for general non-holonomic connections, ie. which are not only powers of first order connections, we need to replace holonomic jets of paths by non-holonomic jets of what we shall call bunches. The generalisation of (1.6) will be given by (3.12) below.

2. Non-holonomic jets. Let M and N be smooth manifolds. A typical element of $\widetilde{J}^r(M,N)_x$ can be written as

(2.1) $\tilde{j}_x^r f =$

$$j_x^1(u_1 \mapsto j_{u_1}^1(u_2 \mapsto j_{u_2}^1(u_3 \mapsto \ldots \mapsto j_{u_{r-1}}^1(u_r \mapsto f(u_1,u_2,\ldots,u_r))\ldots),$$

or briefly, as $\tilde{j}_x^r f = \tilde{j}_x^r(\mapsto f(u_1,u_2,\ldots,u_r))$, where $f : M \times \ldots \times M \rightsquigarrow N$ is defined in a neighbourhood of $(x,\ldots,x)$ and has the appropriate derivatives at $(x,\ldots,x)$. Note that f need not be even continuous at $(x,\ldots,x)$.

In particular, $\tilde{j}_x^r f \in \widetilde{J}^r(\mathbb{R},\mathbb{R})_0$ is uniquely determined by the 2^r derivatives

$$(2.2) \qquad \lambda_{\iota_1,\ldots,\iota_r} = (D_1^{\iota_1} \circ \delta_1 \circ D_2^{\iota_2} \circ \delta_2 \circ \ldots \circ D_r^{\iota_r})\lambda(0,\ldots,0),\ \iota_1,\ldots,\iota_r = 0 \text{ or } 1,$$

where D_i^ι is the identity operator if $\iota = 0$ and the partial derivative D_i if $\iota = 1$. The operator

$$\delta_s : \mathrm{Map}(\mathbb{R}^{s+1},\mathbb{R}) \to \mathrm{Map}(\mathbb{R}^s,\mathbb{R}), \quad s = 1,\ldots,r-1,$$

assigns to the function g the function $\delta_s[g] : (u_1,\ldots,u_s) \mapsto g(u_1,\ldots,u_s,u_s)$. In particular, $(\delta_1 \circ \ldots \circ \delta_r)(\lambda) = \lambda \circ \Delta : u \mapsto (u,\ldots,u)$.

Recall that $Z = \tilde{j}_x^r f \in \widetilde{J}^r(M,N)_x$ is semi-holonomic iff (c.f. (9) in [6] including notation)

$$\pi_q^r(Z) = \mathbf{J}(\pi_{q-1}^{k-1})\pi_k^r(Z) \in \widetilde{J}^q(M,N)_x \text{ whenever } 1 \leqq q < k \leqq r,$$

ie. iff

$$(2.3) \quad \tilde{j}_x^q(\mapsto f(x,\ldots,x,u_{r-q+1},u_{r-q+2},\ldots,u_r)) =$$
$$\tilde{j}_x^q(\mapsto f(x,\ldots,x,u_{r-k+1},u_{r-k+1},\ldots,,u_{r-k+1},u_{r-q+2},\ldots,u_r)) \quad \text{whenever } 1 \leqq q < k \leqq r.$$

Here on the left hand side x is repeated $(r-q)$ -times, on the right hand side x is repeated $(r-k)$-times and u_{r-k+1} is repeated $(k-q+1)$-times. Also, the left hand side is a special case of the right hand side with $k=q$. In coordinates, $\tilde{j}_0^r\lambda \in \widetilde{J}^r(\mathbb{R},\mathbb{R})_0$ is semi-holonomic iff the coordinates (2.2) depend only on the number of zeros and ones among $\iota_1,\dots,\iota_r$ but not on their position.

In particular, (2.3) gives for $q=1$ the r maps

$$\begin{aligned}(2.4)\quad \mathbf{J}(\pi_0^{k-1})\pi_k^r &: \widetilde{J}^r(M,N)_x \to J^1(M,N)_x \\ \tilde{j}_x^r f &\mapsto j_x^1(u \mapsto f(x,\dots,x,u,\dots,u)) \qquad \text{for } k=1,\dots r,\end{aligned}$$

where x is repeated $(r-k)$-times and $u_{r-k+1}=u$ is repeated k-times. The jet $\tilde{j}_x^r f$ is holonomic if f can be chosen so that $f(u_1,\dots,u_r)$ depends only on u_r. An inspection of (2.3) reconfirms the fact that holonomic jets are necessarily semi-holonomic. In particular, if $M=\mathbb{R}$ then there is no distinction between holonomic and semi-holonomic jets.

Recall also the rule for the composition of non-holonomic jets (c.f. [2]) defined recurrently as follows. If $Z=j_x^1\rho \in \widetilde{J}^r(M,N)_x$ and $W=j_y^1\sigma \in \widetilde{J}^r(N,Q)_x$, where $\rho : M \rightsquigarrow \widetilde{J}^{r-1}(M,N)$ and $\sigma : N \rightsquigarrow \widetilde{J}^{r-1}(N,Q)$ are local sections in a neighbourhood of $x\in M$ and $y=\pi_0^r Z \in N$ respectively, then their composition $W\circ Z$ is given by (c.f.[1])

$$(2.5)\qquad W\circ Z = j_x^1(u \mapsto \sigma(\pi_0^{r-1}\rho(u))\circ\rho(u)).$$

Finally, let us quote without proof the following result, easily verified in coordinates.

Lemma 2.1. *If Z_1, $Z_2 \in \widetilde{J}^r(M,N)_x$ are two non-holonomic jets such that $Z_1\circ \tilde{j}_0^r\lambda = Z_2\circ\tilde{j}_0^r\lambda$ for any $\tilde{j}_0^r\lambda \in \widetilde{J}^r(\mathbb{R},M)_0^x$ then $Z_1=Z_2$.*

Note that the non-holonomity of $\tilde{j}_0^r\lambda \in \widetilde{J}^r(\mathbb{R},M)_0^x$ is essential, ie. that $Z_1\circ j_0^r\lambda = Z_2\circ j_0^r\lambda$ for all $\lambda:\mathbb{R}\underset{0}{\rightsquigarrow} M,\quad \lambda(0)=x$, does not imply $Z_1=Z_2$ (unless Z_1 and Z_2 are also holonomic).

3. Bunch connections. If M is a smooth manifold, $r\geqq 1$ an integer then a (not necessarily continuous) map $\lambda: I^r\to M$ is called *semi-smooth* if the map

$$(3.1a)\qquad \lambda\circ\Delta : I\to M, \qquad t\mapsto \lambda(t,\dots,t)$$

as well as the maps

$$(3.1b)\qquad \lambda(s_1,\dots,s_{i-1},.) : I\to M, \qquad t\mapsto\lambda(s_1,\dots,s_{i-1},t,\dots,t))$$

are smooth for any $s_1,\dots,s_{i-1}\in I^{i-1}, i=2,\dots,r$. The maps (3.1a) and (3.1b) are called the *internal paths* of λ.

A semi-smooth map $\lambda: I^r\to M$ is called an *r-bunch* if also the map

$$\begin{aligned}(3.2)\quad &\tilde{j}^r\lambda : I\to\widetilde{J}^r(\mathbb{R},M)\\ &t\mapsto \tilde{j}_t^r\lambda = j_t^1(s_1\mapsto j_{s_1}^1(s_2\mapsto j_{s_2}^1(s_3\mapsto\dots\mapsto j_{s_{r-1}}^1(s_r\mapsto\lambda(s_1,s_2,\dots,s_r))\dots)\end{aligned}$$

is smooth. Since $\lambda \circ \Delta = \pi_0^r \tilde{j}^r \lambda$ this implies the smoothness of (3.1a) but not of (3.1b). Note that any r-jet $Z \in \widetilde{J}^r(\mathbb{R}, M)_0$ can be written as $Z = \tilde{j}_0^r \lambda = \tilde{j}_0^r(\mapsto \lambda(s_1, s_2, \dots, s_r))$, where $\lambda : I^r \to M$ is an r-bunch.

An r-bunch need not be continuous. For instance, the function $\lambda : I^2 \to \mathbb{R}$, given by $\lambda(x, y) = \chi(x)(y - x)^2$, where χ is the Dirichlet function, defines a map which is a two-bunch – ie. for any $s_1 \in I$ the functions $t \mapsto \lambda(t, t), t \mapsto \lambda(s_1, t)$ as well as the function $t \mapsto \frac{d}{dt} D_2\lambda(t, t)$ are smooth – but $\lambda : I^2 \to \mathbb{R}$ is obviously not continuous. Neither does every semi-smooth $\lambda : I^r \to M$ have to be a bunch ie. make also (3.2) smooth. In fact, if $\lambda(x, y) = \chi(x)(y - x)$ then λ is clearly again semi-smooth, however $D_2\lambda(x, y) = \chi(x)$, so $t \mapsto \frac{d}{dt} D_2\lambda(t, t)$ is not smooth. Of course, for $r = 1$, a smooth path, a semi-smooth path and a one-bunch are all the one and the same thing. We shall say that an r-bunch $\lambda : I^{r-1} \times I \to M$ is *stationary on the boundary* if $\lambda(\dot{u}, t)$ is independent of t whenever $\dot{u} \in \dot{I}^{r-1}$. Note that λ is stationary on the boundary iff the internal paths $\lambda(s_1, \dots, s_{r-1}, .) : I \to M$ are constant for any $(s_1, \dots, s_{r-1}) \in \dot{I}^{r-1}$. If the preceding example is modified to $\lambda(x, y) = (\chi(x) - 1)(y - x)^2$ we obtain a two-bunch which is stationary on the boundary, ie. $\lambda(0, y)$ and $\lambda(1, y)$ are independent of y.

We shall say that an r-bunch $\lambda : I^r \to M$ is *piecewise smooth* if $\lambda \circ \theta : I \to M$ is a piecewise smooth path for any piecewise smooth $\theta : I \to I^r$. If $\lambda : I^r \to M$ is semi-smooth then for any $(s_1, \dots, s_r) \in I^r$ there is a unique path starting at $\lambda(0, \dots, 0)$, ending at $\lambda(s_1, \dots, s_r)$, and consisting of a sequence of smooth internal paths of λ. This path is

$$\lambda\langle s_1, \dots, s_r\rangle = \lambda[s_1, \dots, s_r] * \dots * \lambda[s_1, s_2] * \lambda[s_1] : I \to M, \tag{3.3}$$

where (with a suitable reparametrisation to $I \to M$) these paths are defined as (c.f. (3.1a), (3.1b))

$$\begin{aligned}
&\lambda[s_1] \text{ is the path } \lambda \circ \Delta : t \mapsto \lambda(t \dots, t), \text{ for } 0 \le t \le s_1 \\
&\lambda[s_1, s_2] \text{ is the path } \lambda(s_1, .) : t \mapsto \lambda(s_1, t \dots, t), \\
&\qquad \text{for } t \text{ from } s_1 \text{ to } s_2 \\
&\dots \\
&\lambda[s_1, \dots, s_i] \text{ is the path } \lambda(s_1, \dots, s_{i-1}, .) : t \mapsto \lambda(s_1, \dots, s_{i-1}, t \dots, t), \\
&\qquad \text{for } t \text{ from } s_{i-1} \text{ to } s_i \\
&\dots \\
&\lambda[s_1, \dots, s_r] \text{ is the path } \lambda(s_1, \dots, s_{r-1}, .) : t \mapsto \lambda(s_1, \dots, s_{r-1}, t \dots, t), \\
&\qquad \text{for } t \text{ from } s_{r-1} \text{ to } s_r.
\end{aligned} \tag{3.4}$$

Let $\Lambda^r(M)$ denote the set of r-bunches $\lambda : I^r \to M$. If $P(M, G)$ is a principal bundle write again $\Lambda^r(M) \times_M P = \{(\lambda, h) : \lambda \in \Lambda^r(M), h \in P_{\lambda(0,\dots,0)}\}$.

An r-th order *bunch connection* in $P(M, G)$ is a map

$$\begin{aligned}
\Sigma : \Lambda^r(M) \times_M P &\to \Lambda^r(P) \\
(\lambda, h) &\mapsto [(s_1, \dots, s_r) \mapsto \Sigma_\lambda(h; s_1, \dots, s_r)]
\end{aligned} \tag{3.5}$$

satisfying

$$
\begin{array}{lll}
(3.6) & p\Sigma_\lambda(h; s_1, \dots, s_r) = \lambda(s_1, \dots, s_r) & \\
& \text{for all } (s_1, \dots, s_r) \in I^r & \text{(lifting)} \\
(3.7) & \Sigma_\lambda(h; 0, \dots, 0) = h & \text{(initial condition)} \\
(3.8) & \Sigma_\lambda(hg; s_1, \dots, s_r) = \Sigma_\lambda(h; s_1, \dots, s_r)g & \\
& \text{for all } (s_1 \dots, s_r) \in I^r, g \in G & \text{(equivariance)} \\
(3.9) & \text{If one of the internal paths (3.1a) or (3.1b)} & \\
& \text{is constant then so is the corresponding} & \\
& \text{internal path of } \Sigma_\lambda(h; .) : I^r \to P. & \text{(stationary property)}
\end{array}
$$

Note that (3.9) implies that a bunch connection lifts bunches stationary on the boundary to bunches stationary on the boundary. Also, since every $\lambda(s_1, \dots, s_r) \in I^r$ can be reached from $\lambda(0, \dots, 0)$ by a succession of internal paths, namely by $\lambda\langle s_1, \dots, s_r\rangle$, it follows from (3.9) that if $\lambda =$const then so is $\Sigma_\lambda(h; .)$, ie. $\Sigma_\lambda(h; .) = h$.

As in the case of path connections, the condition (3.8) guarantees that a bunch connection defines an equivariant one-to one correspondence between the fibres $P_{\lambda(0,\dots,0)}$ and $P_{\lambda(s_1,\dots,s_r)}$ for any $(s_1, \dots, s_r) \in I^r$. Thus the lift (3.5) is uniquely determined by $\lambda \in \Lambda^r(M)$ and some $h_a \in P$, where $h_a = \Sigma_\lambda(h_0; a_1, \dots, a_r)$. We shall write again $\Sigma_\lambda(h_a; .)$ for such a bunch in P as well. Thus $\Sigma_\lambda(h_a; s_1, \dots, s_r) = \Sigma_\lambda(h_0; s_1, \dots, s_r)$, where $h_0 \in P_{\lambda(0,\dots,0)}$ is given by $\Sigma_\lambda(h_0; a_1, \dots, a_r) = h_a$. In this sense (3.7) can be generalised to

$$\Sigma_\lambda(h_a; a_1, \dots, a_r) = h_a, \text{ where } ph_a = \lambda(a_1, \dots, a_r).$$

The bunch connection (3.5) is called *piecewise smooth* (or *continuous*) if it lifts a piecewise smooth (or continuous) r-bunch λ into a piecewise smooth (or continuous) r-bunch $\Sigma_\lambda(h; .)$ respectively, for any $h \in P_{\lambda(0,\dots,0)}$. We shall see later that for $r > 1$ a bunch connections is usually not continuous. A bunch connection (3.5) of order $r \geq 1$ is *infinitesimal* if there exists a smooth map $\Gamma : P \to \widetilde{J}_r P$ such that

$$
\begin{array}{ll}
(3.11) & \tilde{j}^r_t \Sigma_\lambda(h; .) = \Gamma(\Sigma_\lambda(h; \Delta(t))) \circ \tilde{j}^r_t \lambda \\
& \text{for any } (\lambda, h) \in \Lambda^r(M) \times_M P, \, t \in I,
\end{array}
$$

in particular (c.f. (1.6)),

$$(3.12) \qquad \tilde{j}^r_0 \Sigma_\lambda(h; .) = \Gamma(h) \circ \tilde{j}^r_0 \lambda \text{ for any } (\lambda, h) \in \Lambda^r(M) \times_M P.$$

Since any $\tilde{j}^r_0 \lambda \in \widetilde{J}^r(\mathbb{R}, M)^x_0$ can be expressed by means of an r-bunch, it follows from Lemma 2.1 that if a Γ exists which satisfies (3.11) then it is unique. It follows also easily from (3.7) and (3.12) that $h = \pi^r_0 \Gamma(h)$.

On the other hand, the left hand side of $\tilde{j}^r_0 \Sigma_\lambda(hg; .) = \Gamma(hg) \circ \tilde{j}^r_0 \lambda$ equals by (3.8)

$$
\begin{array}{ll}
(3.13) & \tilde{j}^r_0(\mapsto \Sigma_\lambda(hg; s_1, \dots, s_r)) = \\
& \tilde{j}^r_0(\mapsto \Sigma_\lambda(h; s_1, \dots, s_r)) \cdot \tilde{j}^r_0(\mapsto g),
\end{array}
$$

where $\tilde{j}_0^r(\mapsto g) = j_0^r[g]$ is the r-jet of the constant g. Thus $\tilde{j}_0^r\Sigma_\lambda(h;.) \cdot j_0^r[g] = \Gamma(hg) \circ \tilde{j}_0^r\lambda$, and substitution from (3.12) gives $(\Gamma(h) \circ \tilde{j}_0^r\lambda) \cdot j_0^r[g] = \Gamma(hg) \circ \tilde{j}_0^r\lambda$. Since $j_0^r[g] = j_x^r[g] \cdot \tilde{j}_0^r\lambda$, where $x = ph = \lambda(0, \dots, 0)$, we have derived that $\Gamma(hg) \circ \tilde{j}_0^r\lambda = \{\Gamma(h) \cdot j_x^r[g]\} \circ \tilde{j}_0^r\lambda$ for any $\tilde{j}_0^r\lambda \in \widetilde{J}^r(\mathbb{R}, M)_0$. Again in view of Lemma 2.1 this implies

$$\Gamma(hg) = \Gamma(h) \cdot j_x^r[g] \tag{3.14}$$

Thus if a bunch connection of order r admits a $\Gamma : P \to \widetilde{J}_r P$ satisfying (3.12) for each $(\lambda, h) \in \Lambda^r(M) \times_M P$ then this Γ is unique and it is necessarily an r-th order connection in $P(M, G)$. On the other hand, an r-th order connection Γ will not necessarily determine a bunch connection satisfying (3.13). And if, this bunch connection is not unique, since $t \mapsto \tilde{j}_t^r\lambda$ determines the bunch $\lambda : I^r \to M$ only in a neighbourhood of the diagonal.

Note that a one-bunch is always piecewise smooth and it is the same thing as a smooth path. A path connection is the same thing as a first order piecewise smooth bunch connection, which is also parameter independent since in this case the infinitesimality property implies parameter independence.

The fact that a holonomic r-jet $j_x^r f$ is a special case of (2.1), where $f(u_1, \dots, u_r) = f(u_r)$, leads us to call an r-bunch $\lambda : I^r \to M$ *holonomic* if it is given by a smooth path $I \to M$, ie. if $\lambda(s_1, \dots, s_r) = \lambda(u_r)$. Clearly, any holonomic r-bunch is piecewise smooth. The infinitesimality condition (3.12) now has the same right hand side as (1.6) so we get from Theorem 1.3 directly

Theorem 3.1. *An infinitesimal bunch connection* (3.5) *lifts holonomic bunches to holonomic bunches iff it is associated with the Ehresmann prolongation* $\xi \star \dots \star \xi$ *of a first order connection*ξ *in* $P(M, G)$*, ie. iff in* (3.12) *one can replace* $\Gamma(h)$ *by* $(\xi \star \dots \star \xi)(h), \xi = \pi_1^r \circ \Gamma$*, for any holonomic* $\lambda \in \Lambda^r(M)$ *– in fact,* $\lambda \in \Lambda(M)$ *– and* $h \in p^{-1}(\lambda(0))$.

Let $\xi_k = \mathbf{J}(\pi_0^{k-1})\pi_k^r \circ \Gamma : P \to J_1 P$, $k = 1, \dots, r$, be the r first order connections determined by Γ. Applying $\mathbf{J}(\pi_0^{k-1})\pi_k^r$ to (3.11) and taking into account (2.4) we obtain

$$j_t^1(s \mapsto \Sigma_\lambda(h; t, \dots, t, s, \dots, s)) = \xi_k(h_t) \circ j_t^1(\lambda(t, \dots, t, s, \dots, s)) \tag{3.15}$$

where t is repeated $(r - k)$-times, s is repeated k-times and $h_t = \Sigma_\lambda(h; \Delta(t))$. Thus the one jet at $t \in I$ of the internal path of $\Sigma_\lambda(h; t, \dots, t, .)$ is the lift of the one jet at $t \in I$ of the internal path $\lambda(t, \dots, t, .)$ via the connection $\xi_k : P \to J_1 P$ as in the case of infinitesimal path connections.

This suggests a bunch connection associated with an r-tuple $\xi_1, \xi_2, \dots, \xi_r$ of first order connections in $P(M, G)$ defined by means of lifts along internal paths. This bunch connection will be related to $\Gamma = \xi_1 \star \xi_2 \star \dots \star \xi_r$ in the sense of (3.11). With a slight abuse of language we shall refer to this bunch connection – to be defined explicitly in the next Section – as *the* bunch connection associated with $\xi_1 \star \xi_2 \star \dots \star \xi_r$, though it does not follow from (3.11) that it is the only one satisfying the infinitesimality condition (3.11) with $\Gamma = \xi_1 \star \xi_2 \star \dots \star \xi_r$.

4. The bunch connection associated with the connection $\xi_1\star\xi_2\star\ldots\star\xi_r$. Let now $\Gamma = \xi_1\star\xi_2\star\ldots\star\xi_r : P \to \widetilde{J}_r P$, where $\xi_1, \xi_2, \ldots, \xi_r$ are first order connections. Recall that $\xi_1\star\xi_2\star\ldots\star\xi_r = \mathbf{J}(\xi_1\star\ldots\star\xi_{r-1})\circ\xi_r$. We have also $\xi_k = \mathbf{J}(\pi_0^{k-1})\pi_k^r\Gamma : P \to J_1P$, for $k = 1, \ldots, r$.

Denote by $\Sigma^k : \Lambda(M)\times_M P \to \Lambda(P)$ the path connection corresponding to ξ_k for $k = 1, \ldots, r$. Thus by (1.7)

$$
\begin{aligned}
(4.1)\quad & j_t^1\Sigma_\lambda^k(h;.) = \xi_k(\Sigma_\lambda^k(h;t))\circ j_t^1\lambda \\
& \qquad\text{for any } h\in P_{\lambda(0)},\ t\in I;\ \text{in particular, } j_0^1\Sigma_\lambda^k(h;.) = \xi_k(h)\circ j_0^1\lambda.
\end{aligned}
$$

Recall also that $\Sigma_\lambda^k(h;.)$ is well defined for any h such that $ph = \lambda(s)$ even if $s \neq 0$. Define the bunch connection $\Sigma^{12\ldots r}$ corresponding to $\xi_1\star\xi_2\star\ldots\star\xi_r$ recurrently by

$$
\begin{aligned}
(4.2)\qquad & \Sigma_\lambda^{12}(h;s_1,s_2) := \Sigma^1_{\lambda(s_1,.)}(\Sigma^2_{\lambda\circ\Delta}(h;s_1);s_2) \quad\text{and}\\
& \Sigma_\lambda^{12\ldots r}(h;s_1\ldots,s_r) :=\\
& \Sigma^{12\ldots r-1}_{\lambda(s_1,.)}(\Sigma^r_{\lambda\circ\Delta}(h;s_1);s_2,\ldots,s_r),
\end{aligned}
$$

where $\Delta : I\to I^r$ is the diagonal map $s\mapsto(s,\ldots,s)$.

We need to verify that (4.2) take semi-smooth maps into semi-smooth maps, and that (3.6), (3.7), (3.8) and (3.9), as well as (3.11), are satisfied.

Let us start with (3.9), which is certainly true for $r = 1$. Assume it holds for $\xi_1\star\ldots\star\xi_{r-1}$, and let first $\lambda\circ\Delta$ be constant. Then $\Sigma_\lambda^{12\ldots r}(h;s_1\ldots,s_1) = \Sigma^{12\ldots r-1}_{\lambda(s_1,.)}(h;s_1\ldots,s_1) = h$ since $ph = \lambda(s_1,s_1,\ldots,s_1)$. This shows that if (3.1a) is constant then so is the corresponding internal path of $\Sigma_\lambda^{12\ldots r}(h;.)$. As for (3.1b), let $\lambda(s_1,\ldots,s_{i-1},t,\ldots,t) = \lambda(s_1,\ldots,s_{i-1},0,\ldots,0)$ for some $2\le i\le r$ and all $t\in I$. This means that the internal path (3.1b) of $\lambda(s_1,.)$ corresponding to Δ (if $i = 2$) or $s_2,\ldots,s_{i-1}$ (if $i > 2$) is constant. By the induction assumption, also the corresponding internal path of $\Sigma^{12\ldots r-1}_{\lambda(s_1,.)}(h;.)$ is constant. Thus

$$
\begin{aligned}
& \Sigma_\lambda^{12\ldots r}(h;s_1\ldots,s_{i-1},t,\ldots,t) =\\
& \Sigma^{12\ldots r-1}_{\lambda(s_1,.)}(\Sigma^r_{\lambda\circ\Delta}(h;s_1);s_2,\ldots,s_{i-1},t,\ldots,t) =\\
& \Sigma^{12\ldots r-1}_{\lambda(s_1,.)}(\Sigma^r_{\lambda\circ\Delta}(h;s_1);s_2,\ldots,s_{i-1},0,\ldots,0) =\\
& \Sigma_\lambda^{12\ldots r}(h;s_1\ldots,s_{i-1},0\ldots,0),
\end{aligned}
$$

ie. also the corresponding internal path of $\Sigma_\lambda^{12\ldots r}(h;.)$ is constant as required. This proves (3.9).

To show that (4.2) takes r-bunches into r-bunches, we need to show that the internal paths (3.1a), (3.1b) as well as (3.2) corresponding to $\Sigma_\lambda^{12\ldots r}(h;.)$ are smooth if the ones corresponding to λ are smooth. We shall first verify (3.1a) and (3.1b) starting with $r = 2$. Thus we need to show that $t\mapsto\Sigma^1_{\lambda(s_1,.)}(\Sigma^2_{\lambda\circ\Delta}(h;s_1);t)$ as well as $t\mapsto\Sigma^1_{\lambda(t,.)}(\Sigma^2_{\lambda\circ\Delta}(h;t);t)$ are smooth. The first map is smooth because it is the lift correspoonding to ξ_1 of a smooth path $t\mapsto\lambda(s_1,t)$. The second map is by the generalised form of (1.3) equal to $t\mapsto\Sigma^2_{\lambda\circ\Delta}(h;t)$ which is the lift corresponding to ξ_2 of the smooth $t \mapsto \lambda(t,t)$. Proceed now by induction, ie. assume that $t \mapsto \Sigma^{12\dots r-1}_{\lambda(s_1,.)}(h; s_2, \dots, s_{i-1}, t, \dots, t)$ are smooth for $i = 2, \dots, r$. In particular

$$t \mapsto \Sigma^{12\dots r}_{\lambda}(h; s_1 \dots, s_{i-1}, t, \dots, t) = \Sigma^{12\dots r-1}_{\lambda(s_1,.)}(\Sigma^r_{\lambda\circ\Delta}(h; s_1); s_2, \dots, s_{i-1}, t, \dots, t)$$

are smooth for $i = 2, \dots, r$. It only remains to show that also $t \mapsto \Sigma^{12\dots r}_{\lambda}(h; t, \dots, t)$ is smooth. The last expression equals $t \mapsto \Sigma^{12\dots r-1}_{\lambda(s_1,.)}(\Sigma^r_{\lambda\circ\Delta}(h;t); t, \dots, t)$ which is $t \mapsto \Sigma^r_{\lambda\circ\Delta}(h;t)$ by (3.10) and thus smooth as well. This verifies (3.1a) and (3.1b).

As for (3.2), assume again first $r = 2$. Then we have $\tilde{j}^2_t(\mapsto \Sigma^{12}_{\lambda}(h; s_1, s_2)) = j^1_t(s_1 \mapsto j_{s_1}(s_2 \mapsto \Sigma_{\lambda\circ\Delta}(h; s_1); s_2))$ which by (4.1) is

$$j^1_t(s_1 \mapsto \xi_1(\Sigma^1_{\lambda(s_1,.)}(\Sigma^2_{\lambda\circ\Delta}(h; s_1); s_1)) \circ j^1_{s_1}\lambda(s_1, .))$$

and this by the generalised form of (1.3) gives

$$\tilde{j}^2_t(\mapsto \Sigma^{12}_{\lambda}(h; s_1, s_2)) = j^1_t(s_1 \mapsto \xi_1(\Sigma^2_{\lambda\circ\Delta}(h; s_1)) \circ j^1_{s_1}\lambda(s_1, .)). \tag{4.3}$$

Since $\Sigma^2_{\lambda\circ\Delta}(h;t) \in P_{\lambda(t,t)}$, we have $\xi_2(\Sigma^2_{\lambda\circ\Delta}(h;t)) = j^1_{\lambda(t,t)}\eta$ for some section $\eta : M \rightsquigarrow P$, and by (4.1) also$\xi_2(\Sigma^2_{\lambda\circ\Delta}(h;t)) \circ j^1_t(\lambda \circ \Delta) = j^1_t \Sigma^2_{\lambda\circ\Delta}(h; .)$. We conclude that $j^1_{\lambda(t,t)}\eta \circ j^1_t(\lambda \circ \Delta) = \Sigma^2_{\lambda\circ\Delta}(h; .)$, ie. that $j^1_t(s_1 \mapsto \eta(\lambda(s_1, s_1)) = j^1_t(s_1 \mapsto \Sigma^2_{\lambda\circ\Delta}(h; s_1))$. Since the expression in (4.3) depends only on the one-jet at t of $s_1 \mapsto \Sigma^2_{\lambda\circ\Delta}(h; s_1)$, we can write it as $j^1_t(s_1 \mapsto \xi_1(\eta(\lambda(s_1, s_1))) \circ j^1_{s_1}\lambda(s_1, .))$ and this by (2.5) is $j^1_{\lambda(t,t)}(\xi_1 \circ \eta) \circ \tilde{j}^2_t\lambda$.

Thus we have proved for $t \in I$ that

$$\begin{aligned}\tilde{j}^2_t(\mapsto \Sigma^{12}_{\lambda}(h; s_1, s_2)) &= \\ \mathbf{J}(\xi_1)\{\xi_2(\Sigma^2_{\lambda\circ\Delta}(h;t))\} \circ \tilde{j}^2_t\lambda &= (\xi_1 \star \xi_2)(\Sigma^2_{\lambda\circ\Delta}(h;t)) \circ \tilde{j}^2_t\lambda,\end{aligned} \tag{4.4}$$

whence the smoothness follows from that of $\Sigma^2_{\lambda\circ\Delta}(h; .)$ and of (3.2) for λ.

Assume now that a generalisation of (4.4) holds for $r - 1$, ie. assume that

$$\begin{aligned}&\tilde{j}^{r-1}_{s_1}(\mapsto \Sigma^{12\dots r-1}_{\lambda}(h; s_2, \dots, s_r)) = \\ &(\xi_1 \star \dots \star \xi_{r-1})(\Sigma^{r-1}_{\lambda\circ\Delta}(h; s_1)) \circ \tilde{j}^{r-1}_{s_1}\lambda\end{aligned}$$

for all $s_1 \in I$ and that this map $I \to P$ is smooth. Then for each $t \in I$ we have

$$\begin{aligned}&\tilde{j}^r_t(\mapsto \Sigma^{12\dots r}_{\lambda}(h; s_1, \dots, s_r)) = \\ &j^1_t(s_1 \mapsto \tilde{j}^{r-1}_{s_1}(\mapsto \Sigma^{12\dots r-1}_{\lambda(s_1,.)}(\Sigma_{\lambda\circ\Delta}(h; s_1); s_2, \dots, s_r))) = \\ &j^1_t(s_1 \mapsto (\xi_1 \star \dots \star \xi_{r-1})(\Sigma^{r-1}_{\lambda\circ\Delta}(\Sigma^r_{\lambda\circ\Delta}(h; s_1); s_1)) \circ \tilde{j}^{r-1}_{s_1}\lambda(s_1, .)) = \\ &j^1_t(s_1 \mapsto (\xi_1 \star \dots \star \xi_{r-1})(\Sigma^r_{\lambda\circ\Delta}(h; s_1) \circ \tilde{j}^{r-1}_{s_1}\lambda(s_1, .)).\end{aligned} \tag{4.5}$$

Since $\Sigma^r_{\lambda\circ\Delta}(h;t) \in P_{\lambda(t,\dots,t)}$, we have $\xi_r(\Sigma^r_{\lambda\circ\Delta}(h;t)) = j^1_{\lambda(t,\dots,t)}\eta$, for some section $\eta : M \rightsquigarrow P$, and by (4.1) also $\xi_r(\Sigma^r_{\lambda\circ\Delta}(h;t)) \circ j^1_t(\lambda \circ \Delta) = j^1_t \Sigma^r_{\lambda\circ\Delta}(h; .)$. We conclude that $j^1_{\lambda(t,\dots,t)}\eta \circ j^1_t(\lambda \circ \Delta) = \Sigma^r_{\lambda\circ\Delta}(h;.)$, ie. that $j^1_t(s_1 \mapsto \eta(\lambda(s_1,\dots,s_1)) = j^1_t(s_1 \mapsto \Sigma^r_{\lambda\circ\Delta}(h;s_1))$. Since the expression in (4.5) depends only on the one-jet at t of $s_1 \mapsto \Sigma^r_{\lambda\circ\Delta}(h;s_1)$, we can write it as $j^1_t(s_1 \mapsto (\xi_1\star\dots\star\xi_{r-1})(\eta(\lambda(s_1,\dots,s_1)))\circ j^1_{s_1}\lambda(s_1,.))$ and this by (2.5) is $j^1_{\lambda(t,\dots,t)}((\xi_1\star\dots\star\xi_{r-1})\circ\eta)\circ \tilde{j}^r_t\lambda$. Thus we have proved for $t \in I$ that

$$
\begin{aligned}
(4.6)\qquad & \tilde{j}^r_t(\mapsto \Sigma^{12\dots r}_\lambda(h;s_1,\dots,s_r)) = \\
& \mathbf{J}(\xi_1\star\dots\star\xi_{r-1})[\xi_r(\Sigma^r_{\lambda\circ\Delta}(h;t))]\circ \tilde{j}^r_t\lambda = \\
& (\xi_1\star\dots\star\xi_r)(\Sigma^r_{\lambda\circ\Delta}(h;t))\circ \tilde{j}^r_t\lambda,
\end{aligned}
$$

whence the smoothness follows from that of $\Sigma^r_{\lambda\circ\Delta}(h;.)$ and of (3.2) for λ. This proves completely the semi-smoothness of (4.2).

By (4.2), $\Sigma^{12\dots r}_\lambda(h;s_1\dots,s_1) = \Sigma^r_{\lambda\circ\Delta}(h;s_1)$ since $p\Sigma^r_{\lambda\circ\Delta}(h;s_1) = \lambda(s_1,\dots,s_1)$. Thus (4.6) can also be written as

$$
(4.7)\qquad \tilde{j}^r_t\Sigma^{12\dots r}_\lambda(h;.) = (\xi_1\star\dots\star\xi_r)(\Sigma^{12\dots r}_\lambda(h;t,\dots,t))\circ \tilde{j}^r_t\lambda,
$$

which is the required infinitesimality condition, ie. the relation (3.11) between $\Sigma^{12\dots r}_\lambda(h;.)$ and $\xi_1\star\dots\star\xi_r$.

The remaining conditions are easy. We have $p\Sigma^{12}_\lambda(h;s_1,s_2) = \lambda(s_1,s_2)$ and $\Sigma^{12}_\lambda(h;0,0) = \Sigma^2_{\lambda\circ\Delta}(h;0) = h$. This verifies (3.6) and (3.7) for $r=2$ and similarly for $r>2$ by induction. As for (3.8), assuming it for $\Sigma^{12\dots r-1}$ one easily concludes that $\Sigma^{12\dots r}_\lambda(hg;s_1\dots,s_r) = \Sigma^{12\dots r-1}_{\lambda(s_1,.)}(\Sigma^r_{\lambda\circ\Delta}(hg;s_1);s_2,\dots,s_r) = \Sigma^{12\dots r}_\lambda(h;s_1\dots,s_r)g$. Thus we have proved

Theorem 4.1. *If $\xi_1,\xi_2\dots,\xi_r$ are first order connections in a principal bundle $P(M,G)$ then the recurrent formula* (4.2) *defines an infinitesimal r-th order bunch connection. The associated r-th order connection $\Gamma : P \to \widetilde{J}_rP$ is the product $\Gamma = \xi_1\star\dots\star\xi_r$, ie.* (4.7) *holds.*

Observe that (4.2) says that the value $\Sigma^{12\dots r}_\lambda(h;s_1,\dots,s_r)$ is obtained by transporting h along $\lambda\langle s_1,\dots,s_r\rangle$, c.f. (3.3). More precisely, first ξ_r transports h along $\lambda[s_1]$, then ξ_{r-1} transports the result along $\lambda[s_1,s_2]$, then ξ_{r-2} continues along $\lambda[s_1,s_2,s_3]$, etc., until ξ_1 ends the transport along $\lambda[s_1,\dots,s_r]$.

Thus $\Sigma^{12\dots r}_\lambda(h;s_1,\dots,s_r) = h_r$ is defined recurrently by

$$
\begin{aligned}
(4.8)\qquad & h_0 = h \\
& h_k = \Sigma^{12\dots r}_\lambda(h;s_1,s_2,\dots,s_k,\dots,s_k) = \Sigma^{r-k+1}_{\lambda[s_1,\dots,s_k]}(h_{k-1};s_k), \\
& \quad \text{for } 1 \le k \le r.
\end{aligned}
$$

Consider now the special case of a holonomic r-bunch, ie. when $\lambda(s_1,\dots,s_r) = \lambda(s_r)$. In this case $\lambda\circ\Delta = \lambda$, the internal paths (3.1b) also coincide with λ, and the paths (3.4) are just suitable restrictions of λ, e.g. $\lambda[s_1,\dots,s_i]$ is the portion of λ between $t = s_{i-1}$ and $t = s_i$. It is therefore easily seen from the above that in case of a holonomic λ, the value $\Sigma^{12\dots r}_\lambda(h;s_1,\dots,s_r)$ is obtained by transporting h along λ to $\lambda(s_1)$ using ξ_r, then continuing along λ to $\lambda(s_{2)}$ using ξ_{r-1}, etc. until ξ_1 ends the transport along λ from the point $\lambda(s_{r-1})$ to the point $\lambda(s_r)$. This agrees with Theorem 3.1 which says in particular that (4.2) will lift holonomic bunches to holonomic bunches if all the connections $\xi_1,\dots,\xi_r$ coincide.

5. Bunch loops or bloops. An r-bunch (continuous or piecewise smooth) $\lambda : I^r \to M$ is called a (continuous or piecewise smooth) *bunch loop* or *bloop*, if it is constant on the boundary of I^r, i.e. if $\lambda(s_1,\dots,s_r) = \lambda(0,\dots,0)$ whenever $(s_1,\dots,s_r) \in \dot{I}^r$, ie. whenever one of the $(s_1,\dots,s_r)$ is 0 or 1. Note that a bloop is necessarily stationary on the boundary and that the only holonomic bloop is the constant bloop. If $\lambda(0,\dots,0) = x$ we say that λ is a *bloop at* x. For example, $\lambda(x,y) = [\chi(x)-1](y-x)^2(1-y)y$defines a bloop $\lambda : I^2 \to \mathbb{R}$ which is not continuous. Note that the homotopy classes rel $\dot{I}^r$ of all continuous r-bloops at x form the r-th homotopy group of M at x.

Let $x \in M$ and $r \geq 1$ be fixed, and let Σ be an r-th order bunch connection in $P(M,G)$. If λ is an r-bloop at x then by (3.6) $\Sigma_\lambda(h; s_1,\dots,s_r) \in P$ for each $h \in P_x$ and each $(s_1,\dots,s_r) \in \dot{I}^r$. Thus we get a map $g_\lambda(h;.) : \dot{I}^r \to G$ defined by

$$\Sigma_\lambda(h; s_1,\dots,s_r) = h g_\lambda(h; s_1,\dots,s_r) \text{ for } (s_1,\dots,s_r) \in \dot{I}^r$$

satisfying

$$g_\lambda(h\gamma; s_1,\dots,s_r) = \gamma^{-1} g_\lambda(h; s_1,\dots,s_r)\gamma \text{ for any } \gamma \in G.$$

We shall call this map $g_\lambda(h;,.) : \dot{I}^r \to G$ the *holonomy map of the bloop* λ (corresponding to the given bunch connection and $h \in P$). It is continuous or piecewise smooth iff λ is. For $r = 1$ we have $\dot{I} = \{0,1\}$ with $g_\lambda(h;0) = e$ and $g_\lambda(h;1) = H_h(\lambda)$, where $H_h : \Omega(M) \to G$ is the holonomy map of the path connection at h. Thus the holonomy map of a loop is in fact constant and equals the value at this loop of the standard holonomy map associated with the path connection.

Let us calculate the values of the holonomy map $g_\lambda^{12\dots r}(h;,.) : \dot{I}^r \to G$ of the bunch connection $\Sigma_\lambda^{12\dots r}(h;.)$ that we associated with $\xi_1 \star \dots \star \xi_r$ in the preceding Section. Starting with $r = 2$, in the formula $\Sigma_\lambda^{12}(h; s_1, s_2) = \Sigma^1_{\lambda(s_1,.)}(\Sigma^2_{\lambda\circ\Delta}(h; s_1); s_2)$ we have to consider four cases:

(i) $s_1 = 0$: $\Sigma_\lambda^{12}(h; 0, s_2) = \Sigma^1_{\lambda(0,.)}(\Sigma^2_{\lambda\circ\Delta}(h; 0); s_2) = \Sigma^2_{\lambda\circ\Delta}(h;0)$ since $\lambda(0,.)$ is a constant loop. Thus by (3.7) this is h, hence $g_\lambda^{12}(h; 0, s_2) = e$.

(ii) $s_1 = 1$: $\Sigma_\lambda^{12}(h; 1, s_2) = \Sigma^1_{\lambda(1,.)}(\Sigma^2_{\lambda\circ\Delta}(h; 1); s_2) = \Sigma^2_{\lambda\circ\Delta}(h;1)$ since $\lambda(1,.)$ is a constant loop. Thus $g_\lambda^{12}(h; 1, s_2) = H^2_{h/}(\lambda \circ \Delta)$, where $H^2_h : \Omega(M) \to G$ is the holonomy map of ξ_2.

(iii) $s_2 = 0$: $\Sigma_\lambda^{12}(h; s_1, 0) = h\, g_\lambda^{12}(h; s_1, 0)$.

(iv) $s_2 = 1$: $\Sigma_\lambda^{12}(h; s_1, 1) = h\, g_\lambda^{12}(h; s_1, 1)$. This element is obtained by transporting $= h\, g_\lambda^{12}(h; s_1, 0)$ along the loop $\lambda(s_1,.)$ from $s_2 = 0$ to $s_2 = 1$. Thus it equals $\Sigma^1_{\lambda(s_1,.)}(h\, g_\lambda^{12}(h; s_1, 0); 1) = \Sigma^1_{\lambda(s_1,.)}(h, 1)\, g_\lambda^{12}(h; s_1, 0)$.

Therefore

$$g_\lambda^{12}(h; s_1, 1) = H^1_h(t \mapsto \lambda(s_1, t))\, g_\lambda^{12}(h; s_1, 0). \tag{5.1}$$

So for $r = 2$ the holonomy map of the bloop is uniquely determined by the holonomy maps of the connections ξ_1, ξ_2 and the values of $g_\lambda^{12}(h;.,0) : I \to G$.

For a general $r \geq 2$ we obtain similarly

$$g_\lambda^{12\dots r}(h; 0, s_2,\dots,s_r) = e \text{ and } g_\lambda^{12\dots r}(h; 1, s_2,\dots,s_r) = H^r_h(\lambda \circ \Delta). \tag{5.2}$$

Because of (3.10) one can replace in (4.2) the expression $\Sigma^r_{\lambda\circ\Delta}(h;s_1)$, which is the value of $\Sigma^{12\ldots r-1}_{\lambda(s_1,.)}(\Sigma^r_{\lambda\circ\Delta}(h;s_1);t,\ldots,t)$ for $t=s_1$, by its value for $t=0$, ie. by $\Sigma^{12\ldots r-1}_{\lambda(s_1,.)}(\Sigma^r_{\lambda\circ\Delta}(h;s_1);0,\ldots,0)=\Sigma^{12\ldots r}_{\lambda}(h;s_1,0,\ldots,0)$. Thus we get

$$\Sigma^{12\ldots r}_{\lambda}(h;s_1,\ldots,s_r)=\Sigma^{12\ldots r-1}_{\lambda(s_1,.)}(\Sigma^{12\ldots r}_{\lambda}(h;s_1,0,\ldots,0);s_2,\ldots,s_r).$$

If $(s_2,\ldots,s_r)\in \dot{I}^{r-1}$ then this can be written as

$$\begin{aligned}&h\,g^{12\ldots r}_{\lambda}(h;s_1,\ldots,s_r)=\\ &\Sigma^{12\ldots r-1}_{\lambda(s_1,.)}(h\,g^{12\ldots r}_{\lambda}(h;s_1,0,\ldots,0);s_2,\ldots,s_r)=\\ &h\,g^{12\ldots r-1}_{\lambda(s_1,.)}(h;s_2,\ldots,s_r)\;g^{12\ldots r}_{\lambda}(h;s_1,0,\ldots,0).\end{aligned}$$

Thus

$$g^{12\ldots r}_{\lambda}(h;s_1,\ldots,s_r)=g^{12\ldots r-1}_{\lambda(s_1,.)}(h;s_2,\ldots,s_r)\;g^{12\ldots r}_{\lambda}(h;s_1,0,\ldots,0)\tag{5.3}$$

for $(s_2,\ldots,s_r)\in \dot{I}^{r-1}$ and $s_1\in I$. It is easily seen that (5.2) are a special case of (5.3): If $(s_2,\ldots,s_r)\in I^{r-1}$ and $s_1=0$ then $\lambda(s_1,.)$ is the constant bloop and so $g^{12\ldots r-1}_{\lambda(0,.)}(h;s_2,\ldots,s_r)=e$, as well as $e=g^{12\ldots r}_{\lambda}(h;0,0,\ldots,0)$. If, instead, $s_1=1$ then $\lambda(s_1,.)$ is again the constant bloop, so $g^{12\ldots r-1}_{\lambda(1,.)}(h;s_2,\ldots,s_r)=e$, whereas $g^{12\ldots r}_{\lambda}(h;1,0,\ldots,0)=H^r_h(\lambda\circ\Delta)$. This proves the recurrence formula (5.3) for any $(s_1,\ldots,s_r)\in \dot{I}^r$ and leads easily to the following explicit formulae.

Theorem 5.1. *If s_{i+1} is the first 0 in the sequence $s_1,\ldots,s_r$, $(i\geq 1)$ then*

$$g^{12\ldots r}_{\lambda}(s_1,\ldots,s_r)=\prod_{k=1}^{i} g^{12\ldots r-k+1}_{\lambda(s_1,\ldots,s_{k-1},.)}(s_k,0,\ldots,0).\tag{5.4}$$

If s_{i+1} is the first 1 in the sequence $s_1,\ldots,s_r$, $(i\geq 1)$ then

$$\begin{aligned}&g^{12\ldots r}_{\lambda}(s_1,\ldots,s_r)=\\ &H^{r-i}_h(t\mapsto\lambda(s_1,\ldots,s_i,t,\ldots,t))\prod_{k=1}^{i} g^{12\ldots r-k+1}_{\lambda(s_1,\ldots,s_{k-1},.)}(s_k,0,\ldots,0).\end{aligned}\tag{5.5}$$

Here $\lambda(s_1,\ldots,s_{k-1},.)=\lambda$ if $k=1$. The formulae (5.4) and (5.5) include $i=0$, ie. (5.2), if the empty product is interpreted as $e\in G$. Also note that (5.4) and (5.5) do not contradict each other since $\lambda(s_1,\ldots,s_{k-1},.)$ is the constant bloop if one of $s_1,\ldots,s_{k-1}$ is 0 or 1.

Thus the holonomy map of an r-bloop is completely determined by the holonomy maps of the connections $\xi_1,\ldots,\xi_r$ and the values $g^{12\ldots r-k+1}_{\lambda(s_1,\ldots,s_{k-1},.)}(s_k,0,\ldots,0)$ for $k=1,\ldots,r-1$.

If $\Sigma^k_\gamma(h;.)=\Sigma^1_\gamma(h;.)\,\Delta^k_\gamma(h;.)$, where $\gamma:I\to M$ and $\Delta^k_\gamma(h;.):I\to G$, $k=1,\ldots,r$, then clearly $\Delta^k_\gamma(ha;.)=\mathrm{ad}(a^{-1})\Delta^k_\gamma(h;.)$ for any $a\in G$. In (4.8) we get then recurrently for $k=1,\ldots,r$

$$h_k=\Sigma^1_{\lambda[s_1,\ldots,s_k]}(h_{k-1};s_k)\,\Delta^{r-k+1}_{\lambda[s_1,\ldots,s_k]}(h_{k-1};s_k).\tag{5.6}$$

This could be used to express $g^{12\ldots r-k+1}_{\lambda(s_1,\ldots,s_{k-1},.)}(s_k,0,\ldots,0)$ by means of $\Sigma^1_\gamma(h;.)$ and $\Delta^q_\gamma(h;.)$ for $q=2,\ldots,r$ and various loops γ. We shall do this only for $r=2$ and $k=1$ to obtain easily the formula

$$g^{12}_{\lambda}(h;s_1,0)=H^1_h(\lambda\langle s_1,0\rangle)\,\Delta^2_{\lambda\circ\Delta}(h;s_1).\tag{5.7.}$$

Theorem 5.2. *Let $r > 1$, let M be simply connected, and let (4.2) be the bunch connection associated with connections $\xi_1, \dots, \xi_r$. Consider the three conditions:*

(a) *$\xi_1 = \dots = \xi_r = \xi$ and ξ is curvature-free.*

(b) *The bunch connection (4.2) is piecewise smooth.*

(c) *The holonomy map $g_\lambda^{12\dots r}(;.) : \dot{I}^r \to G$ is trivial (ie. the constant $e \in G$) at any $x \in M$.*

Then (a) *implies* (b) *and* (b) *implies* (c).

Proof. (b)$\Rightarrow$(c): If λ is a piecewise smooth r-bloop at x then for each $(s_1, \dots, s_r) \in \dot{I}^r$ there is a unique piecewise smooth loop at x consisting of a sequence of internal paths (3.3), namely the loop $\lambda\langle s_1, \dots, s_r\rangle$. If $r > 1$ then each one of these loops associated with a piecewise smooth λ is nullhomotopic (ie. contractible to the constant loop). So if (4.2) defines a piecewise smooth bunch connection then the lift of $\lambda\langle s_1, \dots, s_r\rangle$ must also be a piecewise smooth loop, ie. h transported along $\lambda\langle s_1, \dots, s_r\rangle$ remains h, thus $g_\lambda^{12\dots r}(h; s_1, \dots, s_r) = e$ for any $(s_1, \dots, s_r) \in \dot{I}^r$.

(a)$\Rightarrow$(b) If $\xi_1 = \dots = \xi_r = \xi$ then, as observed in (4.8), if λ is an r-bloop at x then the value $\Sigma_\lambda^{12\dots r}(h; s_1, \dots, s_r)$ is simply the result of the transport of h along $\lambda\langle s_1, \dots, s_r\rangle$. Since ξ is curvature-free and M is simply connected this will also be the result of the transport of h along the path $\lambda \circ \theta$, where $\theta : I \to I^r$ is any piecewise smooth path starting at $(0, \dots, 0)$ and ending at $(s_1, \dots, s_r) \in I^r$. In paricular, $\Sigma_\lambda^{12\dots r}(h; .) \circ \theta : I \to P$ will be piecewise smooth for any piecewise smooth $\theta : I \to I^r$ proving that (4.2) defines a piecewise smooth bunch connection.

Recall that a space is called n-connected, if it is path connected and all its homotopy groups of orders $1, \dots, n$ are trivial.

Theorem 5.3. *If M is $(r-1)$-connected, $(r > 1)$, then the three conditions of* Theorem 5.2 *are equivalent.*

Proof. It remains to show that (c)$\Rightarrow$(a). First assume $r = 2$. Since $g_\lambda^{12}(h; 1, s_2) = H_h^2(\lambda \circ \Delta)$ by (ii), the assumption $g_\lambda^{12}(h; 1, s_2) = e$ will imply that ξ_2 is curvature free if we can find for any smooth loop $\gamma : I \to M$ a 2-bloop λ such that $\lambda \circ \Delta = \gamma$. Since M is simply connected, there is a homotopy $F : \gamma \simeq \mathrm{const}_x \,\mathrm{rel}\{0, 1\}$, which we can assume to be smooth, ie.

$$F : I \times I \to M \tag{5.8}$$

satisfying

$$F(s, 0) = \gamma(s), F(s, 1) = F(0, t) = F(1, t) = x \text{ for all } s, t \in I.$$

Now it is easily verified, that $\lambda : I^2 \to M$ defined as

$$\begin{aligned} \lambda(s_1, s_2) =& F(s_1, 1 - \frac{s_2}{s_1}) \qquad \text{if } s_2 \le s_1 \\ & F(s_1, \frac{s_2 - s_1}{1 - s_1}) \qquad \text{if } s_1 \le s_2 \end{aligned} \tag{5.9}$$

is a piecewise smooth 2-bloop such that $\lambda(t, t) = \gamma(t)$. Thus ξ_2 is curvature free. On the other hand, it follows from (5.1) that e.g. $H_h^1(\mapsto \lambda(\frac{1}{2}, t)) = e$. This will imply that also ξ_1 is curvature free if we can find for any smooth loop $\gamma : I \to M$ a 2-bloop λ such that $\lambda(\frac{1}{2}, t) = \gamma(t)$. Using (5.8) again, we see that

$$\begin{aligned}\lambda(s_1, s_2) = & F(s_2, 1-2s_1) \qquad \text{if } s_1 \le \tfrac{1}{2}\\ & F(s_2, 2s_1 - 1) \qquad \text{if } s_1 \ge \tfrac{1}{2}\end{aligned} \tag{5.10}$$

defines the required 2-bloop. It remains to show that $\xi_1 = \xi_2$. If $\Sigma^2_\gamma(h; .) = \Sigma^1_\gamma(h; ,) \circ \Delta^2_\lambda(h; , .)$ then formula (5.7) shows that $\Delta^2_{\lambda\circ\Delta}(h; s_1) = e$ for any 2-bloop λ and any $s_1 \in I$. Since, as we have seen, any loop γ can be represented as some $\lambda \circ \Delta$, we conclude that $\Sigma^2_\gamma(h; .) = \Sigma^1_\gamma(h; ,)$ for any loop at x whence $\xi_1 = \xi_2$. This completes the proof of (c)$\Rightarrow$(a) for $r = 2$.

Proceed by induction, ie. assume (c)$\Rightarrow$(a) for some $r-1$. By (5.3) our assumption implies that $g^{12\ldots r-1}_{\lambda(s_1,.)}(h; s_2, \ldots, s_r) = e$ for any r-bloop λ. Since M is $(r-1)$-connected, any $(r-1)$-bloop γ can be seen as $\lambda(\frac{1}{2}, .)$, where λ is an r-bloop, in a similar manner as (5.8) was used to construct (5.10). Explicitly, given an $(r-1)$-bloop γ we have now a smooth

$$F : I^{r-1} \times I \to M \tag{5.11}$$

satisfying

$$\begin{aligned}&F(s, 0) = \gamma(s), F(s, 1) = x \text{ for all } s \in I^{r-1} \text{ and}\\ &F(s, t) = x \text{ for all } t \in I \text{ and } s \in \dot{I}^{r-1}\end{aligned}$$

which gives rise to the r-bloop

$$\begin{aligned}\lambda(s_1, \ldots, s_r) = & F(s_2, \ldots, s_r, 1-2s_1) \qquad \text{if } s_1 \le \tfrac{1}{2}\\ & F(s_2, \ldots, s_r, 2s_1 - 1) \qquad \text{if } s_1 \ge \tfrac{1}{2}\end{aligned}$$

such that $\lambda(\frac{1}{2}, s_2, \ldots, s_r) = F(s_2, \ldots, s_r, 0) = \gamma(s_2, \ldots, s_r)$. We conclude that also $g^{12\ldots r-1}_\gamma(h; .) : \dot{I}^{r-1} \in G$ is trivial for any $(r-1)$-bloop and so by the induction assumption, $\xi_1 = \ldots = \xi_{r-1} = \xi$ is one and the same curvature free connection.

Next we need to show that for any loop $\gamma : I \to M$ there is an r-bloop $\lambda : I^r \to M$ such that $\gamma = \lambda \circ \Delta$. Since M is $(r-1)$-connected there are smooth maps $F_q : I^{q-1} \times I \to M$, $q = 2, \ldots, r$ (c.f. (5.11)) which are homotopies from $(q-1)$-bloops to the constant bloop rel$\{0, 1\}$ constructed as follows: F_2 is the hotopy (5.8) from $\gamma = \lambda_1$ to const$_x$ rel$\{0, 1\}$ and recurrently, F_q is the homotopy from the bloop $\lambda_{q-1} : I^{q-1} \to M$ to the constant bloop at x rel $\dot{I}^{q-1}$ for $q = 3, \ldots, r$. Explicitly, we have for $q = 2, \ldots, r$

$$F_q : I^{q-1} \times I \to M,$$

satisfying

$$\begin{aligned}&F_q(s, 0) = \lambda_{q-1}(s), F_q(s, 1) = x \text{ for all } s \in I^{q-1} \text{ and}\\ &F_q(s, t) = x \text{ for all } t \in I \text{ and } s \in \dot{I}^{q-1}\end{aligned}$$

where $\lambda_1 = \gamma$, λ_2 is given by (5.9) and recurrently for $q = 3, \dots, r$

$$\begin{aligned} \lambda_q(s_1, \dots, s_q) = & F_q(s_1, \dots, s_{q-1}, 1 - \frac{s_q}{s_{q-1}}) \qquad \text{if } s_q \le s_{q-1} \\ & F_q(s_1, \dots, s_{q-1}, \frac{s_q - s_{q-1}}{1 - s_{q-1}}) \qquad \text{if } s_{q-1} \le s_q . \end{aligned}$$

It is easily verified that each λ_q is a piecewise smooth bloop. Especially $\lambda = \lambda_r$ is an r-bloop such that $\gamma = \lambda \circ \Delta$. Thus we can conclude again from (5.2) that ξ_r is curvature free.

It remains to show that $\xi_r = \xi$. Since now $\xi_1 = \dots = \xi_{r-1} = \xi$ we have also $\Delta^{r-k+1}_{\lambda[s_1,\dots,s_k]}(h_{k-1}; .) = e$ for $k = 2, \dots, r$, and so the recurrence formula (5.6) will give for $\Sigma^{12\dots r}_\lambda(s_1, \dots, s_r) = h_r$ the recurrence formulae

$$\begin{aligned} h_r &= \Sigma_{\lambda[s_1,\dots,s_r]}(h_{r-1}; s_r \\ h_{r-1} &= \Sigma_{\lambda[s_1,\dots,s_{r-1}]}(h_{r-2}; s_{r-1} \\ &\dots \\ h_2 &= \Sigma_{\lambda[s_1,s_2]}(h_1; s_2) \\ h_1 &= \Sigma_{\lambda[s_1]}(h, s_1)\, \Delta_{\lambda[s_1]}(h, s_1). \end{aligned}$$

Denoting by $\lambda *$ the composition of paths $\lambda[s_1, \dots, s_r] * \dots * \lambda[s_1, s_2]$ and observing that $\lambda[s_1] = \lambda \circ \Delta$ (c.f. (3.4)), we get

$$\Sigma^{12\dots r}_\lambda(h; s_1, \dots, s_r) = \Sigma^1_{\lambda *}(h_1; s_r), \text{ where } h_1 = \Sigma^1_{\lambda\circ\Delta}(h; s_1)\, \Delta^r_{\lambda\circ\Delta}(h; s_1).$$

Thus denoting by $< \lambda >$ the path in (3.3), we get

$$\Sigma^{12\dots r}_\lambda(h; s_1, \dots, s_r) = \Sigma^1_{<\lambda>}(h; s_r)\, \Delta^r_{\lambda\circ\Delta}(h; s_1).$$

If $s_r = 0$, this says $h\, g^{12\dots r}_\lambda(h; s_1 \dots, s_{r-1}, 0) = h\, \Delta^r_{\lambda\circ\Delta}(h; s_1)$ because $< \lambda >$ is a loop and $\xi_1 = \xi$ is curvature free. Since the holonomy map of λ was assumed trivial we conclude that $\Delta^r_{\lambda\circ\Delta}(h; s_1) = e$ for all $s_1 \in I$, and since any loop γ can be seen as $\lambda \circ \Delta$ for some r-bloop λ, we conclude further that $\xi_r = \xi$ as required. This completes the proof of Theorem 5.3.

References

[1] Ehresmann C., *Extension du calcul des jets aux jets non holonomes*, C.R.A.S. Paris 239 (1954), 1762–1764.

[2] Ehresmann C., *Sur les connexions d'ordre supérieur*, Atti V^0 Cong. Un. Mat. Italiana, Pavia-Torino, 1956, pp.326–328.

[3] Kolář I., *On some operations with connections*, Math. Nachrichten 69(1975), 297-306.

[4] Telemann C., *Connection and bundles I*, Proc. Koninkljike Nederlandse Akad. van Wettenschappen (Ser A) 72(1969), 89-112.

[5] Virsik J. (George), *On the holonomity of higher order connections*, Cahiers Topol. Géom. Diff. 12(1971), 197–212.

[6] Virsik G., *Total connections in Lie groupoids*, Arch. Math. (Brno) **31** (1995), 183–200.

Department of Mathematics, Monash University
Clayton,Victoria 3168, Australia