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A NOTE ON ENDOMORPHISMS OF MODULES OVER REDUCTIVE LIE GROUPS AND ALGEBRAS

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Abstract. We show how to apply the basic ideas of linear algebra to endomorphisms of modules over reductive Lie groups and algebras, thus obtaining notions of determinant, trace and more generally characteristic coefficients of such endomorphisms which are adapted to the situation. Finally, we show how these ideas can be applied to study locally trivial bundles of such modules over smooth manifolds.

When studying representations of reductive Lie groups and Lie algebras one usually quickly restricts to irreducible representations. Nevertheless, in many situations, in particular in applications of representation theory to differential geometry, one naturally meets reducible representations. The purpose of this note is to show, that in this situation one can apply basic ideas of linear algebra to construct adapted versions of characteristic coefficients, and in particular determinants and traces. The basic background in representation theory that we will use can be found in any standard text book, for example in [Fulton–Harris].

1. Let G be a reductive Lie group with Lie algebra $\mathfrak{g}$. In this note, we will always deal with modules over G, but in the case when G is connected and simply connected, modules and their homomorphisms over G are the same as modules and homomorphisms over $\mathfrak{g}$, so we cover these at the same time. Let V be a vector space over $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$ and let $\lambda : G \to GL(V)$ be a representation. Then it is well known that V splits into a direct sum of irreducible representations. Let $\{V_1,\dots,V_k\}$ be the set of different irreducible representations which occur in a decomposition of V and for $1\le j\le k$ let i_j be the multiplicity of the representation V_j in V. This means just, that $V\cong\oplus_{j=1}^k V_j^{i_j}$.

Now let us fix a decomposition of V into irreducibles, i.e. a family of projections $\pi_\alpha : V\to V$ for $\alpha=1,\dots,n$, such that the image of any π_α is an irreducible subrepresentation of V, and such that $V=\oplus_\alpha \operatorname{im}(\pi_\alpha)$. Moreover we assume that $\operatorname{im}(\pi_\alpha)\cong V_1$ for $1\le\alpha\le i_1$, $\operatorname{im}(\pi_\alpha)\cong V_2$ for $i_1+1\le\alpha\le i_1+i_2$ and so on. We

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say that two indices α, β between 1 and n are *close to each other*, if and only if they both lie between $i_1 + \dots + i_j + 1$ and $i_1 + \dots + i_{j+1}$ for some $j = 1, \dots, k-1$, i.e. if and only if the images of π_α and π_β are isomorphic.

Moreover, we choose a *cocycle of isomorphisms*, i.e. a family of module homomorphisms $\varphi_{\alpha,\beta} : \mathrm{im}(\pi_\beta) \to \mathrm{im}(\pi_\alpha)$ satisfying the following conditions

(1) $\varphi_{\alpha,\alpha}$ is the identity for all α.
(2) If α and β are close to each other, then $\varphi_{\alpha,\beta}$ is an isomorphism.
(3) If either α and β or β and γ are close to each other, then $\varphi_{\alpha,\beta} \circ \varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$.

Note that obviously $\varphi_{\alpha,\beta} = 0$ if α and β are not close. Such a cocycle always exists. To see this, just fix an isomorphism ψ_α of $\mathrm{im}(\pi_\alpha)$ with the appropriate V_j for each α and define $\varphi_{\alpha,\beta}$ as $\psi_\beta \circ \psi_\alpha^{-1}$ if α and β are close to each other, and $\varphi_{\alpha,\beta} = 0$ if α and β are not close. Obviously this satisfies all three conditions.

2. Having chosen the projections π_ℓ and the cocycle $\varphi_{\alpha,\beta}$ of isomorphisms we can associate to any G–homomorphism $\Phi : V \to V$ a block diagonal $n \times n$ matrix $[\Phi]$ with blocks of sizes $i_1, \dots, i_k$ as follows: For $1 \leq \alpha, \beta \leq n$ we can consider the restriction of the homomorphism $\pi_\beta \circ \Phi$ to the image of π_α, thus obtaining a G–homomorphism $\mathrm{im}(\pi_\alpha) \to \mathrm{im}(\pi_\beta)$. If α and β are not close to each other, then these are non–isomorphic irreducible representations, so this must be the zero homomorphism. If on the other hand α and β are close, then by Schur's lemma the space of G–homomorphisms $\mathrm{im}(\pi_\alpha) \to \mathrm{im}(\pi_\beta)$ is one dimensional, so there is a number $a_{\beta,\alpha} \in \mathbb{K}$ such that this restriction equals $a_{\beta,\alpha}\varphi_{\beta,\alpha}$. The defining equation for the number $a_{\beta,\alpha}$ can be rewritten as

$$\Phi(v) = \sum_{\alpha,\beta} a_{\beta,\alpha}\varphi_{\beta,\alpha}(\pi_\beta(v)),$$

so we actually get an isomorphism between the space of G–homomorphisms $\Phi : V \to V$ and the space of all block diagonal $n \times n$–matrices $(a_{\alpha,\beta})$ with blocks of sizes $i_1, \dots, i_k$.

3. Next we claim that another choice of the decomposition and the cocycle of isomorphisms amounts just to conjugation with an invertible block diagonal matrix. So let π'_α be another family of projections and let $\varphi'_{\alpha,\beta}$ be another cocycle of isomorphisms. For $\ell = 1, i_1 + 1, i_1 + i_2 + 1, \dots$ choose an isomorphism $\psi_{\ell,\ell} : \mathrm{im}(\pi'_\ell) \to \mathrm{im}(\pi_\ell)$ and define homomorphisms $\psi_{\alpha,\beta} : \mathrm{im}(\pi'_\beta) \to \mathrm{im}(\pi_\alpha)$ as follows: If α and β are close to each other, then there is a unique ℓ from the list above which is close to both α and β, and we put $\psi_{\alpha,\beta} := \varphi_{\alpha,\ell} \circ \psi_{\ell,\ell} \circ \varphi'_{\ell,\beta}$ in this case, while we set $\psi_{\alpha,\beta} = 0$ if α and β are not close to each other.

Thus $\psi_{\alpha,\beta}$ is an isomorphism, whenever the representations $\mathrm{im}(\pi'_\beta)$ and $\mathrm{im}(\pi_\alpha)$ are isomorphic, and clearly we have $\varphi_{\alpha,\beta} \circ \psi_{\beta,\gamma} = \psi_{\alpha,\gamma}$, whenever either α and β or β and γ are close to each other. Moreover, if α and β are close to each other, consider $\psi_{\beta,\alpha}^{-1} \circ \psi_{\beta,\gamma}$. If also γ is close to α and β, then from the definition of $\psi_{\alpha,\beta}$ one immediately gets that $\psi_{\beta,\alpha}^{-1} \circ \psi_{\beta,\gamma} = \varphi'_{\alpha,\gamma}$, and if γ is not close to α and β, this equation is trivially satisfied.

Now for any α and β, we may restrict π_α to the image of π'_β, thus obtaining a G–homomorphism $\mathrm{im}(\pi'_\beta) \to \mathrm{im}(\pi_\alpha)$ and we define a block diagonal $n \times n$ matrix $(\lambda_{\alpha,\beta})$ with blocks of sizes $i_1, \dots, i_k$ with the nonzero entries fixed by requiring the the above homomorphism equals $\lambda_{\alpha,\beta}\psi_{\alpha,\beta}$. This can be rewritten as

$$\pi_\alpha(v) = \sum_\beta \lambda_{\alpha,\beta}\psi_{\alpha,\beta}(\pi'_\beta(v)), \text{ or as } v = \sum_{\alpha,\beta} \lambda_{\alpha,\beta}\psi_{\alpha,\beta}(\pi'_\beta(v)).$$

Note that in the first expression we may as well sum only over the β which are close to α while in the second expression we only have to sum over pairs of indices which are close to each other.
Similarly, we may write $\pi'_\alpha(v) = \sum_\beta \mu_{\alpha,\beta}\psi^{-1}_{\beta,\alpha}(\pi_\beta(v))$, where $(\mu_{\alpha,\beta})$ is the inverse matrix to $(\lambda_{\alpha,\beta})$, and the sum goes over all pairs of indices which are close to each other. Now suppose that $\Phi : V \to V$ is an endomorphism, and the matrix representation of Φ with respect to the family π_i of projections and the cocycle $\varphi_{i,j}$ is given by the matrix $(a_{\alpha,\beta})$. By definition, this means that $\Phi(v) = \sum_{\alpha,\beta} a_{\alpha,\beta}\varphi_{\alpha,\beta}(\pi_\beta(v))$, where the sum goes over all pairs of indices which are close to each other. We can rewrite $\pi_\beta(v)$ as $\sum_\gamma \lambda_{\beta,\gamma}\psi_{\beta,\gamma}(\pi'_\gamma(v))$, and since $\varphi_{\alpha,\beta} \circ \psi_{\beta,\gamma} = \psi_{\alpha,\gamma}$ for α and β close to each other, we see that $\Phi(v) = \sum_{\alpha,\beta,\gamma} a_{\alpha,\beta}\lambda_{\beta,\gamma}\psi_{\alpha,\gamma}(\pi'_\gamma(v))$. Moreover, $\Phi(v) = \sum_\delta \pi'_\delta(\Phi(v))$, and using the above formula for $\pi'_\delta(v)$, we finally arrive at

$$\Phi(v) = \sum_{\delta,\alpha,\beta,\gamma} \mu_{\delta,\alpha}a_{\alpha,\beta}\lambda_{\beta,\gamma}\psi^{-1}_{\alpha,\delta}\psi_{\alpha,\gamma}(\pi'_\gamma(v)),$$

where we sum over all quadruples $(\delta, \alpha, \beta, \gamma)$ such that δ and α are close to each other. But in this case we have seen above, that $\psi^{-1}_{\alpha,\delta} \circ \psi_{\alpha,\gamma} = \varphi'_{\delta,\gamma}$, which immediately shows that the matrix representation with respect to the family π'_i of projections and the cocycle $\varphi'_{i,j}$ is obtained from the matrix $(a_{\alpha,\beta})$ by conjugation by the matrix $(\lambda_{\alpha,\beta})$.

4. It is now clear that having given an endomorphism $\Phi : V \to V$ the characteristic coefficients of the matrix of Φ with respect to any family of projections and any cocycle of isomorphisms are the same. In particular, we get a well defined version of the determinant, which we denote by $\Delta(\Phi)$ and the trace. For example, the trace of the identity mapping in this case gives the number of irreducible components of V, while the trace of the identity as a linear mapping gives the dimension of V.

Since the matrix representations are obviously compatible with the composition of endomorphisms, all characteristic coefficients vanish on commutators, i.e. if Φ and Ψ are endomorphisms and c is any characteristic coefficient, then $c(\Psi \circ \Phi) = c(\Phi \circ \Psi)$. Since we have the (adapted) trace of an endomorphism, we may as well construct a form of trace coefficients, by just taking the trace of powers of Φ, and the classical relations between characteristic coefficients and trace coefficients (see e.g. [Greub–Halperin–Vanstone, Appendix A]) continue to hold.

Another construction from linear algebra which can be carried over to this setting is the classical adjoint, or matrix of cofactors. For an endomorphism Φ :

$V \to V$ take a matrix representation with respect to a family π_i of projections and a cocycle $\varphi_{i,j}$ of isomorphisms. Then form the classical adjoint of this matrix, i.e. the matrix of determinants of $(n-1)\times(n-1)$ submatrices of the given one with an appropriate sign, and denote by $\mathcal{C}(\Phi)$ the corresponding endomorphism of V. Clearly, this satisfies the equation $\mathcal{C}(\Phi)\circ\Phi = \Phi\circ\mathcal{C}(\Phi) = \Delta(\Phi)\cdot\mathrm{id}_V$. In particular, if Φ is invertible, then $\mathcal{C}(\Phi) = \Delta(\Phi)\cdot\Phi^{-1}$.

It is easy to see in examples that the constructions presented here are better adapted to the setting then viewing endomorphisms simply as linear mappings. For example assume that $\Phi : V \to V$ is an endomorphism, whose kernel is an irreducible subrepresentation W of V of dimension > 1 which occurs only once in V. Then viewing Φ as a linear map and forming the classical adjoint one clearly gets the zero mapping, while $\mathcal{C}(\Phi)$ is a nonzero scalar multiple of the projection from V onto W.

5. Applications to bundles of modules. As before let $\lambda : G \to GL(V)$ be a representation of the group G on a $\mathbb{K}$–vector space V which splits into irreducibles as $V \cong V_1^{i_1}\oplus\dots\oplus V_k^{i_k}$, and put $n = i_1+\dots+i_k$. Then from the above discussion it is clear, that the group of G–module automorphisms of V is actually isomorphic to the subgroup $GL(i_1,\mathbb{K})\times\ldots\times GL(i_k,\mathbb{K})$ of $GL(n,\mathbb{K})$.

Now let M be a smooth manifold and let $p : E \to M$ be a locally trivial bundle of G–modules with standard fiber V, i.e. a locally trivial vector bundle with standard fiber V such that all transition functions are homomorphisms of G–modules. This means that the bundle E is an associated bundle to a principal bundle with structure group the group of G–module automorphisms of V, which by the above is isomorphic to $GL(i_1,\mathbb{K})\times\ldots\times GL(i_k,\mathbb{K})$. Taking the associated bundle with respect to the (restriction of the) defining representation of $GL(n,\mathbb{K})$ we get an associated $\mathbb{K}$–vector bundle with standard fiber $\mathbb{K}^n$, which can be written as a Whitney sum of vector bundles of dimensions $i_1,\dots,i_k$. Thus the classification up to isomorphism of bundles of G–modules with standard fiber V is the same as the classification of rank n $\mathbb{K}$–vector bundles which can be written as a Whitney sum of bundles of ranks $i_1,\dots,i_k$.

Things become particularly simple, if the representation V is simply reducible, i.e. if it splits as a direct sum of different irreducible representations. In this case, the classification of G–module bundles with fiber V amounts just to the classification of $\mathbb{K}$–vector bundles which can be written as a direct sum of line bundles. In particular, if M is a manifold such that any $\mathbb{K}$–line bundle over M is trivial, then any G–module bundle with fiber V over M is trivial as well. Now line bundles are classified by their first Chern class in the complex case and by their first Stiefel–Whitney class in the real case. Thus we obtain

Proposition. *(1) Let M be a smooth manifold with $H^1(M,\mathbb{Z}_2)=0$. Then any G–module bundle with standard fiber a simply reducible real representation of G is trivial.*
(2) Let M be a smooth manifold with $H^2(M,\mathbb{Z})=0$. Then any G–module bundle with standard fiber a simply reducible complex representation of G is trivial.

6. Let us finally remark that the equivalence of G–module bundles with certain $\mathbb{K}$–vector bundles developed above gives the possibility to define characteristic cohomology classes for G–module bundles. Thus for a G–module bundle $E \to M$ with fiber a complex G–representation one obtains Chern classes $\mathrm{Ch}_k(E) \in H^{2k}(M, \mathbb{Z})$ for k between zero and the number of irreducible components of V. In the real case, one obtains analogs of Stiefel–Whitney classes, Pontryagin classes and so on.

It is also clear that the characteristic classes in the real cohomology of M can be computed using the characteristic coefficients developed in 4 along the following lines: Start with a G–equivariant connection on the bundle E and construct its curvature, which is then a two form with values in the bundle of G–module automorphisms of the bundle E. Then follow the usual development of characteristic classes from linear connections (see e.g. [Greub–Halperin–Vanstone]), replacing all characteristic and trace coefficients by their adapted versions.

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