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LIFTINGS OF TENSOR FIELDS TO THE COTANGENT BUNDLE

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ABSTRACT. We study geometrical constructions transforming tensor fields of type (r,s) to the cotangent bundle. We present the full classification of all such natural operators in the cases $(r,s)=(0,1)$, $(r,s)=(0,2)$ and $(r,s)=(1,1)$. We also study some geometrical constructions of (p,q)–tensor fields on T^*M by means of (r,s)–tensor fields on M, $(r,s)\neq(p,q)$.

1. Introduction

Constructions transforming tensor fields to natural bundles play an important role in differential geometry. From a general point of view, geometrical constructions are natural differential operators, cf. [5]. In some particular cases it is possible to give the full classification of all natural operators of certain type, i.e. to give the full list of all geometrical constructions in question.

By a tensor field of type (r,s) we shall understand a smooth section of the vector bundle $T^{(r,s)}M=\overset{r}{\otimes}TM\otimes\overset{s}{\otimes}T^*M$ over a smooth manifold M. There are many papers which are devoted to the geometrical constructions transforming tensor fields of type (r,s) on M into tensor fields of type (p,q) on FM, where F is a natural bundle, cf. [2], [3] and [4]. Such geometrical constructions are in fact natural operators $T^{(r,s)}\rightsquigarrow T^{(p,q)}F$. The aim of this paper is to study natural operators $T^{(r,s)}\rightsquigarrow T^{(r,s)}T^*$, where T^* is a cotangent bundle. The full classification of all such natural operators was given by Mikulski in the case of $(r,s)=(0,0)$ (functions) and by Kobak and Kolář in the case $(r,s)=(1,0)$ (vector fields). We shall present classification theorems for other important cases of (r,s): $(r,s)=(0,1)$ (1–forms), $(r,s)=(0,2)$ (2–forms, Riemannian and symplectic structures) and $(r,s)=(1,1)$ (almost tangent structures, almost complex structures and f–structures). In some particular cases we shall also study geometrical constructions

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of the type $T^{(r,s)} \rightsquigarrow T^{(p,q)}T^*$, $(r,s) \neq (p,q)$, i.e. liftings which do not preserve the type of a tensor field.

We remark that liftings of tensor fields to the cotangent bundle are used in the geometric approach to analytical mechanics, see [1] and [7]. All natural operators transforming classical linear connections on M into classical linear connections on T^*M were determined by Kureš in [6]. We shall use the concept of a natural bundle and of a natural operator from [4]. All manifolds and maps are assumed to be infinitely differentiable and all manifolds are paracompact.

2. Liftings of functions and vector fields

We shall denote by $q_M : T^*M \to M$ the cotangent bundle of a smooth m-dimensional manifold M. The canonical coordinates on T^*M will be denoted by (x^i, p_i). Further, denote by

$$\lambda_M = p_i dx^i \tag{1}$$

the classical Liouville 1–form on T^*M and by

$$\Lambda_M = d\lambda_M = dp_i \wedge dx^i \tag{2}$$

the canonical symplectic 2–form on T^*M. Then Λ_M defines a bijection $S_M : T^{(1,0)}T^*M \to T^{(0,1)}T^*M$ transforming $A \in TT^*M$ into its inner product with Λ_M. This isomorphism can be extended into a vector bundles isomorphism (also denoted by S_M)

$$S_M : T^{(1,k-1)}T^*M \to T^{(0,k)}T^*M \tag{3}$$

by

$$(S_M K)(X_1, \dots, X_k) = \langle K(X_1, \dots, X_{k-1}), S_M(X_k)\rangle$$

for all vector fields $X_1, \dots, X_k$ on T^*M, where $<,>: TE \times_E T^*E \to \mathbb{R}$ means the evaluation mapping. Further, each vector field X on M defines a function $\langle X, -\rangle : T^*M \to \mathbb{R}$. If $X = X^i \frac{\partial}{\partial x^i}$ and $\alpha = p_i dx^i$, then $\langle X, \alpha\rangle = X^i p_i$. In general, we have an operator

$$i_M : T^{(1,k)}M \to T^{(0,k)}T^*M \tag{4}$$

defined by

$$(i_M K)(X_1, \dots, X_k) = \langle K_M(Tq_M X_1, \dots, Tq_M X_k), -\rangle$$

for all vector fields $X_1, \dots, X_k$ on T^*M. In coordinates, if

$$K = K^i_{j_1,\dots,j_k} \frac{\partial}{\partial x^i} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_k}, \quad \text{then} \quad i_M K = K^i_{j_1,\dots,j_k} p_i dx^{j_1} \otimes \dots \otimes dx^{j_k}.$$

Finally, we have an operator

$$\gamma_M : T^{(1,k)}M \to T^{(1,k-1)}T^*M \tag{5}$$

given by $\gamma_M = S_M^{-1} \circ i_M$. If $K = K^i_{j_1,\dots,j_k} \frac{\partial}{\partial x^i} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_k}$, then $\gamma_M K = K^i_{j_1,\dots,j_k} p_i \frac{\partial}{\partial p_{j_1}} \otimes dx^{j_2} \otimes \cdots \otimes dx^{j_k}$. In particular, if $F = F^i_j \frac{\partial}{\partial x^i} \otimes dx^j$, then $i_M F = F^i_j p_i dx^j$ and $\gamma_M F = F^i_j p_i \frac{\partial}{\partial p_j}$.

Let $f : M \to \mathbb{R}$ be a function on M. The *vertical lift* f^V of f to T^*M is a function $f^V : T^*M \to \mathbb{R}$ defined by $f^V = f \circ q_M$. We have $(fg)^V = f^V g^V$ and $(f+g)^V = f^V + g^V$. Further, Mikulski has in [8] proved

Proposition 1. *All natural operators $T^{(0,0)} \rightsquigarrow T^{(0,0)}T^*$ transforming functions on M into functions on T^*M are of the form $f \mapsto \phi \circ f^V$, where $\phi : \mathbb{R} \to \mathbb{R}$ is an arbitrary smooth function.*

For comparison's sake, in the case of a tangent bundle TM we have also a complete lift f^C defined by $f^C(y) = df_x(y)$, $x = p_M(y)$, where $p_M : TM \to M$ is the canonical projection.

Now let X be an arbitrary vector field on M. The *vertical lift* X^V of X to T^*M is defined as the vector field on T^*M generated by the homotheties of the vector bundle T^*M. In other words, X^V is the well-known Liouville vector field L_M on T^*M. The *complete lift* X^C of X to T^*M is the flow prolongation of X (denoted also by $\mathcal{T}^*X$). Taking into account the operator i_M defined in (4), if X is a vector field on M, then $i_M X$ is a function on T^*M, so that the exterior differential $d(i_M X)$ is a 1–form on T^*M. Then $S_M^{-1}(d(i_M X))$ is exactly the complete lift X^C. We have (cf. [4] and [5])

Proposition 2. *All natural operators $T^{(1,0)} \rightsquigarrow T^{(1,0)}T^*$ are of the form*

$$X \mapsto f(\langle X, \lambda_M \rangle) X^V + g(\langle X, \lambda_M \rangle) X^C$$

where $f, g \in C^\infty(\mathbb{R}, \mathbb{R})$ are arbitrary smooth functions.

In coordinates, if $X = X^i \frac{\partial}{\partial x^i}$, then $L_M = X^V = p_i \frac{\partial}{\partial p_i}$ and

$$X^C = X^i \frac{\partial}{\partial x^i} - \frac{\partial X^j}{\partial x^i} p_j \frac{\partial}{\partial p_i}. \tag{6}$$

Each vector field is completely determined by means of its action on functions. Moreover, we have

Lemma 1. *Let Y_1 and Y_2 be vector fields on T^*M such that $Y_1(i_M X) = Y_2(i_M X)$ for all vector fields X on M. Then $Y_1 = Y_2$.*

and

Lemma 2. *Let G_1 and G_2 be tensor fields on T^*M of the type $(0,k)$ or $(1,k)$, $k > 0$ such that*

$$G_1(X_1^C, \dots, X_k^C) = G_2(X_1^C, \dots, X_k^C)$$

for all vector fields $X_1, \dots, X_k$ on M. Then $G_1 = G_2$.

For the proofs of Lemma 1 and Lemma 2 see e.g. [9].

3. Liftings of 1–forms

We first draw our attention to one geometrical construction transforming 1–forms on M into vector fields on T^*M. This is a lifting which changes the type of an original tensor field. Let $\omega = \omega_i dx^i$ be a 1–form on M and $q_M^*\omega = \omega_i dx^i$ be its pull-back to T^*M. Then $S_M^{-1}(q_M^*\omega)$ is a vertical vector field on T^*M, called the *vertical lift of* ω *to* T^*M and denoted by ω^V. In coordinates,

$$\omega^V = \omega_i \frac{\partial}{\partial p_i}, \tag{7}$$

so that also $(dx^i)^V = \frac{\partial}{\partial p_i}$. One verifies easily

Lemma 3. *Let ω, θ be 1–forms on M and f be a function on M. Then*

I. $\omega^V(f^V) = 0$,
II. $(\omega + \theta)^V = \omega^V + \theta^V$,
III. $(f\omega)^V = f^V \omega^V$,
IV. $\left[\omega^V, \theta^V\right] = 0$.

Obviously,

$$\omega^V(i_M X) = (\langle \omega, X \rangle)^V , \tag{8}$$

so that by Lemma 1 one can also define the vertical lift ω^V by means of the condition (8). In the following assertion we describe the main properties of the complete lift of vector fields.

Lemma 4. *Let X, Y be vector fields on M, ω be a 1–form on M, F be a $(1,1)$–tensor field on M and let S be a $(1,2)$–tensor field on M. Then*

I. $(X + Y)^C = X^C + Y^C$,
II. $(fX)^C = f^V X^C - (i_M X)(df)^V$,
III. $\left[X^C, \omega^V\right] = (\mathcal{L}_X \omega)^V$, *where $\mathcal{L}_X$ denotes the Lie derivative with respect to X,*
IV. $\left[X^C, \gamma F\right] = \gamma(\mathcal{L}_X F)$,
V. $\left[X^C, Y^C\right] = [X, Y]^C$,
VI. $(\gamma S)(X^C) = \gamma(S_X)$, *where S_X is the tensor field of the type $(1,1)$ defined by $S_X(Z) = S(X, Z)$ for any vector field Z on M,*
VII. $X^C f^V = (Xf)^V$,
VIII. $X^C(i_M Y) = i_M [X, Y]$.

By Lemma 1, the complete lift X^C can be also defined by VIII. The aim of this section is to prove

Proposition 3. *All natural operators $T^{(0,1)} \rightsquigarrow T^{(0,1)}T^*$ transforming 1–forms on M into 1–forms on T^*M are of the form*

$$\omega \mapsto k_1 \lambda_M + k_2 q_M^* \omega \tag{9}$$

where λ_M is the Liouville 1–form (1), $q_M^*\omega$ is the pull-back of ω to T^*M with respect to the bundle projection $q_M : T^*M \to M$ and k_1 and k_2 are arbitrary real numbers.

Proof. Let G_m^r be the group of all invertible r–jets of $\mathbb{R}^m$ into $\mathbb{R}^m$ with source and target zero. We shall denote by $(a_j^i, a_{jk}^i, \dots, a_{j_1,\dots,j_r}^i)$ the canonical coordinates on G_m^r and by tilde the coordinates of the inverse element. By [5], r–th order natural operators $T^{(0,1)} \rightsquigarrow T^{(0,1)}T^*$ are in a canonical bijection with G_m^{r+1}–equivariant maps $(J^rT^{(0,1)})_0\mathbb{R}^m \oplus T_0^*\mathbb{R}^m \to (T^{(0,1)}T^*)\mathbb{R}^m$ of the corresponding standard fibres. Each 1–form on T^*M has the coordinate expression $A_i dx^i + B^i dp_i$. The canonical coordinates on the standard fibre $(J^rT^{(0,1)})_0\mathbb{R}^m$ will be denoted by $\omega_i, \omega_{ij}, \dots, \omega_{i_1,\dots,i_r}$. The rest of the proof will proceed in three steps.

I. Consider first the zero order natural operators. We have to find all G_m^1–equivariant maps of the form $A_i = A_i(p_i, \omega_i)$, $B^i = B^i(p_i, \omega_i)$ which express the coordinate form of the natural operators in question. Using standard evaluations one computes easily the following transformations laws

$$\begin{aligned}
\overline{p}_i &= \tilde{a}_i^j p_j, \\
\overline{\omega}_i &= \tilde{a}_i^j \omega_j, \quad \overline{B}^i = a_j^i B^j, \\
\overline{A}_i &= \tilde{a}_i^j A_j + a_{jk}^\ell \tilde{a}_i^k \overline{p}_\ell B^j .
\end{aligned}$$

Then the homotheties $\tilde{a}_i^j = t\delta_i^j$, $\tilde{a}_{jk}^i = 0$ yield

$$tA_i = A_i(tp_i, t\omega_i) \quad \text{and} \quad \frac{1}{t}B^i = B^i(tp_i, t\omega_i).$$

Hence A_i are linear in p_i and in ω_i, so that

$$A_i = k_1 p_i + k_2 \omega_i \tag{10}$$

which is the coordinate expression of (9). Further, $B^i = tB^i(tp_i, t\omega_i)$. Setting $t \to 0$ we get $B^i = 0$. Finally, the full equivariance does not change the expression (10).

II. The r–th order operators correspond to the equivariant maps of the form

$$\begin{aligned}
A_i &= A_i(p_i, \omega_i, \omega_{ij}, \dots, \omega_{i_1,\dots,i_r}) \\
B^i &= B^i(p_i, \omega_i, \omega_{ij}, \dots, \omega_{i_1,\dots,i_r}).
\end{aligned}$$

Applying homotheties, one verifies easily that

$$\begin{aligned}
tA_i &= A_i(tp_i, t\omega_i, t^2\omega_{ij}, \dots, t^r\omega_{i_1,\dots,i_r}) \\
\frac{1}{t}B^i &= B^i(tp_i, t\omega_i, t^2\omega_{ij}, \dots, t^r\omega_{i_1,\dots,i_r}).
\end{aligned}$$

Therefore r–th order natural operators are reduced to the case I. for any $r > 0$.

III. By [5], every natural operator $T^{(0,1)} \rightsquigarrow T^{(0,1)}T^*$ has finite order. □

4. Natural operators $T^{(r,s)} \rightsquigarrow T^{(p,q)}T^*$

In this section we shall solve the following problem: For which r, s, p, q does there exist a natural operator $T^{(r,s)} \rightsquigarrow T^{(p,q)}T^*$? Such natural operators are examples of geometrical constructions which change the type of a tensor field (if $(p,q) \neq (r,s)$).

Definition. An operator $A : T^{(r,s)} \rightsquigarrow T^{(p,q)}T^*$ is said to be absolute, if $A_M X = A_M O_M$ for all tensor fields $X : M \to T^{(r,s)}M$, where O_M means the zero tensor field of type (r,s).

By the following assertion, if $(p,q) \neq (0,0)$, then we can associate an absolute (p,q)–tensor field on T^*M to each (r,s)–tensor field on M.

Lemma 5. *If $(p,q) \neq (0,0)$, then there is an absolute operator $T^{(r,s)} \rightsquigarrow T^{(p,q)}T^*$ for all r, s, p, q.*

Proof. Obviously, $X \mapsto L_M$ is an absolute operator $T^{(r,s)} \rightsquigarrow T^{(1,0)}T^*$ and $X \mapsto \lambda_M$ is an absolute operator $T^{(r,s)} \rightsquigarrow T^{(0,1)}T^*$. Evaluating the tensor products of L_M and λ_M, we can construct a tensor of an arbitrary type (p,q) on T^*M. $\square$

Lemma 6. *A natural operator $T^{(r,s)} \rightsquigarrow T^{(0,0)}T^*$ transforming (r,s)–tensor fields on M into functions on T^*M exists if and only if $r \geq s$.*

Proof. The coordinate expression of a natural operator $T^{(r,s)} \rightsquigarrow T^{(0,0)}T^*$ is $f(X^{i_1,\dots,i_r}_{j_1,\dots,j_s}, p_i)$. By homotheties,

$$f\left(X^{i_1,\dots,i_r}_{j_1,\dots,j_s}, p_i\right) = f\left(k^{r-s}X^{i_1,\dots,i_r}_{j_1,\dots,j_s}, kp_i\right).$$

Hence if $s > r$, then the only function f is zero. For $r \geq s$ we can get a function on T^*M by contraction. $\square$

Lemma 5 and Lemma 6 describe the most trivial constructions of a (p,q)–tensor field on T^*M by means of an (r,s)–tensor field on M. There are also nontrivial natural operators $T^{(r,s)} \rightsquigarrow T^{(p,q)}T^*$ in many particular cases. For example, $X \mapsto d(i_M X)$ is an operator $T^{(1,0)} \rightsquigarrow T^{(0,1)}T^*$. On the other hand, $f \mapsto S_M^{-1}(df^V)$ is an operator $T^{(0,0)} \rightsquigarrow T^{(1,0)}T^*$. In what follows we give the full classification of natural operators transforming 1–forms on M into vector fields and into $(1,1)$–tensor fields on T^*M. Considering the isomorphism $S_M : T^{(1,0)}T^*M \to T^{(0,1)}T^*M$ we see that $(S_M^{-1})\lambda_M$ is exactly the Liouville vector field L_M on T^*M. We have

Proposition 4. *All natural operators $T^{(0,1)} \rightsquigarrow T^{(1,0)}T^*$ transforming 1–forms on M into vector fields on T^*M are of the form*

$$\omega \mapsto k_1 L_M + k_2 \omega^V \tag{11}$$

where k_1, k_2 are arbitrary real numbers.

Proof. Each vector field on T^*M is of the form $Y^i \frac{\partial}{\partial x^i} + P_i \frac{\partial}{\partial p_i}$. The transformation formulas for Y^i and P_i are

$$\overline{Y}^i = a^i_j Y^j, \quad \overline{P}_i = \tilde{a}^j_i P_j + \tilde{a}^j_{i\ell} a^\ell_k Y^k p_j.$$

The rest of the proof is quite analogous to that of Proposition 3. $\square$

Proposition 5. *All natural operators $T^{(0,1)} \rightsquigarrow T^{(1,1)}T^*$ transforming 1–forms on M into $(1,1)$–tensor fields on T^*M are of the form*

$$(12) \quad \omega \mapsto k_1 Id_{T^*M} + k_2 L_M \otimes \lambda_M + k_3 \omega^V \otimes q_M^* \omega + k_4 L_M \otimes q_M^* \omega + k_5 \omega^V \otimes \lambda_M$$

where Id_{T^*M} is the identity $(1,1)$–tensor field on T^*M and $k_1, \dots, k_5$ are arbitrary real numbers.

Proof. The coordinate expression of a $(1,1)$–tensor field on T^*M is

$$(13) \qquad A^i_j \frac{\partial}{\partial x^i} \otimes dx^j + B^{ij} \frac{\partial}{\partial x^i} \otimes dp_j + C_{ij} \frac{\partial}{\partial p_i} \otimes dx^j + D^j_i \frac{\partial}{\partial p_i} \otimes dp_j .$$

The action of G^2_m on the standard fibre of $T^{(1,1)}T^*$ is

$$(14) \qquad \begin{aligned} \overline{A}^i_j &= a^i_k \tilde a^\ell_j A^k_\ell + a^i_k \tilde a^r_{\ell s} \tilde a^s_j \overline{p}_r B^{k\ell} \\ \overline{B}^{ij} &= a^i_k a^j_\ell B^{k\ell} \\ \overline{C}_{ij} &= \tilde a^r_{is} a^s_k \tilde a^\ell_j p_r A^k_\ell + \tilde a^r_{is} a^s_k a^t_{\ell u} \tilde a^u_j p_r \overline{p}_t B^{k\ell} + \tilde a^k_i \tilde a^\ell_j C_{k\ell} + \tilde a^k_i a^r_{\ell s} \tilde a^s_j \overline{p}_r D^\ell_k \\ \overline{D}^j_i &= \tilde a^r_{is} a^s_k a^j_\ell p_r B^{k\ell} + \tilde a^k_i a^j_\ell D^\ell_k \end{aligned}$$

I. The first order natural operators correspond to the G^2_m– equivariant maps

$$\begin{aligned} A^i_j &= A^i_j(p_i, \omega_i, \omega_{ij}), \quad & B^{ij} &= B^{ij}(p_i, \omega_i, \omega_{ij}) \\ C_{ij} &= C_{ij}(p_i, \omega_i, \omega_{ij}), \quad & D^j_i &= D^j_i(p_i, \omega_i, \omega_{ij}) \end{aligned}$$

where $\overline{\omega}_{ij} = \tilde a^k_i \tilde a^\ell_j \omega_{k\ell} + \tilde a^r_{ij} \omega_r$. Using homotheties we find $A^i_j = k_1 \delta^i_j$, $B^{ij} = 0$, $C_{ij} = k_2 p_i p_j + k_3 \omega_i \omega_j + k_4 p_i \omega_j + k_5 \omega_i p_j + k_6 \omega_{ij}$, $D^j_i = k_7 \delta^j_i$. Then the full equivariance yields $k_6 = 0$ and $k_7 = k_1$, which is nothing else but the coordinate form of (12). The remaining steps II. and III. are quite similar to those in the proof of Proposition 3. □

5. Liftings of $(0,2)$–tensor fields

Let $G = g_{ij} dx^i \otimes dx^j$ be a $(0,2)$–tensor field on M. Then G induces another $(0,2)$–tensor field $G' = g_{ji} dx^i \otimes dx^j$ on M given by $\langle G, X \otimes Y \rangle = \langle G', Y \otimes X \rangle$. We prove

Proposition 6. *All natural operators $T^{(0,2)} \rightsquigarrow T^{(0,2)}T^*$ transforming $(0,2)$–tensor fields on M into $(0,2)$–tensor fields on T^*M are of the form*

$$(15) \qquad G \mapsto k_1 \Lambda_M + k_2 \lambda_M \otimes \lambda_M + k_3 q_M^* G + k_4 q_M^* G'$$

where k_1, k_2, k_3 and k_4 are arbitrary real numbers.

Proof. A $(0,2)$–tensor field on T^*M is of the form

$$P_{ij} dx^i \otimes dx^j + Q^j_i dx^i \otimes dp_j + R^i_j dp_i \otimes dx^j + S^{ij} dp_i \otimes dp_j .$$

A direct calculation leads to the formulas

$$\begin{aligned}
\overline{P}_{ij} &= \tilde a_i^k \tilde a_j^\ell P_{k\ell} + \tilde a_j^k a_{\ell r}^m \tilde a_j^r \overline{p}_m Q_k^\ell + a_{kr}^m \tilde a_i^r \tilde a_j^\ell \overline{p}_m R_\ell^k + a_{kr}^m \tilde a_i^r a_{\ell s}^n \tilde a_j^s \overline{p}_m \overline{p}_n S^{k\ell},\\
\overline{Q}_i^j &= \tilde a_i^k \tilde a_\ell^j Q_k^\ell + a_{kr}^m \tilde a_i^r a_\ell^j \overline{p}_m S^{k\ell},\\
\overline{R}_j^i &= a_k^i \tilde a_j^\ell R_\ell^k + a_k^i a_{\ell s}^n \tilde a_j^s \overline{p}_n S^{k\ell},\\
\overline{S}^{ij} &= a_k^i a_\ell^j S^{k\ell}.
\end{aligned}$$

I. Zero order natural operators correspond to the G_m^1–equivariant maps $P_{ij} = P_{ij}(p_i, g_{ij})$, $Q_i^j = Q_i^j(p_i, g_{ij})$, $R_j^i = R_j^i(p_i, g_{ij})$, $S^{ij} = S^{ij}(p_i, g_{ij})$. Evaluating equivariance with respect to the homotheties we find $S^{ij} = 0$, $R_j^i = k_1\delta_j^i$, $Q_j^i = k_2\delta_j^i$, $P_{ij} = k_3 g_{ij} + k_4 g_{ji} + k_5 p_i p_j$. Then the full equivariance leads to such relations among $k_1, \dots, k_5$ which correspond to (15). The rest of the proof is similar to steps II. and III. in the proof of Proposition 2. □

6. Liftings of $(1,1)$–tensor fields

Let F be a tensor field of type $(1,1)$ on M, $F = F_j^i \frac{\partial}{\partial x^i} \otimes dx^j$. Then $i_M F = F_i^k p_k dx^i$ is a 1–form on T^*M, so that the exterior differential $d(i_M F)$ is a 2–form on T^*M. The *complete lift* of F to T^*M is defined as the $(1,1)$– tensor field on T^*M given by $F^C = S_M^{-1}(d(i_M F))$. In coordinates,

$$F^C = F_j^i \frac{\partial}{\partial x^i} \otimes dx^j + \left(\frac{\partial F_j^k}{\partial x^i} - \frac{\partial F_i^k}{\partial x^j}\right) p_k \frac{\partial}{\partial p_i} \otimes dx^j + F_j^i \frac{\partial}{\partial p_i} \otimes dp_j. \tag{16}$$

Let Id_M be the identity $(1,1)$–tensor field on M and $\widetilde F = F_i^i$ be the function on T^*M given by the full contraction of F. We prove

Proposition 7. *All first order natural $\mathbb{R}$–linear operators $T^{(1,1)} \rightsquigarrow T^{(1,1)}T^*$ are of the form*

$$F \mapsto k_1 F^C + k_2 Id_{T^*M} + k_3 L_M \otimes \lambda_M + k_4 \widetilde F \cdot Id_{T^*M} + k_5 \widetilde F \cdot L_M \otimes \lambda_M \tag{17}$$

where $k_1, \dots, k_5$ are arbitrary real numbers.

Proof. According to the general theory, we have to find all G_m^2–equivariant maps $A_j^i = A_j^i(p_i, F_j^i, F_{jk}^i)$, $B^{ij} = B^{ij}(p_i, F_j^i, F_{jk}^i)$, $C_{ij} = C_{ij}(p_i, F_j^i, F_{jk}^i)$ and $D_i^j = D_i^j(p_i, F_j^i, F_{jk}^i)$. The action of G_m^2 on the standard fibre of $T^{(1,1}T^*$ is given by (14). Further,

$$\begin{aligned}
\overline{F}_j^i &= a_\ell^i \tilde a_j^k F_k^\ell,\\
\overline{F}_{jk}^i &= a_m^i \tilde a_j^n \tilde a_k^p F_{np}^m + a_m^i \tilde a_{jk}^n F_n^m + a_{mp}^i a_j^n \tilde a_k^p F_n^m.
\end{aligned}$$

Using homotheties we find easily $A_j^i = a_1 F_j^i + a_2 \delta_j^i F_\ell^\ell + a_9 \delta_j^i$, $B^{ij} = 0$, $C_{ij} = (a_7 F_\ell^\ell + a_8) p_i p_j + a_5 p_\ell F_{ij}^\ell + a_6 p_\ell F_{ji}^\ell$, $D_j^i = a_3 F_j^i + a_4 \delta_j^i F_\ell^\ell + a_{10} \delta_j^i$. Finally, the full equivariance reads $a_1 = a_3 = a_5 = -a_6 = k_1$, $a_2 = a_4 = k_4$, $a_9 = a_{10} = k_2$, $a_8 = k_3$ and $a_7 = k_5$, which corresponds to (17). □

Some properties of the complete lift F^C are expressed in the following assertion (cf. [9]).

Lemma 7. *Let X be a vector field on M, ω be a 1–form and let F and G be $(1,1)$–tensor fields on M. Then*

I. $F^C\omega^V = (\omega\circ F)^V$ *where $\omega\circ F$ is a 1–form defined by $(\omega\circ F)(Z)=\omega(FZ)$ for any vector field Z on M,*

II. $F^C(\gamma G) = \gamma(G\circ F)$,

III. $F^C(X^C) = (F(X))^C + \gamma(\mathcal{L}_X F)$,

IV. $(F^C)^2 = (F^2)^C + \gamma N_F$ *where N_F is the Nijenhuis tensor of F,*

V. *If F is an almost complex structure on M, then F^C is an almost complex structure on T^*M if and only if F is integrable,*

VI. $(Id_M)^C = Id_{T^*M}$.

By Lemma 2, the complete lift F^C can be also defined by III. from the above assertion.

Remark 1. If $\mathcal{E}$ is a product preserving functor, then the definition of the complete lift of a $(1,1)$–tensor field F from a manifold M to $\mathcal{E}(M)$ is quite different, cf. [3]. In this case we interpret F as a linear mapping $F : TM \to TM$ over the identity of M and we define $F^C = \eta_M\circ\mathcal{E}(F)\circ\eta_M^{-1}$, where $\eta_M : \mathcal{E}(TM)\to T(\mathcal{E}M)$ is a natural isomorphism from [5]. Then the property III from Lemma 7 is of the form

$$F^C(X^C) = (F(X))^C. \tag{*}$$

Moreover, there is another approach to the definition of the complete lift, where the relation (*) even plays an essential role, see [2]. Let $\mathcal{E}$ be a natural bundle (not necessarily a product preserving functor) and F be a tensor field of type $(1,1)$ on M. F is said to admit the complete lift to $\mathcal{E}(M)$ if there is a tensor field $\overline{F}$ of type $(1,1)$ on $\mathcal{E}(M)$ such that $\overline{F}(X^C) = (F(X))^C$ for all vector fields X on M. Gancarzewicz proved in [2] that F admits the complete lift to T^*M (in the sense of the above definition) if and only if $F = f\cdot \mathrm{Id}_M$ for some function f on M.

Remark 2. There are also nontrivial natural operators $T^{(1,1)} \rightsquigarrow T^{(p,q)}T^*$, $(p,q)\neq(1,1)$. For example, $F\mapsto iF$ and $F\mapsto \gamma F$ are examples of natural operators $T^{(1,1)} \rightsquigarrow T^{(0,1)}T^*$ and $T^{(1,1)} \rightsquigarrow T^{(1,0)}T^*$, respectively. Further, the contractions F^i_i, $F^i_kF^k_i, \dots$ are functions on T^*M naturally generated by a $(1,1)$–tensor field F on M.

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