DIFFERENTIAL GEOMETRY AND APPLICATIONS
Proc. Conf., Aug. 28 – Sept. 1, 1995, Brno, Czech Republic
Masaryk University, Brno 1996, 133–140

ALMOST COMPLEX STRUCTURES AND CONNECTIONS ON TM

Anton Dekrét

Abstract. Let α be a vector (1,1)-form on TM. In this paper natural operators transforming α into connections on TM, especially when α is an almost complex structure, are studied and some their properties are found.

Introduction

All manifolds and maps are assumed to be infinitely differentiable.

Let (x^i) be a local chart on a manifold M and (x^i, x_1^i), (x^i, x_1^i, x_j^i) be induced charts on TM and on the space JTM of all 1-jets of local cross - section of $\pi : TM \to M$.

There are different equivalent definitions of connections on a fibre bundle, see [7]. We will use, in the case of TM, the following ones. The connection on TM is

a) a global section $\Gamma : TM \to JTM, (x^i, x_1^i) \to (x^i, x_1^i, x_j^i = \Gamma_j^i(x, x_1)$,
b) a decomposition $\Gamma : T(TM) = VTM \oplus H\Gamma$ on the vertical subbundle VTM and on the so-called Γ-horizontal subbundle $H\Gamma$ satisfying an equation $dx_1^i = \Gamma_j^i dx^j$,
c) a (1,1)-form h_Γ on TM (horizontal form of Γ) such that $T\pi \cdot h_\Gamma = T\pi$, $h_\Gamma(VTM) = 0$, $h_\Gamma = dx^i \otimes \partial/\partial x^i + \Gamma_j^i dx^j \otimes \partial/\partial x_1^i$,
d) a (1,1)-form v_Γ (vertical form of Γ) such that $v_\Gamma\big|_{VTM} = Id_{VTM}$, $v_\Gamma(TTM) = VTM$, $v_\Gamma = (dx_1^i - \Gamma_j^i dx^i) \otimes \partial/\partial x_1^i$.

A connection Γ on TM is linear if $\Gamma : TM \to JTM$ is linear. Then $\Gamma_j^i(x, x_1) = \Gamma_{kj}^i(x) x_1^k$.

Recall some notions and well known facts:

A scalar 1-form ω or a vector (1,1)-form α on TM is semibasic if $\omega(VTM) = 0$ or $\alpha(VTM) = 0$.

1991 *Mathematics Subject Classification*. 53C05, 58A20.
Key words and phrases. Tangent bundle, tangent value (1,1)-form, almost complex structure, connection, semispray.
This paper is in final form and will not be submitted for publication elsewhere.

If h_Γ is the horizontal form of Γ and $\beta = b^i_j dx^j \otimes \partial/\partial x^i_1$ is a semibasic (1,1)-form with values in VTM then $h_\Gamma + \beta$ is the horizontal form of the other connection denoted by $\Gamma + \beta$.

On TM there are the canonical semibasic (1,1)-form $J = dx^i \otimes \partial/\partial x^i_1$ induces by the identification $VTM = TMx_M TM$ and the Liouville vector field $V = x^i_1 \partial/\partial x^i_1$ generated by the homotheties in the individual fibres of TM.

A semispray is a vector field $S : TM \to TTM$ on TM such that $J(S) = V$, $S = x^i_1 \partial/\partial x^i + \eta^i(x, x_1)\partial/\partial x^i_1$. If Γ is a connection on TM then there is a unique semispray S_Γ which is Γ-horizontal, i.e. $S_\Gamma \subset H\Gamma$. It is said a Γ-semispray, $S_\Gamma = x^i_1 \partial/\partial x^i + \Gamma^i_j x^j_1 \partial/\partial x^i_1$.

A vertical prolongation of a given connection Γ on TM is the so-called connection $V\Gamma$ on the fibre bundle $VTM \to M$. In a local chart (x^i, x^i_1, η^i) on VTM it is of the form $d\eta^i = \Gamma^i_{jk_1}\eta^k dx^i$, $dx^i_1 = \Gamma^i_j dx^j$, see [6], where we use throughout this paper the denotation $\frac{\partial f}{\partial x^i} = f_i$, $\frac{\partial f}{\partial x^i_1} = f_{i_1}$. By the identification $VTM = TMx_M TM$ a connection $\overline{V\Gamma}$, $dx^i_1 = \Gamma^i_{jk_1} x^k_1 dx^j$, is determined on TM.

Every semispray $X = x^i_1 \partial/\partial x^i + \eta^i \partial/\partial x^i_1$ induces a connection Γ_X, $dx^i_1 = \frac{1}{2}\eta^i_{j_1} dx^j$, [4].

Remark. Let $X = x^i_1 \partial/\partial x^i + \Gamma^i_j x^j_1 \partial/\partial x^i_1$ be a Γ-semispray. Then Γ_X is given by the equations

$$dx^i_1 = \frac{1}{2}(\Gamma^i_{jk_1} x^j_1 + \Gamma^i_k) dx^k.$$

This mean that if Γ is without torsion $(\Gamma^i_{jk_1} = \Gamma^i_{kj_1})$ then $\Gamma_X = \frac{1}{2}(\overline{V\Gamma} + \Gamma)$.

An almost complex structure on a manifold M, dim M=2 m, is a (1,1)-form α on M such that $\alpha^2 = -Id_M$. Janyška, [3], has proved that there is not any linear connection on TM which can be constructed from α by natural polynomial operators. We are interested in this question when α is an almost complex structure on TM.

Every connection Γ on TM generates the canonical almost complex structure α_Γ satisfying $\alpha_\Gamma J = h_\Gamma$, $\alpha_\Gamma h_\Gamma = -J$, see for example [5]. In coordinates, if Γ is given by $dx^i_1 = \Gamma^i_j dx^j$ then

$$(1) \qquad \alpha_\Gamma = (-\Gamma^i_j dx^j + dx^i_1) \otimes \partial/\partial x^i + [(\delta^i_j - \Gamma^i_k \Gamma^k_j) dx^j + \Gamma^i_j dx^j_1] \otimes \partial/\partial x^i_1 \ .$$

Connections and almost complex structures on TM

Let $\alpha = (a^i_j dx^j + b^i_j dx^j_1) \otimes \partial/\partial x^i + (c^i_j dx^j + h^i_j dx^j_1) \otimes \partial/\partial x^i_1$ be an arbitrary (1,1)-form on TM. It is called vertical if $b = J\alpha J = 0$, i.e. if $\alpha(VTM) \subset VTM$.

In coordinates α is an almost complex structure on TM iff

$$(2) \qquad \begin{aligned} A^i_j &\equiv a^i_s a^s_j + b^i_s c^s_j = -\delta^i_j, & B^i_j &\equiv a^i_s b^s_j + b^i_s h^s_j = 0, \\ C^i_j &\equiv c^i_s a^s_j + h^i_s c^s_j = 0, & H^i_j &\equiv c^i_s b^s_j + h^i_s h^s_j = -\delta^i_j. \end{aligned}$$

Let us recall that according to the identification $VTM \equiv TM \times_M TM$ any semibasic (1,1)-form $K = K^i_j dx^j \otimes \partial/\partial x^i_1$ with values in VTM can be interpreted as a section $TM \to T^*M \otimes_{TM} TM$ or as a section $TM \to V^*TM \otimes VTM$, i.e. as a vector bundle morphism $VTM \to VTM$ over Id_{TM}.

There are two cases of α:

1. $b = J\alpha J = b^i_j dx^j \otimes \partial/\partial x^i_1 \neq 0$.
   Author, [2], has shown, that only in the case when b is regular there are natural operators Φ of zero order transforming α into connections on TM and Φ are of the form $\Gamma = \Phi(\alpha) = \Gamma_\alpha + \varphi(b, J\alpha^2 J,\ \alpha - \alpha b^{-1} T\pi \cdot \alpha)$, where Γ_α is the connection the horizontal subbundle $H\Gamma_\alpha$ of which is spanned on such vectors X that $\alpha(X) \in VTM$, i.e. $\Gamma^i_j = -\tilde{b}^i_k a^k_j$, $\tilde{b}^i_s b^s_j = \delta^i_j$, and φ is a natural operator of zero order determining by the semibasic (1,1)-forms $b, J\alpha^2 J,\ \alpha - \alpha b^{-1} T\pi \cdot \alpha \equiv \gamma$ with values in VTM the other semibasic VTM valued (1,1)-form on TM.

Lemma 1. *Let α be a (1,1)-form on TM. Then $\alpha(VTM)$ is a horizontal vector bundle of a unique connection Γ iff $b = J\alpha J$ is regular.*

Proof. Let Γ be a connection on TM with the equation $dx^i_1 = \Gamma^i_j dx^j$. Then $\alpha(Y) = b^i_j \eta^j \partial/\partial x^i + h^i_j \eta^j \partial/\partial x^i_1$ is Γ-horizontal for every vertical vector $Y = \eta^i \partial/\partial x^i_1$ iff $\Gamma^i_k b^k_j \eta^j = h^i_j \eta^j$, i.e. iff $\Gamma^i_k b^k_j = h^i_j$. This equation has a unique solution Γ^i_k iff $\det b^i_j \neq 0$. Then $\Gamma^i_j = h^i_k \tilde{b}^k_j$.

The connection the horizontal subbundle $H\Gamma$ of which is $\alpha(VTM)$ will be denoted by Γ^2_α. It is of the form $\Phi(\alpha)$, $\Gamma^2_\alpha = \Gamma_\alpha + b^{-1} J\alpha^2 J b^{-1}$. It can be described by the scalar forms. Let $\omega = y_i dx^i + z_i dx^i_1$ be a scalar form on TM. Then the form $i_\alpha \omega = \omega\alpha = (y_i a^i_j + z_i c^i_j) dx^j + (b^i_j y_i + z_i h^i_j) dx^j_1$ is semibasic iff $b^i_j y_i + z_i h^i_j = 0$, i.e. $y_i = -h^u_k \tilde{b}^k_i z_u$, i.e. iff $\omega\big|_{H\Gamma^2_\alpha} = 0$.

Lemma 2. *Let S_{Γ_α} or $S_{\Gamma^2_\alpha}$ be the horizontal semispray of the connection Γ_α or Γ^2_α. Then $\alpha(S_{\Gamma_\alpha}) = \gamma(V)$ and $\alpha(b^{-1}(V)) = S_{\Gamma^2_\alpha}$.*

Proof. In coordinates $\alpha(S_{\Gamma_\alpha}) = (c^i_j - h^i_k \tilde{b}^k_s a^s_j) x^j_1 \partial/\partial x^i_1 = \gamma(V)$, $\alpha(b^{-1}(V)) = x^i_1 \partial/\partial x^i + h^i_s \tilde{b}^s_j x^j \partial/\partial x^i_1 = S_{\Gamma^2_\alpha}$.

A (1,1)-form α is called a Lagrange form if $\alpha(V)$ is a semispray.

Lemma 3. *A (1,1)-form α on TM which is not vertical is the Lagrange form iff $b(V) = V$.*

Proof follows from the coordinate form of the image $\alpha(V) = b^i_j x^j_1 \partial/\partial x^i + h^i_j x^j_1 \partial/\partial x^i_1$.

Proposition 1. *The canonical almost complex structure α_Γ of a connection Γ is a Lagrange form and $\alpha_\Gamma(V)$ is Γ-semispray.*

Proof is clair from the relation (1).

Proposition 2. *If α is a Lagrange (1,1)-form and b is regular then $\alpha(V)$ is the Γ^2_α-semispray.*

Proof is the consequence of Lemma 2.

Lemma 4. *If $b = J\alpha J$ is regular then there exists a unique such semispray S_α that $\alpha(S_\alpha)$ is a semispray.*

Proof. Let $S = x_1^i \partial/\partial x^i + \eta^i \partial/\partial x_1^i$ be an arbitrary semispray. Then $\alpha(S) = (a_j^i x_1^j + b_j^i \eta^j)\partial/\partial x^i + (c_j^i x_1^j + d_j^i \eta^j)\partial/\partial x_1^i$ is a semispray iff

$$a_j^i x_1^j + b_j^i \eta^j = x_1^i.$$

Because of regularity of b this system has a unique solution

$$\eta^i = \tilde{b}_k^i x_1^k - \tilde{b}_k^i a_j^k x_1^j = (\tilde{b}_j^i - \tilde{b}_k^i a_j^k) x_1^j. \tag{3}$$

Definition. If b is regular then S_α is called the semispray of α.

Lemma 5. *If S_α is the semispray of α then $S_\alpha = b^{-1}(V) + S_{\Gamma_\alpha}$, i.e. S_α is the $(\Gamma_\alpha + b^{-1})$-semispray and $\alpha(S_\alpha)$ is the $(\Gamma_\alpha^2 + \gamma)$-semispray.*

Proof. Our first assertion follows immediately from (3). The second one is the consequence of the relations (4) and (5):

$$\alpha(S_\alpha) = x_1^i \partial/\partial x^i + (c_s^i + h_k^i \tilde{b}_s^k - h_k^i \tilde{b}_u^k a_s^u) x_1^s \partial/\partial x_1^i \tag{4}$$

$$\gamma = \alpha - \alpha b^{-1} T\pi \cdot \alpha = (c_j^i - h_s^i \tilde{b}_k^s a_j^k) dx^j \otimes \partial/\partial x_1^i. \tag{5}$$

Proposition 3. *Let b be regular. Then α^2 is a vertical (1,1)-form on TM iff $\Gamma_\alpha = \Gamma_\alpha^2$.*

Proof. By the relation (2) α^2 is vertical, $J\alpha^2 J = 0$, iff

$$B_j^i \equiv a_k^i b_j^k + b_k^i h_j^k = 0, \text{ i.e. iff } h_t^i = -\tilde{b}_s^i a_k^s b_t^k.$$

Then ${}^2\Gamma_j^i = h_t^i \tilde{b}_j^t = -\tilde{b}_s^i a_j^s$ are the Christoffels of the connection Γ_α. Conversely, if $h_k^i \tilde{b}_j^k = -\tilde{b}_s^i a_j^s$ then $b_s^i h_k^s \tilde{b}_j^k = -a_j^i$, i.e. $b_s^i h_j^s + a_s^i b_j^s = 0$, i.e. α^2 is vertical.

Proposition 4. *Let b be regular and α^2 be vertical. Then α is an almost complex structure iff $J\alpha^2 = -Id\big|_{VTM}$.*

Proof. In coordinates

$$J\alpha^2 = ((a_s^i a_j^s + b_s^i c_j^s) dx^j + (a_s^i b_j^s + b_s^i h_j^s) dx^j) \otimes \partial/\partial x_1^i.$$

The form α^2 is vertical iff $a_s^i b_j^s + b_s^i h_j^s = 0$. It means that if α^2 is vertical then $J\alpha^2$ is a semibasic (1,1)-form with values in VTM and determines a vector bundle morphism on VTM over Id_{TM} which is equal to $-Id_{VTM}$ iff $a_s^i a_j^s + b_s^i c_j^s = -\delta_j^i$. Then $h_j^i = -\tilde{b}_s^i a_k^s b_j^k$, $c_j^i = -\tilde{b}_j^i - \tilde{b}_s^i a_k^s a_j^k$. Then

$$C_j^i = c_s^i a_j^s + h_s^i c_j^s = (-\tilde{b}_s^i - \tilde{b}_t^i a_k^t a_s^k) a_j^s - \tilde{b}_t^i a_k^t b_s^k (-\tilde{b}_j^s - \tilde{b}_v^s a_u^v a_j^u) = 0$$

$$H_j^i = c_s^i b_j^s + h_s^i h_j^s = (-\tilde{b}_s^i - \tilde{b}_t^i a_k^t a_s^k) b_j^s - \tilde{b}_t^i a_k^t b_s^k (-\tilde{b}_u^s a_r^u b_j^r) = -\delta_j^i.$$

The relations (2) are satisfied and $\alpha^2 = -Id_{TTM}$. The converse is clear.

Corollary. If α is a such almost complex structure that b is regular then $\Gamma_\alpha = \Gamma^2_\alpha$.

Lemma 6. *If b is regular and α is an almost complex structure then $\gamma = -b^{-1}$.*

Proof. Using (2) and (5) we get $\gamma = -\tilde{b}^i_j - \tilde{b}^i_k a^k_s a^s_j + \tilde{b}^i_k a^k_u b^u_r \tilde{b}^r_s a^s_j = -\tilde{b}^i_j$.

Proposition 5. *Let b be regular and α be an almost complex structure. Let S_α be the semispray of α. Then $\alpha(S_\alpha)$ is the semispray of the connection $\Gamma_\alpha - b^{-1}$ and $S_\alpha - \alpha(S_\alpha) = 2b^{-1}(V)$.*

Proof. It immediately follow from Lemmas 5 and 6 that $\alpha(S_\alpha)$ is the $(\Gamma_\alpha - b^{-1})$-semispray. It can be written in the form $\alpha(S_\alpha) = S_{\Gamma_\alpha} - b^{-1}(V)$. Then $S_\alpha - \alpha(S_\alpha) = b^{-1}(V) + S_{\Gamma_\alpha} - S_{\Gamma_\alpha} + b^{-1}(V) = 2b^{-1}(V)$.

Proposition 6. *Let α^2 be vertical and b be regular. Then the connection $\Gamma_\alpha = \Gamma^2_\alpha$ is without torsion iff the Γ_α-horizontalization of the Nijenhuis-Frolicher bracket $[\alpha, J]$ is a (1,2)-form with values in VTM.*

Proof. Using the induced coordinates we get

$$h_{\Gamma_\alpha} = dx^i \otimes \partial/\partial x^i + h^i_k \tilde{b}^k_j dx^j \otimes \partial/\partial x^i_1,$$

$$[h_{\Gamma_\alpha}, J] = (h^i_{uj_1}\tilde{b}^u_k + h^i_u \tilde{b}^u_{kj_1}) dx^j \wedge dx^k \otimes \partial/\partial x^i_1, \text{ torsion of } \Gamma_\alpha,$$

$$[\alpha, J] = (a^i_{sj_1} dx^j \wedge dx^s - b^i_{js_1} dx^j_1 \wedge dx^s) \otimes \partial/\partial x^i + [(c^i_{sj_1} + a^i_{js}) dx^j \wedge dx^s +$$

$$+ (b^i_{js} - d^i_{js_1} - a^i_{sj_1}) dx^j_1 \wedge dx^s + b^i_{js_1} dx^j_1 \wedge dx^s_1] \otimes \partial/\partial x^i_1.$$

Then $[\alpha, J] \cdot h_{\Gamma_\alpha}$ is a 2-form with values in VTM iff

$$\omega \equiv (a^i_{sj_1} + b^i_{uj_1} h^u_k \tilde{b}^k_s) dx^j \wedge dx^s = 0.$$

If α^2 is vertical then $a^i_j = -b^i_u h^u_r \tilde{b}^r_s$, and $\omega = -b^i_u(h^u_{rj_1}\tilde{b}^r_s + h^u_r \tilde{b}^r_{sj_1}) dx^j \wedge dx^s$. Now the regularity of b completes our proof.

Proposition 7. *Let α be an almost complex structure on TM with the regular form b. Let α_{Γ_α} be the canonical almost complex structure determined by the connection Γ_α. Then $\alpha_{\Gamma_\alpha} = \alpha$ iff $b = Id\big|_{VTM}$.*

Proof. If $b = Id_{VTM}$, $b^i_j = \delta^i_j$, then the relations (2) imply $h^i_j = -a^i_j$, $a^i_s a^s_j = -\delta^i_j - c^i_j$. As $\Gamma^i_j = -a^i_j$ then using (1) we get $\alpha_{\Gamma_\alpha} = \alpha$. Conversely, if $\alpha_{\Gamma_\alpha} = \alpha$ then the relation (1) immediately gives $b^i = \delta^i_j$.

Remark. If $\alpha, \overline{\alpha}$ are arbitrary almost complex structure then $(\alpha \cdot \overline{\alpha})^{-1} = \overline{\alpha} \cdot \alpha$.

Proposition 8. *Let α be an almost complex structure on TM the form b of which is regular. Then $\alpha \cdot \alpha_{\Gamma_\alpha}$ is vertical, $J \cdot (\alpha \cdot \alpha_{\Gamma_\alpha}) = -b$ and $(\alpha \cdot \alpha_{\Gamma_\alpha}) \cdot J = -b^{-1}$.*

Proof. In the case of the connection $\Gamma_\alpha, \Gamma^i_j = -\tilde{b}^i_k a^k_j$, the relation (1) gives

$$\alpha_{\Gamma_\alpha} = (\tilde{b}^i_k a^k_j dx^j + dx^i_1) \otimes \partial/\partial x^i + [(-\delta^i_j - \tilde{b}^i_s a^s_k \tilde{b}^k_u a^u_j) dx^j - \tilde{b}^i_k a^k_j dx^j_1] \otimes \partial/\partial x^i_1.$$

Using the relations (2) we calculate

$$\alpha \cdot \alpha_{\Gamma_\alpha} = -b^i_j dx^j \otimes \partial/\partial x^i + [(-\tilde{b}^i_r \tilde{b}^r_k a^k_j + h^i_j) dx^j - \tilde{b}^i_j dx^j_1] \otimes \partial/\partial x^i_1.$$

Then $J(\alpha \cdot \alpha_{\Gamma_\alpha}) \cdot J = 0$, $J(\alpha \cdot \alpha_{\Gamma_\alpha}) = -b^i_j dx^j \otimes \partial/\partial x^i_1 = -b$, $(\alpha \cdot \alpha_{\Gamma_\alpha}) \cdot J = -\tilde{b}^i_j dx^j \otimes \partial/\partial x^i_1 = -b^{-1}$.

Corollary. If Y is a vertical vector on TM then $\alpha \cdot \alpha_{\Gamma_\alpha}(Y) = -b^{-1}(Y)$ and $\alpha_{\Gamma_\alpha} \cdot \alpha(Y) = -b(Y)$.

2. $b = J\alpha J = 0$, i.e. α is vertical, i.e. $\alpha(VTM) \subset VTM$.
The forms $a \equiv J\alpha = a^i_j dx^j \otimes \partial/\partial x^i_1$, $h = \alpha J = h^i_j dx^j \otimes \partial/\partial x^i_1$ are semibasic with values in VTM. Then $\mathcal{H}_\alpha : y \to ya - hy$, $(y^s_k) \to (y^s_u a^u_k - h^s_u y^u_k)$, is a vector bundle morphism on $T^*M \otimes_{TM} TM$ over Id_{TM}. Author, [2], has proved that only in the case when $\mathcal{H}_\alpha$ is regular there are natural operators Φ of zero order such that $\Gamma = \Phi(\alpha)$ are connections on TM. These operators are of the form
$$\Phi(\alpha) = \Gamma_\alpha + B(a,h),$$
where Γ_α is the connection the Christoffel functions of which are $\Gamma^i_j = A^{iu}_{jv} c^v_u$, where A^{iu}_{jv} are the coordinates of the inverse map to $\mathcal{H}_\alpha$ and B is a natural operator of zero order such that $B(a,h)$ is a semibasic (1,1)-form on TM with values in VTM.

Proposition 9. *Let Γ be a connection on TM. Let a, h be semibasic (1,1)-forms on TM with values in VTM. Then there exists a unique vertical (1,1)-form on TM such that $J\alpha = a$, $\alpha J = h$ and α preserves the horizontal subbundle $H\Gamma$ of the connection Γ, i.e. $\alpha(H\Gamma) \subset H\Gamma$.*

Proof. Let $X = \xi^i \partial/\partial x^i + \Gamma^i_j \xi^j \partial/\partial x^i_1$ be a Γ-horizontal vector of Γ. Let $a = a^i_j dx^j \otimes \partial/\partial x^i_1$, $h = h^i_j dx^j \otimes \partial/\partial x^i_1$. Let $\alpha = a^i_j dx^j \otimes \partial/\partial x^i + (c^i_j dx^j + h^i_j dx^j_1) \otimes \partial/\partial x^i_1$ be a vertical (1,1)-form on TM such that $J\alpha = a$, $\alpha J = h$. The vector $\alpha(X) = a^i_j \xi^j \partial/\partial x^i + (c^i_j \xi^j + h^i_k \Gamma^k_j \xi^j)\partial/\partial x^i_1$ is Γ-horizontal iff
$$(c^i_j + h^i_k \Gamma^k_j)\xi^j = \Gamma^i_k a^k_j \xi^j .$$
It is true for any Γ-horizontal vector X iff
$$(\Gamma^i_k a^k_j - h^i_k \Gamma^k_j) = c^i_j . \tag{6}$$
It completes our proof.

The relation (6) gives

Corollary. If α is a such vertical (1,1)-form on TM that the map $\mathcal{H}_\alpha$ is regular then Γ_α is a unique connection Γ on TM such that $\alpha(H\Gamma) \subset H\Gamma$.

It is clear from the relations (2) that if α is a vertical almost complex structure on TM then $a = J\alpha$, $h = \alpha J$ are almost complex structures on VTM and hence dim M is even.

Proposition 10. *Let α be a vertical (1,1)-form on TM and Γ be a such connection on TM that $\alpha(H\Gamma) \subset H\Gamma$. Let $a = J\alpha$, $h = \alpha J$ be almost complex structures on VTM. Then α is an almost complex structure on TM.*

Proof. It is the consequence of Proposition 9 that the Christoffel functions Γ^i_j of the connection Γ satisfy the relation (6). Then using $a^i_j a^j_k = -\delta^i_k$, $h^i_j h^j_k = -\delta^i_k$ we get $c^i_s a^s_j + h^i_s c^s_j = (\Gamma^i_k a^k_p - h^i_k \Gamma^k_p) a^p_j + h^i_s(\Gamma^s_k a^k_j - h^s_k \Gamma^k_j) = -\Gamma^i_j + \Gamma^i_j = 0$. It means that α satisfies the relations (2) and then α is an almost complex structure on TM.

Proposition 11. *Let α be a vertical almost complex structure. Then the map $\mathcal{H}_\alpha : y \to ya - hy$ on $T^*M \otimes_{TM} TM$ is not regular.*

Proof. Denote by h^-, h^+ the maps $h^- : y \to hy,\ h^+ : y \to yh$ on $T^*M \otimes_{TM} TM$. The maps a^-, a^+ are defined analogously. Then $\mathcal{H}_\alpha = a^+ - h^-$ and using $hh = -Id = aa$ we get

$$h^-\mathcal{H}(y) = hya - hhy = hya + y$$
$$a^+\mathcal{H}(y) = (ya - hy)a = -y - hya.$$

Then $h^-\mathcal{H} + a^+\mathcal{H} = 0$, i.e. $(h^- + a^+)\mathcal{H} = 0$. If $\mathcal{H}$ is regular then $h^- + a^+ = 0$, i.e. $hy + ya = 0$ for all $y \in T^*M \otimes_{TM} TM$. It is possible iff $h = -a = kId\big|_{VTM}$. Then $k^2 = -1$. It is contradictory with the fact that h and a are real forms.

Corollary. If α is a vertical almost complex structure then there is not a such natural operator Φ of zero order that $\Phi(\alpha)$ is a connection on TM. It can exist an operator of higher order.

Proposition 12. *If α is a vertical (1,1)-form on TM such that $\alpha^2 = Id\big|_{TTM}$ then the maps $\mathcal{H}_\alpha : y \to ya - hy$ on $T^*Mx_{TM}TM$ is regular iff $a = Id = -h$ or $a = -Id = -h$.*

Proof. If $\alpha^2 = Id$ then $a^2 = Id = h^2$ and $h^-\mathcal{H}(y) = hya - y,\ a^+\mathcal{H}(y) = y - hya$. Therefore $(h^- + a^+)\mathcal{H} = 0$. If $\mathcal{H}$ is regular then $h^- + a^+ = 0$. It is possible iff $h = a = kId$. Then $k = \pm 1$. Conversely, if $h = -a = \pm Id$ then $\mathcal{H} = \pm 2Id$ is regular.

Corollary. If α is such a vertical (1,1)-form on TM that $\alpha^2 = Id, \mathcal{H}$ is regular, $a = Id$, then $\frac{1}{2}(\alpha + Id)$ is the horizontal form of the connection Γ_α with $\Gamma^i_j = \frac{1}{2}c^i_j$.

Remarks.

1. If α is a vertical (1,1)-form on TM then the Lie derivative $L_V\alpha$ is again vertical. If α is an vertical almost complex structure then the map $\mathcal{H}_{L_V\alpha}y \to yJ \cdot L_V\alpha - L_V\alpha \cdot Jy$ on $T^*M \otimes_{TM} TM$ can be regular and hence natural operators of first order transforming α into connections $\varphi(\alpha)$ in TM can exist.
2. If S is a semispray and α is a vertical (1,1)-form on TM then the Lie derivative $L_S\alpha$ is a such (1,1)-form on TM that $JL_S\alpha \cdot J = J\alpha - \alpha J = a - h$.
3. Let $X = A^i\partial/\partial x^i + B^i\partial/\partial x^i_1$ be a vector field and $\omega = \omega^i_j dx^j \otimes \partial/\partial x^i_1$ be a semibasic (1,1)-form with values in VTM. Then

$$L_X\omega = -A^i_{k_1}\omega^k_j dx^j \otimes \partial/\partial x^i + [(\omega^i_{jk}A^k + \omega^i_{jk_1}B^k + \omega^i_k A^k_j - B^i_{k_1}\omega^k_j)dx^j +$$
$$+ \omega^i_k A^k_{j_1} dx^j_1] \otimes \partial/\partial x^i_1$$

is vertical. If X is a semispray then $JL_X\omega = -L_S\omega J = -k\omega$ and $\mathcal{H}_{L_S\omega}$ is regular iff ω has not eingenvalues $\pm\lambda$. The vector field $\alpha(X)$ is a semispray iff $a(X) = V$. If α is a vertical almost complex structure and S is a semispray then $\alpha(S)$ is not a semispray.

Let α be vertical and S be a semispray. Then Lie bracket $[\alpha(S), S]$ is vertical iff

$$(h^i_j - a^i_j - a^i_{kj_1}x^k_1)\eta^j = a^i_{jk}x^j_1x^k_1 - c^i_jx^j_1. \tag{7}$$

Because $a(V) = a^i_jx^j_1\partial/\partial x^i_1$ and $L_{a(V)}J = -(a^i_j + a^i_{kj_1}x^k_1)dx^j \otimes \partial/\partial x^i_1$ then (7) immediately yields

Proposition 13. *If α is a such vertical (1,1)-form on TM that the semibasic (1,1)-form $h + L_{a(V)}J$ is regular then there is a unique semispray S_α such that the Lie bracket $[\alpha(S_\alpha), S_\alpha]$ is vertical.*

Corollary. Under the conditions of Proposition 13 there is a connection Γ_{S_α} determined by the semispray S_α and the operator $\alpha \to \Gamma_{S_\alpha}$ is natural of second order.

Example. Let α be a such vertical (1,1)-form that $a = J\alpha = -\alpha J = -h$ and let α be projectable ($J\alpha$ is a (1,1)-form on M). Then $h + L_{a(V)}J = 2h$. If h is regular then S_α exists and if α is also a VB-form, i.e. $\alpha(X)$ is a projectable linear vector field for every projectable linear vector field X, see [1], then S_α is a spray and the connection Γ_{S_α} is linear. Such a form α can be an almost complex structure on TM.

Remark. Let $A = a^i_jdx^j \otimes \partial/\partial x^i$ be a (1,1)-form on M. Using the theory of natural operators, [6], it is easy to see that all natural operators of zero order transforming a (1,1)-form A on M into (1,1)-forms on TM are of the form

$$\alpha = kdx^i \otimes \partial/\partial x^i + (C^i_j(a)dx^j + kdx^i_1) \otimes \partial/\partial x^i_1, \ k \in R \tag{8}$$

where $C = C^i_j(a)$ is a natural operator of zero order transforming a (1,1)-form A into a (1,1)-form $C(A)$ on M. By the Janyška result, [3], there is not any linear connection $\Gamma(\alpha)$ polynomial in α. It is clear that α given by (8) cannot be an almost complex structure on TM.

References

[1] Cabras, A., Kolář, I., *Special tangent valued forms and the Frölicher-Nijenuis bracket*, Arch. Mathematicum (Brno) **Tom. 29** (1993), 71–82.

[2] Dekrét, A., *(1,1)-forms and connections on a tangent bundle TM*, Acta Univ. M. Bellii, Math. **2** (1994), 3 – 8.

[3] Janyška, J., *Remarks on the Nijenhuis tensor and almost complex connections*, Arch. Math. (Brno) **26 No. 4** (1990), 229 – 240.

[4] Griffone, J., *Structure presque-tangent et connections I.*, Ann. Inst. Fourier (Grenoble) **22** (1972), 287 – 334.

[5] Klein, J., *On variational second order differential equations*, Polynomial case, Proc. Conf. "Diff. Geometry and its Applications 1992" (1993), 449 – 459, Silesian University, Opava.

[6] Kolář, I., Michor, P.W., Slovak, J., *Natural operations in differential geometry*, Springer - Verlag, 1993.

[7] Modugno, M., *New results on the theory of connections, systems, overconnections and prolongations*, Proc. of the conference "Diff. geom and its Appl. 1986", 245 – 269, Brno.

Department of Mathematics, TU Zvolen
Masarykova 24, 960 53 Zvolen, SLOVAKIA
E-mail: dekret@vsld.tuzvo.sk