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AN INFINITE DIMENSIONAL MOTIVATION IN HIGHER ORDER GEOMETRY

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ABSTRACT. Given two fiber bundles Y_1 and Y_2 over the same base, we start with a survey of some geometric constructions on the space $\mathcal{F}(Y_1, Y_2)$ of all smooth maps of a fiber of Y_1 into the fiber of Y_2 over the same base point. In particular, the theory of Weil bundles can be extended from finite dimensional manifolds to $\mathcal{F}(Y_1, Y_2)$. A systematic development of such an approach yields some new operations on the jet prolongations of vertical Weil bundles and on similar objects. The applications to the theory of connections are discussed.

Let $p_1 : Y_1 \to M$ and $p_2 : Y_2 \to M$ be two classical fiber bundles (i.e. locally trivial fibered manifolds of the class C^∞) over the same base. Consider the set of all fiber maps

$$\mathcal{F}(Y_1, Y_2) = \bigcup_{x \in M} C^\infty(Y_{1x}, Y_{2x}) \tag{1}$$

with the canonical projection $p : \mathcal{F}(Y_1, Y_2) \to M$. Starting from the idea of the Schrödinger connection by A. Jadczyk and M. Modugno, [6], A. Cabras and the author developed the theory of connections on $\mathcal{F}(Y_1, Y_2)$, [2]. This generalizes the theory of connections on a classical fibered manifold, because we have $\mathcal{F}(Y_1, Y_2) = Y_2$ in the extreme case Y_1 is the trivial fibering $M \to M$ with one-point fibers. Both the jet approach to connections on $\mathcal{F}(Y_1, Y_2)$ and the tangent one are discussed in [2]. In the present paper, we recall the idea of the r-th jet prolongation of $\mathcal{F}(Y_1, Y_2)$ in Section 1. On the other hand, the construction of the tangent bundle of $\mathcal{F}(Y_1, Y_2)$ is incorporated into the general framework of the theory of Weil bundles. In Section 2 we present a jet-like approach to the Weil bundles over classical manifolds, [11]. In the case of classical manifolds it has been clarified that Weil algebras form an appropriate technique for investigating all

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product preserving bundle functors, [11]. Every natural transformation between two Weil functors can be interpreted as a reparametrization, which corresponds to a homomorphism of the Weil algebras in question. In Section 3 we construct the Weil bundles over $\mathcal{F}(Y_1, Y_2)$ and deduce that those reparametrizations can be applied to the case of $\mathcal{F}(Y_1, Y_2)$ as well.

Such results give rise to a general question: What constructions can be extended from finite dimensional manifolds to the functional bundle $\mathcal{F}(Y_1, Y_2)$? Using systematically this point of view, we define certain operations, which are new even for classical manifolds and can be applied to our functional bundle as well. In Section 4 we introduce a natural identification i of the r-th jet prolongation of a vertical Weil bundle with the vertical Weil bundle of the r-th jet prolongation, which is based on the idea of exchanging the parameters. Using the map i, we can lift a connection on a fibered manifold $Y \to M$ to any vertical Weil bundle of Y and the exchange property of i implies an interesting characterization of the lifted connection in terms of the flow prolongation of vector fields. Section 5 is devoted to a similar operation on the bundles of contact elements. In the last section we study a connection Γ on $\mathcal{F}(Y_1, Y_2)$. Applying the ideas from Section 4, we define the iterated absolute differentiation with respect to Γ and an auxiliary linear connection Λ on the base manifold. It seems that the general Ricci identity, which has been recently deduced by A. Cabras in the finite dimensional case, [1], can be extended to $\mathcal{F}(Y_1, Y_2)$ as well. But a detailed discussion of this problem is beyond the scope of the present paper.

1. Jet prolongations of $\boldsymbol{\mathcal{F}(Y_1, Y_2)}$. The definition of the r-th jet prolongation of $\mathcal{F}(Y_1, Y_2)$ is based on our idea of fiber r-jet, [9], [11]. Let $p : Y \to M$ be a fibered manifold, let $Y_x = p^{-1}(x)$, $x \in M$, let N be a manifold and $f, g : Y \to N$ be two C^∞-maps.

Definition 1. We say that f and g determine the same fiber r-jet $j^r_x f = j^r_x g$ at $x \in M$, if $j^r_y f = j^r_y g$ for all $y \in Y_x$.

For every section $s : M \to \mathcal{F}(Y_1, Y_2)$, we can construct the associated map $\widetilde{s} : Y_1 \to Y_2$, $\widetilde{s}(y) = s(p_1 y)(y)$. We say that s is smooth (in the sense of A. Frölicher, [4]), if $\widetilde{s}$ is a C^∞-map.

Definition 2. Two smooth sections $s_1, s_2 : M \to \mathcal{F}(Y_1, Y_2)$ determine the same r-jet $j^r_x s_1 = j^r_x s_2$ at $x \in M$, if $j^r_x \widetilde{s}_1 = j^r_x \widetilde{s}_2$ in the sense of Definition 1.

The set $J^r\mathcal{F}(Y_1, Y_2)$ of all r-jets of the local sections of $\mathcal{F}(Y_1, Y_2)$ is called the r-th jet prolongation of $\mathcal{F}(Y_1, Y_2)$.

Consider the classical r-th jet prolongation $J^r Y_i$ of $Y_i \to M$, $i = 1, 2$. Since $\widetilde{s} : Y_1 \to Y_2$ is a base-preserving morphism of fibered manifolds, it is extended to a map $J^r\widetilde{s} : J^r Y_1 \to J^r Y_2$. The following assertion can be proved by direct evaluation analogously to [9].

Proposition 1. *We have $j^r_x s_1 = j^r_x s_2$ if and only if the restrictions of $J^r\widetilde{s}_1$ and $J^r\widetilde{s}_2$ to the fiber of $J^r Y_1$ over $x \in M$ coincide.*

The condition

$$J^r\widetilde{s}_1|J^r_xY_1 = J^r\widetilde{s}_2|J^r_xY_2 \tag{2}$$

can be interpreted as a kind of representation formula for $j^r_xs_1 = j^r_xs_2$. Such an approach will be modified to the case of Weil bundles in Section 3.

2. Classical Weil bundles. In the jet-like approach, a Weil functor can be interpreted as a generalization of the (k,r)-velocities functor T^r_k, [11]. Let us restate the definition of $T^r_kM = J^r_0(\mathbb{R}^k, M)$ (in the case $r = k = 1$ we have the tangent bundle) in a more algebraic way. Denote by $E(k)$ the algebra of germs of smooth functions on $\mathbb{R}^k$ at 0. The algebra $\mathbb{R}[t_1,\dots,t_k]$ of all polynomials in k variables can be canonically interpreted as a subalgebra of $E(k)$. By $t_1,\dots,t_k$ we indicate both the canonical variables on $\mathbb{R}^k$ and their germs at 0. Moreover, $\langle t_1,\dots,t_k\rangle$ means the ideal generated by $t_1,\dots t_k$ and $\langle t_1,\dots,t_k\rangle^{r+1}$ is its $(r+1)$-th power, i.e. the ideal of all polynomials vanishing up to order r at 0. Clearly, two maps g, $h:\mathbb{R}^k\to M$, $g(0)=h(0)=x$, satisfy $j^r_0g = j^r_0h$, if and only if

$$\varphi\circ g - \varphi\circ h \in \langle t_1,\dots t_k\rangle^{r+1} \tag{3}$$

for every germ φ of a smooth function on M at x. More generally, consider an ideal $\mathcal{A}\subset\mathbb{R}[t_1,\dots,t_k]\subset E(k)$ satisfying

$$\langle t_1,\dots,t_k\rangle^2 \supset \mathcal{A} \supset \langle t_1,\dots,t_k\rangle^{r+1} \tag{4}$$

The factor algebra $A = \mathbb{R}[t_1,\dots,t_k]/\mathcal{A}$ is called the Weil algebra determined by $\mathcal{A}$ and $\mathcal{A}$ is said to be a Weil ideal.

Definition 3. Two maps $g,h:\mathbb{R}^k\to M$, $g(0)=h(0)=x$, are said to be A-equivalent, if $\varphi\circ g - \varphi\circ h\in\mathcal{A}$ for every germ φ of a smooth function on M at x. Such an equivalence class will be denoted by j^Ag and called an A-velocity on M.

Denote by T^AM the set of all A-velocities on M. It is easy to see that $T^A\mathbb{R} = A$. The map $j^Ag\mapsto g(0)$ is a bundle projection $T^AM\to M$. Further, for every C^∞-map $f:M\to N$ we define $T^Af:T^AM\to T^AN$ by $T^Af(j^Ag) = j^A(f\circ g)$. Hence T^A is a bundle functor on the category $\mathcal{M}f$ of all manifolds, which is called the Weil functor corresponding to A. Conversely, every product preserving bundle functor F on $\mathcal{M}f$ is a Weil functor corresponding to the Weil algebra $F\mathbb{R}$, [11].

Let $B = \mathbb{R}[t_1,\dots,t_l]/\mathcal{B}$ be another Weil algebra and $H:A\to B$ be an algebra homomorphism. Then H is the factor map of an algebra homomorphism ψ : $\mathbb{R}[t_1,\dots,t_k]\to\mathbb{R}[t_1,\dots t_l]$ satisfying

$$\psi(\mathcal{A})\subset\mathcal{B} \tag{5}$$

and ψ is defined by a polynomial map $h:\mathbb{R}^l\to\mathbb{R}^k$ of the form $t_i = \psi(t_i)$, $i=1,\dots,k$. In [11] it is proved that the maps $\tau^H_M : T^AM\to T^BM$,

$$\tau^H_M(j^Ag) = j^B(g\circ h)\,,\quad g:\mathbb{R}^k\to M \tag{6}$$

define a natural transformation. Conversely, every natural transformation $\tau : T^A \to T^B$ is of the form τ^H, where H is the algebra homomorphism $H = \tau_{\mathbb{R}} : T^A\mathbb{R} \to T^B\mathbb{R}$, [11].

An important special case is the iteration $T^B \circ T^A$ of two Weil functors. Let $\langle \mathcal{A}, \mathcal{B} \rangle \subset \mathbb{R}[t_1, \dots, t_{k+l}]$ be the ideal generated by the injection of $\mathcal{A}$ to the first k places and the injection of $\mathcal{B}$ to the last l places. The factor algebra $\mathbb{R}[t_1, \dots t_{k+l}]/\langle \mathcal{A}, \mathcal{B} \rangle$ is canonically identified with the tensor product $A \otimes B$ of Weil algebras. On the other hand, the iteration $T^A \circ T^B$ leads to $\mathbb{R}[t_1, \dots t_{k+l}]/\langle \mathcal{B}, \mathcal{A} \rangle = B \otimes A$. Clearly, the exchange map

$$\mathbb{R}^{k+l} \to \mathbb{R}^{k+l},\ (t_1, \dots, t_k, t_{k+1}, \dots, t_{k+l}) \mapsto (t_{k+1}, \dots, t_{k+l}, t_1, \dots, t_k) \tag{7}$$

satisfies the corresponding modification of (5), so that it induces a canonical natural transformation

$$\kappa : T^A \circ T^B \to T^B \circ T^A \tag{8}$$

Obviously, on the Weil algebra level, (8) is determined by the exchange algebra homomorphism $A \otimes B \to B \otimes A$.

Example. Take $T^A = T^B = T$, so that we have the case of iterated tangent functor TT. The Weil algebra in question is $\mathbb{R}[t_1, t_2]/\langle t_1^2, t_2^2 \rangle$.Thus, our approach reflects the well known fact that the elements of TTM can be generated by maps $g : \mathbb{R}^2 \to M$ and the coordinates on $TT\mathbb{R}^m$ are determined by $g(0,0)$, $\partial g(0,0)/\partial t_1$, $\partial g(0,0)/\partial t_2$, $\partial^2 g(0)/\partial t_1 \partial t_2$. The exchange map $(t_1, t_2) \mapsto (t_2, t_1)$ induces the well known canonical involution $\kappa_M : TTM \to TTM$.

In general, let F be a bundle functor over m-dimensional manifolds, [11], and $X : M \to TM$ be a vector field, $\dim M = m$. The flow prolongation of X is a vector field $\mathcal{F}X : FM \to TFM$, which is constructed as follows. If $\exp tX$ is the flow of X, then $F(\exp tX)$ is a local diffeomorphism of FM for each t and we define

$$\mathcal{F}X = \frac{\partial}{\partial t}\Big|_0 F(\exp tX) \tag{9}$$

In particular, take $F = T^A$. Let $\kappa : TT^A \to T^A T$ be the natural transformation (8). For every vector field $X : M \to TM$ we have the induced map $T^A X : T^A M \to T^A TM$. The following simple assertion, [11], will be used in Sections 4 and 5.

Proposition 2. *We have* $\mathcal{T}^A X = \kappa_M^{-1} \circ T^A X$.

3. Weil bundles over $\mathcal{F}(\mathbf{Y_1}, \mathbf{Y_2})$.

Let N be a classical manifold. Consider a map $f : N \to \mathcal{F}(Y_1, Y_2)$ such that $p \circ f : N \to M$ is of the class C^∞. Then we can construct the induced bundle $(p \circ f)^* Y_1 \to N$. According to [2], f is called smooth, if the associated map

$$\widetilde{f} : (p \circ f)^* Y_1 \to Y_2\,,\ \widetilde{f}(q, y) = f(q)(y)\,,\ (q, y) \in (p \circ f)^* Y_1 \tag{10}$$

is of the class C^∞. In the case $N = \mathbb{R}$ we obtain the concept of a smooth curve on $\mathcal{F}(Y_1, Y_2)$. This endows $\mathcal{F}(Y_1, Y_2)$ with the structure of a smooth space in the sense of Frölicher, [3], [4].

Taking into account the Weil algebra A, we have $T^A p_1 : T^A Y_1 \to T^A M$ and $T^A p_2 : T^A Y_2 \to T^A M$. Let $f : \mathbb{R}^k \to \mathcal{F}(Y_1, Y_2)$ be a smooth map. Write $j^A(p \circ f) = X \in T^A M$ and define $T^A_X Y_1 = \left(T^A p_1\right)^{-1}(X)$ and $T^A_X Y_2 = \left(T^A p_2\right)^{-1}(X)$. Then we can construct the associated map $T^A_0 f : T^A_X Y_1 \to T^A_X Y_2$,

$$T^A_0 f(j^A k(\tau)) = j^A f(\tau)(k(\tau)), \quad \tau \in \mathbb{R}^k, \tag{11}$$

where $k : \mathbb{R}^k \to Y_1$ satisfies $p \circ f = p_1 \circ k$. Let $g : \mathbb{R}^k \to \mathcal{F}(Y_1, Y_2)$ be another smooth map with the same $X = j^A(p \circ g)$.

Definition 4. If $T^A_0 f = T^A_0 g$, then we say that f and g determine the same A-velocity $j^A f = j^A g$.

The set $T^A \mathcal{F}(Y_1, Y_2)$ of all equivalence classes will be called the bundle of all A-velocities on $\mathcal{F}(Y_1, Y_2)$. For every $D = j^A f \in T^A \mathcal{F}(Y_1, Y_2)$ we denote by $\widetilde{D}$ the associated map (11). We have two projections $\pi : T^A \mathcal{F}(Y_1, Y_2) \to \mathcal{F}(Y_1, Y_2)$ and $T^A p : T^A \mathcal{F}(Y_1, Y_2) \to T^A M$. For $A = \mathbb{R}[t_1, \dots, t_k]/\langle t_1, \dots, t_k\rangle^{r+1}$ we write $T^r_k \mathcal{F}(Y_1, Y_2)$ similarly to the classical case. In the case $k = r = 1$ we obtain the tangent bundle $T\mathcal{F}(Y_1, Y_2)$ of $\mathcal{F}(Y_1, Y_2)$.

Consider another Weil algebra B and an algebra homomorphism $H : A \to B$ generated by a polynomial map $h : \mathbb{R}^l \to \mathbb{R}^k$. In [10] it is proved that the idea of reparametrization (6) can be applied to $j^A g \in T^A \mathcal{F}(Y_1, Y_2)$ as well. In other words, we have a well defined map

$$\tau^H : T^A \mathcal{F}(Y_1, Y_2) \to T^B \mathcal{F}(Y_1, Y_2), \ \tau^H(j^A g) = j^B(g \circ h) \tag{12}$$

The correctness of (12) is based on the following characterization of τ^H is terms of the associated maps, [10].

Proposition 3. *For every $D \in T^A \mathcal{F}(Y_1, Y_2)$ over $X = T^A p(D)$, $\tau^H D$ is the unique element of $T^B \mathcal{F}(Y_1, Y_2)$ over $Z = \tau^H_M X$ such that the associated maps $\widetilde{D} : T^A_X Y_1 \to T^A_X Y_2$ and $\widetilde{\tau^H D} : T^B_Z Y_1 \to T^B_Z Y_2$ satisfy $\tau^H_{Y_2} \circ \widetilde{D} = \widetilde{\tau^H D} \circ \tau^H_{Y_1}$.*

The simpliest example of such an operation is the canonical involution on the iterated tangent bundle $TT\mathcal{F}(Y_1, Y_2)$. We remark that A. Cabras and the author introduced this involution in a direct way in [2], when they studied connections on $\mathcal{F}(Y_1, Y_2)$.

4. Exchange maps. The results of Section 3 enable us to define some operations, which are new even in the finite dimensional case and can be applied to $\mathcal{F}(Y_1, Y_2)$ as well. First of all, we discuss a classical fibered manifold $p : Y \to M$. Let VY be the vertical tangent bundle of Y. A well known identification $i : J^r(VY \to M) \to$

$V(J^rY \to M)$ by H. Goldschmidt and S. Sternberg, [5], is defined as follows. If $X = j^r_x\sigma(u) \in J^rVY$, $u \in M$, and $\sigma(u) = \frac{\partial}{\partial t}\big|_0 \varphi(u,t)$, $t \in \mathbb{R}$, we set

$$i(X) = \frac{\partial}{\partial t}\Big|_0 (j^r_x\varphi(u,t)) \in V(J^rY \to M)\,, \tag{13}$$

i.e. we exchange the order of $\partial/\partial t$ and j^r.

More generally, consider the vertical Weil bundle

$$V^AY = \bigcup_{x\in M} T^A(Y_x) \subset T^AY$$

Clearly, V^A is canonically extended to the morphisms of fibered manifolds, so that V^A is a bundle functor on the category $\mathcal{FM}$ of fibered manifolds, [11]. Every $X \in J^r(V^AY \to M)$ is of the form $j^r_x\sigma(u)$, where σ is a local section of $V^AY \to M$, and $\sigma(u) = j^A\varphi(u,\tau)$, $\tau \in \mathbb{R}^k$. Hence we also can apply the idea of exchange by setting

$$i(X) = j^A(j^r_x\varphi(u,\tau)) \tag{14}$$

Proposition 4. *$i : J^rV^AY \to V^AJ^rY$ is a well defined $\mathcal{FM}$-isomorphism.*

Proof. Locally, Y is isomorphic to $U \times S$, $U \subset \mathbb{R}^m$. In such a situation, (14) is reduced to the canonical natural transformation $\kappa_S : T^r_kT^AS \to T^AT^r_kS$, which is well defined by Section 2. □

According to Section 3, (14) can be applied to $\mathcal{F}(Y_1, Y_2)$ as well.

We are going to use (14) in the theory of connections on Y. If $\Gamma : Y \to J^1Y$ is a connection, we have the induced map $V^A\Gamma : V^A\Gamma \to V^AJ^1Y$. Applying i^{-1}, we obtain a connection

$$\mathcal{V}^A\Gamma = i^{-1} \circ V^A\Gamma \tag{15}$$

on $V^AY \to M$. The Γ-lift ΓX of a vector field X on M is a projectable vector field on Y, so that we can construct its flow prolongation $\mathcal{V}^A(\Gamma X)$. On the other hand, we have the lift $\mathcal{V}^A\Gamma(X)$ of X by means of the connection $\mathcal{V}^A\Gamma$.

Proposition 5. *$\mathcal{V}^A\Gamma(X) = \mathcal{V}^A(\Gamma X)$ for every vector field X on M.*

Proof. One verifies easily that the natural transformation κ_Y maps $TV^AY \subset TT^AY$ into $V^A(TY \to TM) \subset T^ATY$. By Proposition 2, $\mathcal{V}^A(\Gamma X) = \kappa_Y^{-1} \circ V^A(\Gamma X)$. If we interpret the elements of the first jet prolongation as linear subspaces of the tangent bundle, we find that κ and i correspond each other. This proves our claim. □

5. The case of contact elements. L. Mangiarotti and M. Modugno introduced a natural map $J^r(TY \to M) \to TJ^rY$, [13]. We are going to apply our previous ideas to such a situation as well. However, since we shall heavily use the fact that J^rY is an open dense subset of the bundle K^r_mY of all contact (m,r)-elements on Y, $m = \dim M$, we start with a construction on the contact elements.

Given an arbitrary manifold M, we recall that every n-dimensional submanifold $N \subset M$ determines a contact (n,r)-element k^r_xN at every point $x \in N$, [11]. The bundle of all contact (n,r)-elements on M is denoted by K^r_nM. Since every local parametrization of N defines a regular (n,r)-velocity at each point, we have a canonical submersion $\varrho_M : \operatorname{reg} T^r_nM \to K^r_nM$. Further, let $\operatorname{Reg} T^r_nT^AM \subset \operatorname{reg} T^r_nT^AM$ denote the set of all regular (n,r)-velocities on T^AM, which are projected into regular (n,r)-velocities on M by the bundle projection $T^AM \to M$. Similarly, let $\operatorname{Reg} K^r_nT^AM \subset K^r_nT^AM$ be the subspace of all contact (n,r)-elements, which are determined by the (n,r)-velocities from $\operatorname{Reg} T^r_nT^AM$.

We are going to define a natural map $\mu : \operatorname{Reg} K^r_nT^AM \to T^AK^r_nM$. Every $X \in \operatorname{Reg} K^r_nT^AM$ is of the form $X = k^r_YN$, where N is a submanifold of T^AM and $Y \in N$. Every $Z \in N$ is of the form $j^A\psi(\tau, Z)$, $\tau \in \mathbb{R}^k$, where $\psi : \mathbb{R}^k \times N \to M$. Locally $\psi(\tau, -)$ is a submanifold $N_\tau \subset M$, $\tau \in \mathbb{R}^k$, by the regularity assumption. Setting $\Psi(\tau) = k^r_{\psi(\tau,Y)}N_\tau$, we obtain a local map $\Psi : \mathbb{R}^k \to K^r_nM$. Then we define

$$\mu(X) = j^A\Psi \tag{16}$$

The correctness of (16) follows from the commutativity of the following diagram, in which we do not indicate explicitely the restrictions of the maps in the top row and the left column.

Proposition 6. *The following diagram commutes*

$$\begin{array}{ccc}
\operatorname{Reg} T^r_nT^AM & \xrightarrow{\kappa_M} & T^A \operatorname{reg} T^r_nM \\
\Big\downarrow{\scriptstyle \varrho_{T^AM}} & & \Big\downarrow{\scriptstyle T^A\varrho_M} \\
\operatorname{Reg} K^r_nT^AM & \xrightarrow{\mu} & T^AK^r_nM
\end{array}$$

Proof. Let $\nu : \mathbb{R}^n \to N$ be a local parametrization of N, so that $\psi \circ (\mathrm{id}_{\mathbb{R}^k} \times \nu) : \mathbb{R}^k \times \mathbb{R}^n \to M$ is a local parametrization of the system N_τ. Then $\kappa_M : T^r_nT^AM \to T^AT^r_nM$ is projectable over μ by the definition of μ. □

Since every section is a submanifold, we have an inclusion $J^r(T^AY \to M) \subset \operatorname{Reg} K^r_mT^AY$, $m = \dim M$. The restriction of (16) is a map

$$J^r(T^AY \to M) \to T^AJ^rY \tag{17}$$

For $r = 1$ and $T^A = T$ one deduces by direct evaluation that this map coincides with that one introduced by L. Mangiarotti and M. Modugno, [13].

In general, every immersion $f : M \to \overline{M}$ induces a map $K^r_nf : K^r_nM \to K^r_n\overline{M}$, $K^r_nf(k^r_xN) = k^r_{f(x)}f(N)$. In particular, if $X : M \to TM$ is a vector field, we have the induced map $K_n^r X : K_n^r M \to K_n^r TM$, the values of which lie in $\operatorname{Reg} K_n^r TM$. On the other hand, we can construct the flow prolongation $\mathcal{K}_n^r X : K_n^r M \to TK_n^r M$.

Proposition 7. *We have* $\mathcal{K}_n^r X = \mu \circ K_n^r X$.

Proof. This follows directly from Propositions 2 and 6.

We remark that the definition (16) can be directly extended to the case of the functional bundle $\mathcal{F}(Y_1, Y_2)$.

6. The iterated absolute differentiation. According to [2], a connection on $\mathcal{F}(Y_1, Y_2)$ is a section $\Gamma : \mathcal{F}(Y_1, Y_2) \to J^1\mathcal{F}(Y_1, Y_2)$, which is smooth in the sense of the theory by A. Frölicher, [4], i.e. smooth curves on $\mathcal{F}(Y_1, Y_2)$ are to be transformed into smooth curves on $J^1\mathcal{F}(Y_1, Y_2)$. If $s : M \to \mathcal{F}(Y_1, Y_2)$ is a smooth section, then the difference $\nabla s = \nabla_\Gamma s = j^1 s - \Gamma \circ s$ is a section of $\mathcal{F}(Y_1, VY_2 \otimes T^*M)$, which is called the absolute differential of s, [2]. Further, the connection Γ is called differentiable, if the tangent map $T\Gamma : T\mathcal{F}(Y_1, Y_2) \to TJ^1\mathcal{F}(Y_1, Y_2)$ is well defined. Since a connection on $\mathcal{F}(Y_1, Y_2)$ is a kind of differential operator, we can characterize the r-th order ones, [2]. Every r-th order connection is differentiable.

Consider a differentiable connection $\Gamma : \mathcal{F}(Y_1, Y_2) \to J^1\mathcal{F}(Y_1, Y_2)$. If we compose the induced map $V\Gamma : V\mathcal{F}(Y_1, Y_2) \to VJ^1\mathcal{F}(Y_1, Y_2)$ with $i^{-1} : VJ^1\mathcal{F}(Y_1, Y_2) \to J^1V\mathcal{F}(Y_1, Y_2)$, we obtain a connection

$$\mathcal{V}\Gamma = i^{-1} \circ V\Gamma : V\mathcal{F}(Y_1, Y_2) \to J^1V\mathcal{F}(Y_1, Y_2)\,, \tag{18}$$

which is called the vertical prolongation of Γ. Let $\Lambda : TM \to J^1TM$ be a classical linear connection on the base manifold and $\Lambda^* : T^*M \to J^1T^*M$ be the dual connection. If Γ is an r-th order connection, we can construct a connection $\mathcal{V}\Gamma \otimes \Lambda^*$ on $\mathcal{F}(Y_1, VY_2 \otimes T^*M)$ analogously to the classical case, [11]. Hence we have the absolute differential

$$\nabla_{\mathcal{V}\Gamma \otimes \Lambda^*}(\nabla_\Gamma s) = \nabla^2_\Lambda s \tag{19}$$

Definition 5. (19) is called the second absolute differential of s by means of Λ.

In [1], A. Cabras deduced the general Ricci identity for connections on a classical fibered manifolds $Y \to M$. It seems that such an identity holds even in the case of Definition 5. But a detailed discussion of this question is beyond the scope of the present paper.

Finally we remark that one can define the space of all r-jets $J^r(N, \mathcal{F}(Y_1, Y_2))$ of a classical manifold N into $\mathcal{F}(Y_1, Y_2)$ analogously to Section 3. It seems that the exchange map of the classical spaces of separated jets by M. Kawaguchi, [7], [8], can be extended to the case of $\mathcal{F}(Y_1, Y_2)$ as well. The same seems to be true for the spaces of iterated jets by G. Virsik, [15].

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