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DiffM DOES NOT ACT TRANSITIVELY ON THE SOLUTIONS OF AN INVARIANT VARIATIONAL PROBLEM ON *FM*

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ABSTRACT. Let $\pi : FM \to M$ be the bundle of linear frames of a C^∞ m−manifold M, and let $\mathcal{L} : J^1(FM) \to \mathbb{R}$ be an invariant under diffeomorphisms function. We set $E(\mathcal{L}) = \{j^2_x s \in J^2(FM);\ \mathcal{E}(\mathcal{L})(j^2_x s) = 0\}$, where $\mathcal{E}(\mathcal{L})$ is the Euler-Lagrange operator associated to the Lagrangian density $\mathcal{L}\theta^1 \wedge \ldots \wedge \theta^m$, $\theta = (\theta^1, \ldots, \theta^m)$ being the canonical form on FM. Let $\mathrm{Diff}_x M$ be the group of germs of diffeomorphisms $\phi : (M, x) \to (M, x)$ and let $\mathcal{S}_x$ be the set of germs of sections of π, defined around $x \in M$, which are extremals of the above density. The following result is proved. If (i) M and $\mathcal{L}$ are real analytic, (ii) $E(\mathcal{L}) \to M$ is a fibered subbundle of $J^2(FM)$, and (iii) $E(\mathcal{L})$ is formally integrable, then: 1) Under the natural action of the diffeomorphisms on linear frames, $\mathrm{Diff}_x M$ leaves invariant $\mathcal{S}_x$. 2) The action of $\mathrm{Diff}_x M$ on $\mathcal{S}_x$ is non-transitive.

1. Introduction

In General Relativity and gauge theories one usually works with Lagrangian densities invariant under a large group of transformations G: the group of all diffeomorphisms of the base manifold in the case of General Relativity and the gauge group (that is, the group of vertical automorphisms of a principal bundle) in the case of gauge theories. If $\mathcal{S}$ is the set of extremals of such an invariant Lagrangian density, the group G acts in a natural way on $\mathcal{S}$, and one is interested in the structure of the quotient space $\mathcal{S}/G$; that is, the space of solutions of the variational problem modulo the gauge group of the (geometric) structure that we consider. Note that it is essential for the Lagrangian density to be G-invariant in order for the group G to act on $\mathcal{S}$, the space of solutions.

In the present work we analyse Lagrangian densities defined on the first jet prolongation of the bundle of linear frames of an m-dimensional manifold M, which are invariant under the group of all diffeomorphisms of M (cf. [GM]). We consider the bundle of linear frames $\pi : FM \to M$ of a C^∞ manifold, and an invariant under diffeomorphisms function $\mathcal{L} : J^1(FM) \to \mathbb{R}$, associated uniquely to a Lagrangian

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density Ω_m; that is, $\Omega_m = \mathcal{L}\theta^1\wedge...\wedge\theta^m$, where $\theta = (\theta^1, ..., \theta^m)$ is the canonical form on FM. For each Lagrangian function $\mathcal{L} \in C^\infty(J^1(FM))$ there exist (globally defined) unique functions $E_{ij}(\mathcal{L}) \in C^\infty(J^2(FM))$, such that the Euler-Lagrange equations of the Lagrangian density Ω_m can be globally written as: $E_{ij}(\mathcal{L})(j^2_x s) = 0,\ \ 1 \leq i, j \leq m$. In theorem 3.3 we prove that these functions $E_{ij}(\mathcal{L})$ are natural respect to the group of diffeomorphisms on M. Therefore, assuming that (1) M and $\mathcal{L}$ are real analytic, (2) in order to be able to construct a fiber subbundle of the 2−jet bundle $E(\mathcal{L}) \to M$, called the Euler-Lagrange equation bundle, we assume that the number of functions $E_{ij}(\mathcal{L})$ that are functionally independent is locally constant, and assuming that (3) the Euler-Lagrange equation bundle $E(\mathcal{L})$ is formally integrable, taking into account that dim $E_x(\mathcal{L}) > dim\ G^3_x(M)$ and that there exists an analytic solution of any formally integrable partial differential equation (cf. [G]), we prove that the group Diff $_xM$ acts on $\mathcal{S}_x$ in a natural and non-transitive way.

The importance of this result is that it ensures that $\mathcal{S}_x/Diff_xM$ is not trivial, so that this space is a good candidate to carry on it a multisymplectic structure. Indeed, in every case we have dealt with (see §5), the quotient space $\mathcal{S}_x/Diff_xM$ is not only non-trivial but also a (germ of) infinite dimensional functional space which we describe in some examples.

2. Invariant Lagrangians on FM

Let M be an m−dimensional connected C^∞ manifold and let $\pi : FM \to M$ be the bundle of linear frames. FM is a principal fiber bundle over M whose structure group is the full linear group $GL(m;\mathbb{R})$. This bundle is endowed with a $\mathbb{R}^m$−valued 1−form $\theta = (\theta^1, ..., \theta^m)$, called the *canonical form* on FM (cf. [KN]). Any coordinate system (x_i), $\ \ 1 \leq i \leq m$, on an open subset U of M induces a coordinate system (x_i, z_{ij}), $\ \ 1 \leq i, j \leq m$, on $\pi^{-1}(U)$ by the formula $u = ((\partial/\partial x_1)_x, ..., (\partial/\partial x_m)_x) \cdot (z_{ij}(u))$, $\ \ u \in \pi^{-1}(x)$. The canonical form is given in terms of an induced system by $\theta^i = \sum_j z^{ij} dx_j$, $\ 1 \leq i \leq m$, where $(z^{ij}) = (z_{ij})^{-1}$ stands for the inverse matrix. Any diffeomorphism ϕ of M induces an automorphism $\tilde{\phi}$ of FM mapping the frame $u = (X_1, ..., X_m)$ at x into the frame $\tilde{\phi}(u) = (\phi_* X_1, ..., \phi_* X_m)$ at $\phi(x)$, such that $\phi \cdot \pi = \pi \cdot \tilde{\phi}$. Given a fibered manifold $p : P \to M$, let us denote by $p^1 : J^1(P) \to M$ the 1−jet bundle of local sections of P. If we consider the bundle of linear frames of a m−dimensional manifold M, then each automorphism $\varphi : FM \to FM$ induces a diffeomorphism $J^1(\varphi) : J^1(FM) \to J^1(FM)$ by setting $J^1(\varphi)(j^1_x s) = j^1_{\varphi_M(x)}(\varphi \cdot s \cdot \varphi_M^{-1})$, where $\varphi_M : M \to M$ is the induced map on the base manifold.

Definition 2.1. *A differentiable function $f : J^1(FM) \to \mathbb{R}$ is said to be* invariant under diffeomorphisms *if $f \cdot J^1(\tilde{\phi}) = f$, $\forall\phi$ diffeomorphism of M.*

Definition 2.2. *Given a fibered manifold $p : P \to M$, with $m = dimM$, $m + n = dimP$, a* Lagrangian density *over P is an m−form Ω_m on $J^1(P)$, that is horizontal on M with respect to the projection $p^1 : J^1(P) \to M$.*

Remark 2.3. *If the manifold M is orientable and ω_m is a volume form on M, then every Lagrangian density Ω_m can be uniquely written as $\Omega_m = L\omega_m$, for some function $L \in C^\infty(J^1P)$, which is called the* Lagrangian function associated to the volume form ω_m. *The Lagrangian function L depends on the volume form ω_m. In the particular case of the bundle of linear frames, a Lagrangian density admits a canonical decomposition, such that there is a natural and 1:1 correspondence between Lagrangian densities and Lagrangian functions, namely*

Proposition 2.4. Let $\pi : FM \to M$ be the principal fiber bundle of linear frames of an $m-$dimensional manifold M, and let $(\theta^1, ..., \theta^m)$ be the components of the canonical form θ on FM. We have:

(i) The $m-$form $\theta^1 \wedge ... \wedge \theta^m$ is $\pi-$horizontal on FM and it does not vanish at every $x \in M$.

(ii) For each Lagrangian density Ω_m on $J^1(FM)$ there is a unique function $\mathcal{L} \in C^\infty(J^1(FM))$, such that: $\Omega_m = \mathcal{L}\theta^1 \wedge ... \wedge \theta^m$.

The function $\mathcal{L}$ will be called the *Lagrangian function associated to the density* Ω_m.

Definition 2.5. *A Lagrangian density Ω_m on $J^1(FM)$ is said to be* invariant under diffeomorphisms *if for every diffeomorphism ϕ of M we have: $(J^1(\tilde{\phi}))^*\Omega_m = \Omega_m$.*

Proposition 2.6. Let $\mathcal{L} \in C^\infty(J^1(FM))$ be the Lagrangian associated to a density Ω_m on $J^1(FM)$. Then Ω_m is invariant under diffeomorphisms if and only if $\mathcal{L}$ is a function invariant under diffeomorphisms.

3. Euler-Lagrange equations for an invariant Lagrangian

Let the $m-$form $\Omega_m = L dx_1 \wedge ... \wedge dx_m$ be a Lagrangian density on $J^1(FM)$. As is well-known ([KMS], XI, §49) there exists a morphism of fibered manifolds over M, called the *Euler-Lagrange morphism* associated to Ω_m, $\mathcal{E}(\Omega_m) : J^2(FM) \to V^*(FM) \otimes \wedge^m T^*M$, whose local expression is given by: $\mathcal{E}(\Omega_m) = \sum_{i,j} \mathcal{E}_{ij}(L) dz_{ij} \otimes dx_1 \wedge ... \wedge dx_m$, where,

$$\mathcal{E}_{ij}(L) = \partial L/\partial z_{ij} - \textstyle\sum_k d/dx_k(\partial L/\partial z_k^{ij}),$$

$d/dx_k = \partial/\partial x_k + z_k^{ij}\partial/\partial z_{ij} + z_{(kl)}^{ij}\partial/\partial z_l^{ij}$ is the formal derivative with respect to x_k, and $(x_k, z_{ij}, z_k^{ij}, z_{(kl)}^{ij})$ is the coordinate system induced by (x_k, z_{ij}) on $J^2(FM)$.

Proposition 3.1. Let $E_{ij} \in \mathfrak{gl}(m;\mathbb{R})$ be the matrix such that the entry at the i-th row and the j-th column is 1 and the others entries are all zero, and let $\varepsilon_{ij} \in V^*(FM)$, $1 \le i, j \le m$, be the dual basis of E_{ij}^*, where E_{ij}^* is the fundamental vector field associated to E_{ij} *(*[KN], *I,§4)*; that is, $\varepsilon_{ij}(E_{kl}^*) = \delta_{ik}\delta_{jl}$. For each Lagrangian function $\mathcal{L} \in C^\infty(J^1(FM))$ associated to a Lagrangian density Ω_m, (i.e. $\Omega_m = \mathcal{L}\theta^1 \wedge ... \wedge \theta^m$), there exist (globally defined) unique functions $E_{ij}(\mathcal{L}) \in C^\infty(J^2(FM))$, such that $\mathcal{E}(\Omega_m)(j_x^2 s) = \sum_{i,j} E_{ij}(\mathcal{L})(j_x^2 s)(\varepsilon_{ij})_{s(x)} \otimes (\theta^1 \wedge ... \wedge \theta^m)_{s(x)}$. Accordingly, the Euler-Lagrange equations of a Lagrangian density

$\Omega_m = \mathcal{L}\theta^1 \wedge ... \wedge \theta^m$ on $J^1(FM)$ can be globally written down as: $E_{ij}(\mathcal{L})(j^2_x s) = 0, \quad 1 \leq i, j \leq m$.

Theorem 3.2. With the above notations, for any diffeomorphism ϕ of M, we have: $E_{ij}(\mathcal{L} \cdot J^1(\tilde{\phi})) = E_{ij}(\mathcal{L}) \cdot J^2(\tilde{\phi})$, where $J^2(\tilde{\phi})$ is the diffeomorphism induced by $\tilde{\phi}$ on $J^2(FM)$.

Proof. On a coordinate neighbourhood $(J^1(FU); (x_i, z_{ij}, z^{ij}_k))$ of $J^1(FM)$, the expression of the Euler-Lagrange equations for the Lagrangian $\mathcal{L}$ is (†) $E_{ij}(\mathcal{L}) = det z_{ab} \sum_k \mathcal{E}_{kj}(L) z_{ki}$ where $L \in C^\infty(J^1(FM))$ is the Lagrangian function associated to $\omega_m = dx_1 \wedge ... \wedge dx_m$, and so $\mathcal{L} = L det(z_{ij})$. Let us calculate the Euler-Lagrange equations of $(J^1(\tilde{\phi}))^* \Omega_m = \bar{L} det(\Phi_{ab}) dx_1 \wedge ... \wedge dx_m$, where $\Phi_{ab} = \partial\phi_a/\partial x_b$, and $\bar{L} = L \cdot J^1(\tilde{\phi})$. Therefore,

$$\begin{aligned}
(*)\ & \mathcal{E}_{ij}(\bar{L} det(\Phi_{ab})) = \partial(\bar{L} det(\Phi_{ab}))/\partial z_{ij} - \textstyle\sum_k d/dx_k(\partial(\bar{L} det(\Phi_{ab}))/\partial z^{ij}_k) \\
&= \partial\bar{L}/\partial z_{ij} det(\Phi_{ab}) - \textstyle\sum_k d/dx_k(det(\Phi_{ab})\partial\bar{L}/\partial z^{ij}_k) \\
&= (\partial\bar{L}/\partial z_{ij} - \textstyle\sum_k d/dx_k(\partial\bar{L}/\partial z^{ij}_k)) det(\Phi_{ab}) - \sum_k \partial\bar{L}/\partial z^{ij}_k \partial(det(\Phi_{ab}))/\partial x_k \\
&= \mathcal{E}_{ij}(\bar{L}) det(\Phi_{ab}) - \textstyle\sum_k \partial\bar{L}/\partial z^{ij}_k \partial(det(\Phi_{ab}))/\partial x_k, \quad 1 \leq i, j \leq m.
\end{aligned}$$

So, in order to know $\mathcal{E}_{ij}(\bar{L} \cdot det(\Phi_{ab}))$, we need to calculate (1) $\partial\bar{L}/\partial z_{ij}$ and (2) $d/dx_k(\partial\bar{L}/\partial z^{ij}_k)$. We have:

$$\begin{aligned}
\phi_i &= x_i \cdot J^1(\tilde{\phi}) = x_i \cdot \phi, \\
\phi_{ij} &= z_{ij} \cdot J^1(\tilde{\phi}) = z_{ij} \cdot \tilde{\phi} = \textstyle\sum_k (\partial\phi_i)/(\partial x_k) z_{kj}, \\
\phi^{ij}_k &= z^{ij}_k \cdot J^1(\tilde{\phi}) = \textstyle\sum_{l,k}((\partial^2\phi_i)/(\partial x_l \partial x_k) z_{kj} + \Phi_{ik} z^{kj}_l)\Phi^{lk},
\end{aligned}$$

where $\Phi^{lk} = \partial(\phi^{-1})_l/\partial x_k$; then we have:

$$(1)\ \partial\bar{L}/\partial z_{ij} = \textstyle\sum_a (\partial L/\partial z_{aj} \cdot J^1(\tilde{\phi}))\Phi_{ai} + \sum_{a,c}(\partial L/\partial z^{aj}_c \cdot J^1(\tilde{\phi}))\partial(\Phi_{ai})/\partial x_l \Phi^{lc}$$

$$\begin{aligned}
(2)\ & d/dx_k(\partial\bar{L}/\partial z^{ij}_k) = \textstyle\sum_{a,c} d/dx_k[(\partial L/\partial z^{aj}_c \cdot J^1(\tilde{\phi}))\Phi_{ai}\Phi^{kc}] \\
&= \textstyle\sum_{a,c}[d/dx_k(\partial L/\partial z^{aj}_c \cdot J^1(\tilde{\phi}))\Phi_{ai}\Phi^{kc} + (\partial L/\partial z^{aj}_c \cdot J^1(\tilde{\phi}))\partial(\Phi_{ai}\Phi^{kc})/\partial x_k] \\
&= \textstyle\sum_{a,c}\sum_l (d/dx_l(\partial L/\partial z^{aj}_c) \cdot J^2(\tilde{\phi}))\Phi_{ai}\delta^c_l + \sum_k(\partial L/\partial z^{aj}_c \cdot J^1(\tilde{\phi}))\partial(\Phi_{ai}\Phi^{kc})/\partial x_k.
\end{aligned}$$

Hence,
$$\begin{aligned}
\partial\bar{L}/\partial z_{ij} - \textstyle\sum_k d/dx_k(\partial\bar{L}/\partial z^{ij}_k) &= \textstyle\sum_a[(\partial L/\partial z_{aj} \cdot J^1(\tilde{\phi})) \\
&\quad - \textstyle\sum_l(d/dx_l(\partial L/\partial z^{aj}_l) \cdot J^2(\tilde{\phi})) - \sum_c(\partial L/\partial z^{aj}_c \cdot J^1(\tilde{\phi}))\partial\Phi^{lc}/\partial x_l)]\Phi_{ai} \\
&= \textstyle\sum_a[(\mathcal{E}_{aj}(L) \cdot J^2(\tilde{\phi})) - \sum_{cl}(\partial L/\partial z^{aj}_c \cdot J^1(\tilde{\phi}))\partial\Phi^{lc}/\partial x_l]\Phi_{ai}.
\end{aligned}$$

Therefore, replacing the above formula on $(*)$ we have:

$$\begin{aligned}
\mathcal{E}_{ij}(\bar{L} det(\Phi_{ab})) &= \mathcal{E}_{ij}(\bar{L}) det(\Phi_{ab}) - \textstyle\sum_l \partial\bar{L}/\partial z^{ij}_l\ \partial(det(\Phi_{ab}))/\partial x_l \\
&= \textstyle\sum_k\{(\mathcal{E}_{kj}(L) \cdot J^2(\tilde{\phi})) det(\Phi_{ab}) \\
&\quad - \textstyle\sum_c[(\partial L/\partial z^{kj}_c \cdot J^1(\tilde{\phi}))\sum_l \partial(\Phi^{lc} det(\Phi_{ab}))/\partial x_l]\}\Phi_{ki},
\end{aligned}$$

but $\sum_l \partial(\Phi^{lc} det(\Phi_{ab}))/\partial x_l = 0$. In fact, we can define m vector fields $X_1, ..., X_m$, by imposing that:

$(**) \quad d\phi_1 \wedge ... \wedge \widehat{d\phi_c} \wedge ... \wedge d\phi_m = i_{X_c}(dx_1 \wedge ... \wedge dx_m)$.

Hence:

$$\textstyle\sum_l \partial(\phi_1,...,\widehat{\phi_c},...,\phi_m)/\partial(x_1,...,\widehat{x_l},...,x_m)\ dx_1\wedge..\wedge\widehat{dx_l}\wedge...\wedge dx_m = \\ = \sum_l (-1)^{l-1}X_c(x_l)dx_1\wedge...\wedge\widehat{dx_l}\wedge...\wedge dx_m,$$

and accordingly, $X_c = (-1)^{c-1}\sum_l((det\Phi)\Phi^{lc})\partial/\partial x_l$. Moreover, $divX_c = \sum_l \partial((det\Phi)\Phi^{lc})/\partial x_l = 0$, since the form on the left hand side of $(**)$ is closed. Consequently, we have:

$$\mathcal{E}_{ij}((L\cdot J^1(\tilde{\phi}))det(\Phi_{ab})) = \textstyle\sum_k(\mathcal{E}_{kj}(L)\cdot J^2(\tilde{\phi}))det(\Phi_{ab})\Phi_{ki},$$

and replacing the above formula on (†) the result follows,

$$\begin{aligned} E_{ij}(\mathcal{L}\cdot J^1(\tilde{\phi})) &= E_{ij}((L\cdot J^1(\tilde{\phi}))det(\Phi_{ab})det(z_{ij})) \\ &= det(z_{ab})\textstyle\sum_l z_{li}\mathcal{E}_{lj}((L\cdot J^1(\tilde{\phi}))det(\Phi_{ab})) \\ &= det((z\cdot J^1(\tilde{\phi}))_{ab})\textstyle\sum_k(\mathcal{E}_{kj}(L)\cdot J^2(\tilde{\phi}))(z_{ki}\cdot J^1(\tilde{\phi})) \\ &= E_{ij}(\mathcal{L})\cdot J^2(\tilde{\phi}). \end{aligned}$$

□

4. The action of $G^3(M)$ on the equation bundle

As is well-known ([KMS], XI, §49.3) a section $s: U \to FM$ of the principal bundle of linear frames FM of a manifold M is an extremal of the variational problem defined by a Lagrangian density Ω_m on $J^1(FM)$ if and only if the 2−jet prolongation of s, satisfies the Euler-Lagrange equations of Ω_m; that is, (see prop. 3.2) if and only if $E_{ij}(\mathcal{L})\cdot j^2_x s = 0,\ \ 1\le i,j\le m,\ \ \forall x\in U$, where $\mathcal{L}\in C^\infty(J^1(FM))$ is the Lagrangian function associated to the Lagrangian density Ω_m.

Definition 4.1. *Let us define for each $x\in M$, the set $E_x(\mathcal{L})$ of 2−jet extension $j^2_x s$ of sections s of FM on x, that satisfy the Euler-Lagrange equations of Ω_m on x; that is,*

$$\forall x\in M,\quad E_x(\mathcal{L}) = \{j^2_x s\in J^2_x(FM);\ \ E_{ij}(\mathcal{L})(j^2_x s) = 0,\ \ \forall i,j\}.$$

We set $E(\mathcal{L}) = \bigcup_{x\in M} E_x(\mathcal{L})$. If $E(\mathcal{L})$ is a fibered submanifold of $\pi^2: J^2(FM)\to M$ (see [G], def.7.1), then $E(\mathcal{L})$ is a partial differential equation of order 2 on FM. In that case, the fiber bundle $E(\mathcal{L})$ over M will be called the *Euler-Lagrange equation bundle.* Let $G^3_x(M)$ be the Lie group (cf. [KMS], IV ,§12) of 3−jets, $j^3_x\phi$, with source $x\in M$, of local diffeomorphisms ϕ of M, satisfying $\phi(x) = x$, with the product given by $j^3_x\phi\cdot j^3_x\phi' := j^3_x(\phi\cdot\phi')$, and differentiable structure given by the global chart:

$$\begin{array}{rcl} G^3_x(M) & \longrightarrow & GL(m;\mathbb{R})\times(S^2(\mathbb{R}^m)^*\otimes\mathbb{R}^m)\times(S^3(\mathbb{R}^m)^*\otimes\mathbb{R}^m) \\ j^3_x\phi & \longmapsto & (D\phi(x), D^2\phi(x), D^3\phi(x)), \end{array}$$

where D, D^2 and D^3 stand for the first, second and third derivatives of ϕ with respect to a coordinate system defined around $x\in M$. Let $J^2_x(FM)$ be the set of all 2−jet of sections s on the frame bundle FM, $j^2_x s$, with source $x\in M$. The Lie group $G^3_x(M)$ acts (on the left) on the manifold $J^2_x(FM)$ by setting:

$j_x^3\phi \cdot j_x^2 s = J^2(\tilde{\phi})(j_x^2 s) = j_x^2(\tilde{\phi} \cdot s \cdot \phi^{-1})$, where $\tilde{\phi}$ is the diffeomorphism induced on FM by the diffeomorphism ϕ of M.

Proposition 4.2. If the function $\mathcal{L} \in C^\infty(J^1(FM))$ is invariant under diffeomorphisms then the Lie group $G_x^3(M)$ acts on $E_x(\mathcal{L})$, for all $x \in M$.

Notation 4.3. 1. *Let* $Diff_x M$ *be the group of germs of diffeomorphisms* $\phi : (M, x) \to (M, x)$.

2. *Let* $\mathcal{S}_x$ *be the set of germs of sections of* π, *defined around* $x \in M$, *which are the extremals of the Lagrangian density* Ω_m.

Theorem 4.4. Let M be an $m-$dimensional $(m > 1)$ connected, analytic manifold and let us consider a variational problem given by a Lagrangian density Ω_m on $J^1(FM)$. Let us assume the following three conditions:

1. The Lagrangian function $\mathcal{L}$ associated to Ω_m is analytic,

2. for every $j_x^2 s \in J^2(FM)$, the number of functions $E_{ij}(\mathcal{L}) : J_x^2(FM) \to \mathbb{R}$ that are functionally independent is locally constant. Hence $E(\mathcal{L}) \to M$ is a fiber subbundle of the $2-$jet bundle of FM, and

3. the Euler-Lagrange equation bundle $E(\mathcal{L})$ is formally integrable *(*[G], *Def. 7.2)*.

Then the group $Diff_x M$ acts on $\mathcal{S}_x$ in a natural and non-transitive way.

Proof. There exist two elements ξ, and ξ' of the space $E_x(\mathcal{L})$ that are not equivalent by the Lie group $G_x^3(M)$. This is true because $G_x^3(M)$ acts on $E_x(\mathcal{L})$, and dim $E_x(\mathcal{L}) > dim\ G_x^3(M)$ for $m > 1$. In fact, we have: $dim\ E_x(\mathcal{L}) \geq dim\ J_x^2(FM) - m^2 = m^2(m+1)(m+2)/2 - m^2 = m^3(m+3)/2$, the equality being true if all functions $E_{ij}(\mathcal{L}) : J_x^2(FM) \to \mathbb{R}$ are functionally independent, and we also have for $m > 1$: $dim\ G_x^3(M) = m^2(m^2 + 6m + 11)/6$, and so, dim $E_x(\mathcal{L}) > dim\ G_x^3(M)$. Therefore, (see [G], th.9.1) there exist two analytic solutions s and s' of the Euler-Lagrange equations, over a neighbourhood U of x, such that: $j_x^2 s = \xi, \quad j_x^2 s' = \xi', \quad \forall x \in U$. Let us assume that there is $\phi \in Diff_x M$ such that $\tilde{\phi} \cdot s \cdot \phi^{-1} = s'$; that is, the diffeomorphism ϕ of M transforms the solution s into the solution s', then we will have: $J^2(\tilde{\phi})(\xi) = \xi'$ and we are led to a contradiction. □

5. The basis of invariant Lagrangians. Examples

In [GM] it is proved that a Lagrangian $\mathcal{L} : J^1(FM) \to \mathbb{R}$ is invariant under diffeomorphisms if and only if $\mathcal{L}$ can be locally written as a differentiable function of a fixed system of invariant Lagrangians $\mathcal{L}_{jk}^i : J^1(FM) \to \mathbb{R}$, $1 \leq i \leq m$, $1 \leq j < k \leq m$; i.e., $\mathcal{L} = F \cdot \mathcal{L}_{jk}^i$, where $F \in C^\infty(\mathbb{R}^{m^2(m-1)/2})$. The Lagrangians $\mathcal{L}_{jk}^i$ have the following geometrical meaning: Given a point $\xi \in J_x^1(FM)$, let $s = (X_1, ..., X_m)$ be a local frame defined on a neighbourhood of $x \in M$, representing the above point; i.e., $\xi = j_x^1 s$. Then, we define $\mathcal{L}_{jk}^i$ by the formula: $[X_j, X_k]_x = \sum_i \mathcal{L}_{jk}^i(\xi)(X_i)_x$. It is easily proved that the definition makes sense,

because $[X_j, X_k]_x$ only depends on $j_x^1 s$. The local expression for $\mathcal{L}^i_{jk}$ is as follows: $\mathcal{L}^i_{jk} = \sum_{h,l}(z_{hj} z_h^{lk} - z_{hk} z_h^{lj}) z^{il}$.
We shall study the variational problems defined by the invariant Lagrangians $\mathcal{L}^i_{jk}$ of the above basis for $m = 2$ or 3.

Proposition 5.1. The densities $\Omega_m = \mathcal{L}^i_{jk}\theta^1 \wedge .. \wedge \theta^m$ only define two essentially different kinds of variational problems according to $i = j$ or $i \neq j, k$.

Proof. Let $\psi_\sigma : FM \to FM$ be the fibre-preserving diffeomorphisms given by $\psi_\sigma(u) = (X_{\sigma_1}, ..., X_{\sigma_m})$, where $\sigma = (\sigma_1, ..., \sigma_m) \in Perm\{1, ..., m\}$ and $u = (X_1, ..., X_m)$. It is not difficult to prove that for every $A \in GL(m; \mathbb{R})$, $u \in FM$, we have: $\psi_\sigma(u \cdot A) = u \cdot A^\sigma$, where A^σ is the matrix $A^\sigma = (A_{\sigma_1}, ..., A_{\sigma_m})$, $A_1, ..., A_m$ being the columns of A. Given two Lagrangians $\mathcal{L}^i_{jk}$ and $\mathcal{L}^a_{bc}$ as above, with $1 \leq i \leq m$, $1 \leq j < k \leq m$, $i \neq j, k$, and $1 \leq a \leq m$, $1 \leq b < c \leq m$, $a \neq b, c$, there is $\sigma \in Perm\{1, ..., m\}$ such that $\sigma_i = a$, $\sigma_j = b$, and $\sigma_k = c$. Then, we have: $(\mathcal{L}^i_{jk} \cdot J^1(\psi_\sigma))(j_x^1 s) = \mathcal{L}^i_{jk}(j_x^1(\psi_\sigma(s(x)))) = \omega_{\sigma_i}([X_{\sigma_j}, X_{\sigma_k}]) = \omega_a([X_b, X_c]) = \mathcal{L}^a_{bc}(j_x^1 s)$. Hence, $\mathcal{L}^i_{jk} \cdot J^1(\psi_\sigma) = \mathcal{L}^a_{bc}$. The diffeomorphisms ψ_σ preserve the fibres and leave invariant the canonical form on $J^1(FM)$, so we know that the Lagrangians $\mathcal{L}^i_{jk}$, with $i \neq j, k$ define the same variational problem. If we consider the Lagrangians $\mathcal{L}^i_{ij}$ and $\mathcal{L}^a_{ab}$, it suffices to take a permutation σ such that $\sigma_i = a$, and $\sigma_j = b$, and the result follows. □

(I) dim M=2. According to the above proposition we only have to study the following Lagrangian density, $\Omega_2 = \mathcal{L}^1_{12}\theta^1 \wedge \theta^2$.

Theorem 5.2. With the above hypotheses and notations, for every local frame $s = (X_1, X_2)$, with dual coframe (ω_1, ω_2), we have:

$$(j^1 s)^*\Omega_2 = \omega_1([X_1, X_2])\omega_1 \wedge \omega_2 = -d\omega_1,$$

or equivalently, Ω_2 is the horizontal part of $d\theta^1$. Consequently, Ω_2 defines a trivial variational problem; i.e., every section is an extremal.

For a general germ of frame $s = (X_1, X_2)$ around $x \in M$, the orbit of s under $\mathrm{Diff}_x M$ can be identified with the germs of non-singular vector fields at $x \in M$.

Proof. Let D be a compact submanifold of M, with $\partial D \neq \emptyset$ and dim$D$=dim$M$ = m, and let us assume that $s : D \to FM$ is a critical section for the Lagrangian density Ω_2, that is, $d/dt \int_D (j^1 s_t)^*\Omega_2 = 0$ for every one-parameter deformation s_t of the section s. By considering the one-parameter family of frames given by: $s_t(x) = s(x) \cdot A_t(x)$, $s_t : D \to FM$, $|t| < \epsilon$, $s_0 = s$, where, $A_t(x) = e^{t\varphi(x)E_{ij}}$, $1 \leq i, j \leq 2$, with $\varphi \in C^\infty(D)$, $\varphi|_{\partial D} = 0$, for E_{11} and taking into account the Stokes theorem, we obtain:

$$0 = \frac{d}{dt}\bigg|_{t=0} \int_D d(e^{t\varphi}\omega_1) = \int_D d(\varphi\omega_1) = \int_{\partial D} \varphi\omega_1.$$

Therefore, using the same argument for every E_{ij}, we have that every section is an extremal of the Lagrangian density Ω_2.

By considering a diffeomorphism ψ of M such that $\psi \cdot X_1 = \partial/\partial x_1$, and $\psi \cdot X_2 = a\partial/\partial x_1 + b\partial/\partial x_2$ with $a, b \in C^\infty(U)$, $U \subset D$, we see that the space of solutions of the variational problem given by the Lagrangian density Ω_2, cocient by the group of diffeomorphisms of M can be identified with the space of vector fields at $x \in M$. $\square$

(II) dim M=3. In this case, we have to study the following two Lagrangian densities, $\Omega_3 = \mathcal{L}^1_{12}\theta^1 \wedge \theta^2 \wedge \theta^3$, $\Omega'_3 = \mathcal{L}^3_{12}\theta^1 \wedge \theta^2 \wedge \theta^3$.

Theorem 5.3. With the above hypotheses and notations, we have

1) A local frame $s = (X_1, X_2, X_3)$, with dual coframe $(\omega_1, \omega_2, \omega_3)$, is a critical section for the Lagrangian density Ω_3 if and only if:

$$\omega_1 \wedge d\omega_3 = 0, \quad d\omega_1 \wedge \omega_3 = 0, \quad \omega_2 \wedge d\omega_3 = 0,$$
$$d\omega_1 \wedge \omega_1 = 0, \quad \omega_3 \wedge d\omega_3 = 0, \quad d\omega_1 \wedge \omega_2 = 0.$$

2) A local frame $s = (X_1, X_2, X_3)$, with dual coframe $(\omega_1, \omega_2, \omega_3)$, is a critical section for the Lagrangian density Ω'_3 if and only if:

$$d\omega_3 \wedge \omega_3 = 0, \; \omega_1 \wedge d\omega_3 + d\omega_3 \wedge \omega_1 = 0, \; \omega_2 \wedge d\omega_3 + d\omega_3 \wedge \omega_2 = 0.$$

Moreover, we have

1') Given a local coordinate system (x_1, x_2, x_3) on a neighbourhood of $x \in M$, for every critical section $s = (X_1, X_2, X_3)$ for the Lagrangian density Ω_3, there exists a local diffeomorphism ψ, with $\psi_M(x) = x$, such that: $\psi \cdot X_1 = \partial/\partial x_1$, $\psi \cdot X_2 = \delta\partial/\partial x_2$, $\delta \in C^\infty(\mathbb{R}^3)$,

$$\psi \cdot X_3 = \alpha\partial/\partial x_1 + \beta\partial/\partial x_2 + \gamma\partial/\partial x_3, \quad \alpha, \gamma \in C^\infty(\mathbb{R}^3), \quad \beta \in C^\infty(\mathbb{R}).$$

2') Given a local coordinate system (x_1, x_2, x_3) on a neighbourhood of $x \in M$, for every critical section $s = (X_1, X_2, X_3)$ for the Lagrangian density Ω'_3, there exists a local diffeomorphism ψ, keeping x fixed, such that: $\psi \cdot X_1 = f_{11}\partial/\partial x_1 + f_{21}\partial/\partial x_2$,

$$\psi \cdot X_2 = f_{12}\partial/\partial x_1 + f_{22}\partial/\partial x_2,$$
$$\psi \cdot X_3 = f_{13}\partial/\partial x_1 + f_{23}\partial/\partial x_2 + g\partial/\partial x_3, \quad f_{ij} \in C^\infty(\mathbb{R}^3), \; g \in C^\infty(\mathbb{R}).$$

Proof. Let D be a compact submanifold of M, with $\partial D \neq \emptyset$ and dim$D$=dim$M = m$, and let us assume that $s : D \to FM$ is a critical section for the Lagrangian density Ω_3, that is, $d/dt \int_D (j^1 s_t)^* \Omega_3 = 0$ for every one-parameter deformation s_t of the section s, where $(j^1 s)^*\Omega_3 = \omega_1([X_1, X_2])\omega_1 \wedge \omega_2 \wedge \omega_3 = d\omega_1 \wedge \omega_3$, (*up to sign*). By considering the one-parameter family of frames given by: $s_t(x) = s(x) \cdot A_t(x)$, $s_t : D \to FM$, $|t| < \epsilon$, $s_0 = s$, where, $A_t(x) = e^{t\varphi(x)E_{ij}}$, $1 \leq i, j \leq 3$, with $\varphi \in C^\infty(D)$, $\varphi|_{\partial D} = 0$, for E_{11} we obtain:

$$0 = \frac{d}{dt}\Big|_{t=0} \int_D d(e^{t\varphi}\omega_1) \wedge \omega_3 = \int_D d(\varphi\omega_1) \wedge \omega_3 = \int_D d(\varphi\omega_1 \wedge \omega_3) + \int_D \varphi\omega_1 \wedge d\omega_3$$

taking into account that φ is an arbitrary function and the Stokes theorem, we have, $\int_D d(\varphi\omega_1 \wedge \omega_3) = \int_{\partial D} \varphi\omega_1 \wedge \omega_3 = 0$. Consequently , $\omega_1 \wedge d\omega_3 = 0$.

Using the same argument for every E_{ij}, we obtain all the Euler-Lagrange equations, and contracting them, we have: $\mathcal{L}^1_{ij} = 0$, and $\mathcal{L}^3_{ij} = 0$, $i \neq j$, $1 \leq i, j \leq 3$. Therefore the section $s = (X_1, X_2, X_3)$ is a critical section of the action if it verifies: $[X_i, X_j] = \mathcal{L}^2_{ij} X_2$, $1 \leq i, j \leq 3$, $i \neq j$. By considering a diffeomorphism ψ of

M such that $\psi \cdot X_1 = \partial/\partial x_1$, and imposing the above conditions we obtain that $\psi \cdot X_2 = \delta\partial/\partial x_2$, where $\delta \in C^\infty(\mathbb{R}^3)$ and $\psi \cdot X_3 = \alpha\partial/\partial x_1 + \beta\partial/\partial x_2 + \gamma\partial/\partial x_3$, where $\alpha, \gamma \in C^\infty(\mathbb{R}^3)$, *and* $\beta \in C^\infty(\mathbb{R})$.

Using the same arguments as in the above case, we obtain for the Lagrangian density $\Omega'_3 = \mathcal{L}^3_{12}\theta^1 \wedge \theta^2 \wedge \theta^3$ the following Euler-Lagrange equations: $d\omega_3 \wedge \omega_3 = 0$, $\omega_1 \wedge d\omega_3 + d\omega_3 \wedge \omega_1 = 0$, and $\omega_2 \wedge d\omega_3 + d\omega_3 \wedge \omega_2 = 0$, or equivalently, $\mathcal{L}^3_{ij} = 0$, $1 \leq i, j \leq 3$, $i \neq j$. Therefore, a section $s = (X_1, X_2, X_3)$ is a critical section for the variational problem defined by Ω'_3 if it verifies: $(*)$ $[X_i, X_j] = \mathcal{L}^k_{ij} X_k$, $k = 1$, *or* 2, $1 \leq i, j \leq 3$, $i \neq j$. From the equation $[X_1, X_2] = \mathcal{L}^1_{12} X_1 + \mathcal{L}^2_{12} X_2$, we know that (X_1, X_2) is involutive and as a calculation shows, there is a coordinate system such that:

$$\begin{aligned} X_1 &= f_{11}(x_1, x_2, x_3)\partial/\partial x_1 + f_{21}(x_1, x_2, x_3)\partial/\partial x_2, \\ X_2 &= f_{12}(x_1, x_2, x_3)\partial/\partial x_1 + f_{22}(x_1, x_2, x_3)\partial/\partial x_2, \quad det(f_{ij}) \neq 0, \\ X_3 &= a(x_1, x_2, x_3)X_1 + b(x_1, x_2, x_3)X_2 + c(x_1, x_2, x_3)\partial/\partial x_3, \end{aligned}$$

By imposing the others two conditions of $(*)$ to s we obtain that $\partial c/\partial x_1 = 0$ and $\partial c/\partial x_2 = 0$, therefore, $X_3 = \alpha\partial/\partial x_1 + \beta\partial/\partial x_2 + \gamma\partial/\partial x_3$, $\alpha, \beta \in C^\infty(\mathbb{R}^3)$ *and* $\gamma \in C^\infty(\mathbb{R})$. □

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