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# THE RICCI IDENTITY FOR GENERAL CONNECTIONS

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Abstract. Given an arbitrary fibered manifold $Y \to M$, we introduce the second order absolute differentiation with respect to a general connection $\Gamma$ on $Y$ and a classical linear connection $\Lambda$ on $M$. Then we generalize the Ricci identity to such a situation. The problem requires a new geometric operation on a subclass of the iterated 2-jets.

## Introduction

It was pointed out by many authors,[KoMiSl],[MaMo],[MaMo],[Mo],[Sau], that the general connections on an arbitrary fibered manifold $Y \to M$ have several properties of the classical principal or linear connections. In the present paper we deduce that even the Ricci identity for the second order absolute differential can be generalized to any fibered manifold. The second order absolute differentiation of sections of $Y$ is based on a general connection $\Gamma$ on $Y$ and on a classical linear connection $\Lambda$ on $M$. In Section 2 we discuss an arbitrary connection $\Gamma$ on a vector bundle. In this case the tensor alternation can be used for formulating the Ricci identity. In the case of a linear connection $\Gamma$ we rededuce a classical result. However, on an arbitrary fibered manifold we need a new geometric operation, which is defined on a subclass of the iterated 2–jets. The basic properties of iterated 2–jets are described in Section 3 and the difference tensor is introduced in Section 4. This concept enables us to formulate and prove the Ricci identity for general connections in Section 5. All manifolds and maps are assumed to be infinitely differentiable.

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## 1. The iterated absolute differentiation

Let $p : Y \to M$ be an arbitrary fibered manifold and $\Gamma : Y \to J^1Y$ be a connection on $Y$, [KoMiSl]. The first absolute differentiation of the sections of $Y$ can

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be based on the fact that $J^1Y \to Y$ is an affine bundle with derived vector bundle $VY \otimes T^*M$, where $VY$ is the vertical tangent bundle of $Y$. The tensor product is over $Y$, i.e. we construct the pullback $p^*T^*M$ and then we tensorize over $Y$. The absolute differential of a section $s : M \to Y$ is a section $\nabla s : M \to VY \otimes T^*M$ defined by

$$\nabla s(x) : j^1_x s - \Gamma(s(x)) \ , \qquad x \in M \tag{1}$$

Let $x^i, y^p$ be some local fiber coordinates on $Y$ and let $\Gamma$ be expressed by

$$dy^p = F^p_i(x, y) dx^i \tag{2}$$

and $s$ by $y^p = s^p(x)$. Then the coordinate form of (1) is

$$\frac{\partial s^p}{\partial x^i} - F^p_i(x, s(x)) \tag{3}$$

There is a canonical isomorphism $i_Y : V(J^1Y \to M) \to J^1(VY \to M)$, [KoMiSl, p.255]. If we compose the vertical tangent map $V\Gamma : VY \to VJ^1Y$ with $i_Y$, we obtain a connection $\mathcal{V}\Gamma := i_Y \circ V\Gamma$ on $VY \to M$, which is called the vertical prolongation of $\Gamma$. If $Y^p$ are the additional coordinates on $VY$, then the equations of $\mathcal{V}\Gamma$ are (2) and

$$dY^p = \frac{\partial F^p_i}{\partial y^q} Y^q dx^i \tag{4}$$

In general, let $q : Z \to Y$ be a fibered manifold, so that we have the total projection $p \circ q : Z \to M$. A connection $\gamma : Z \to J^1Z = J^1(Z \to M)$ is called projectable, if there exists a connection $\Gamma : Y \to J^1Y$ such that $J^1q \circ \gamma = \Gamma \circ q$, where $J^1q : J^1Z \to J^1Y$ is the induced map. Moreover, if $Z \to Y$ is a vector bundle, then $J^1Z$ is a vector bundle over $J^1Y$. A projectable connection $\gamma : Z \to J^1Z$ over $\Gamma$ is called semilinear, if $\gamma$ is a vector bundle morphism of $Z \to Y$ into $J^1Z \to J^1Y$ over the base map $\Gamma : Y \to J^1Y$. If $\bar{Z} \to Y$ is another vector bundle and $\bar{\gamma}$ is another semilinear connection over the same $\Gamma$, then one can construct the tensor product $\gamma \otimes \bar{\gamma}$ analogously to [KoMiSl, p.381]. Also $\gamma \otimes \bar{\gamma}$ is a semilinear connection on $Z \otimes \bar{Z} \to Y \to M$ over $\Gamma$.

In particular, consider a linear connection $\Lambda$ on $TM$, which induces the dual connection $\Lambda^*$ on $T^*M$. The product $\Gamma \times \Lambda^*$ is a connection on $Y \underset{M}{\times} T^*M = p^*T^*M$, which a semilinear connection on $p^*T^*M \to Y \to M$ over $\Gamma$. The tensor product of $\mathcal{V}\Gamma$ and $\Gamma \times \Lambda^*$ is a semilinear connection on $VY \otimes T^*M$, which will be denoted by $\mathcal{V}\Gamma \otimes \Lambda^*$. If $Y^p_i$ are the induced coordinates on $VY \otimes T^*M$ and $\Lambda^i_{jk}(x)$ are the Christoffel symbols of $\Lambda$, then the coordinate expression of $\mathcal{V}\Gamma \otimes \Lambda^*$ is (2) and

$$dY^p_i = \left( \frac{\partial F^p_j}{\partial y^q} Y^q_i - \Lambda^i_{jk} Y^p_k \right) dx^j \tag{5}$$

**Definition 1.1.** *The absolute differential $\nabla^2_\Lambda s$ of $\nabla s$ with respect to $\mathcal{V}\Gamma \otimes \Lambda^*$ is called the second absolute differential of a section $s : M \to Y$ with respect to $\Gamma$ and $\Lambda$.*

Thus, the coordinate form of $\nabla^2_\Lambda s$ is (3) and

$$\frac{\partial}{\partial x^j}\left(\frac{\partial s^p}{\partial x^i} - F^p_i\right) - \frac{\partial F^p_j}{\partial y^q}\left(\frac{\partial s^q}{\partial x^i} - F^q_i\right) + \Lambda^k_{ij}\left(\frac{\partial s^p}{\partial x^k} - F^p_k\right) \tag{6}$$

We recall that the curvature $R$ of $\Gamma$ is a map $Y \underset{M}{\times} \overset{2}{\wedge} TM \to VY$. Its coordinate expression is

$$\left(\frac{\partial F^p_i}{\partial x^j} - \frac{\partial F^p_i}{\partial y^q} F^q_j\right) dx^i \wedge dx^j \tag{7}$$

On the other hand, the torsion $S$ of $\Lambda$ is a section of $TM \otimes \overset{2}{\wedge} T^*M$.

## 2. The Ricci identity for non linear connections

Given a vector bundle $p : E \to M$, a general connection $\Gamma$ on $E$ can be called a non linear connection. Since $VE = E \underset{M}{\times} E$ and similarly for $E \otimes T^*M$, (6) can be considered as a section (denoted by the same symbol)

$$\nabla^2_\Lambda s \,:\, M \to E \otimes T^*M \otimes T^*M \tag{8}$$

Hence we have defined the alternation $\mathrm{Alt}(\nabla^2_\Lambda s) : M \to E \otimes \overset{2}{\wedge} T^*M$. On the other hand, the curvature $R$ of $\Gamma$ can be interpreted as a map

$$R \,:\, E \to E \otimes \overset{2}{\wedge} T^*M \tag{9}$$

Clearly, the contraction $\langle S, \nabla s\rangle$ of the torsion of $\Lambda$ with $\nabla s$ is a section $M \to E \otimes \overset{2}{\wedge} T^*M$. From (6) and (7) we obtain easily

**Proposition 2.1.** *(The Ricci identity for non linear connections.) It holds*

$$\mathit{Alt}\left(\nabla^2_\Lambda s\right) = -R \circ s + \langle S, \nabla s\rangle \tag{10}$$

In particular, consider a linear connection $\Gamma$ on $E$. Its coordinate form is

$$dy^p = \Gamma^p_{qi}(x) y^q \, dx^i \tag{11}$$

By (4) we have $\mathcal{V}\Gamma = \Gamma \times \Gamma$, so that $\nabla^2_\Lambda s$ coincides with the absolute differential of $\nabla s : M \to E \otimes T^*M$ with respect to the classical tensor product $\Gamma \otimes \Lambda^*$. The curvature of a linear connection can be interpreted as a section $M \to E \otimes E^* \otimes \overset{2}{\wedge} T^*M$. Hence the contraction $\langle R, s\rangle$ is a section of $E \otimes \overset{2}{\wedge} T^*M$ and (10) reduces to

$$\mathrm{Alt}\left(\nabla^2_\Lambda s\right) = -\langle R, s\rangle + \langle S, \nabla s\rangle \tag{12}$$

This is the general Ricci identity for linear connections in the sense of [KoMiSl, p.235].

Moreover, if $E = \overset{r}{\otimes} TM \otimes \overset{s}{\otimes} T^*M$ is a classical tensor bundle over $M$, then one starts from a classical linear connection $\Lambda$ on $M$ and put $\Gamma = \overset{r}{\otimes}\Lambda \otimes \overset{s}{\otimes}\Lambda^*$. Then (12) is the Ricci identity from the classical tensor analysis.

However, in the case of an arbitrary fibered manifold $Y \to M$ one must replace the alternation in (10) by a more sophisticated geometric operation. It will be defined in Section 4 below.

## 3. Iterated 2–jets

Let $J^1(M,N)$ be the space of all 1–jets between two manifolds $M,N$ and $\alpha : J^1(M,N) \to M$ or $\beta : J^1(M,N) \to N$ be the source or target projection, respectively. According to Virsik, [Vir], the space $\check{J}^2(M,N)$ of iterated 2–jets of $M$ into $N$ is defined by the iteration $\check{J}^2(M,N) = J^1(M,J^1(M,N))$. Denote by $\alpha_1 : \check{J}^2(M,N) \to M$ or $\beta_1 : \check{J}^2(M,N) \to J^1(M,N)$ the source or target projection of $J^1(M,J^1(M,N))$. By [KoMiSl, p.118], $J^1$ is a bundle functor on the product category $\mathcal{M}f_m \times \mathcal{M}f$, where $\mathcal{M}f$ is the category of manifolds and smooth maps and $\mathcal{M}f_m$ is the category of $m$–dimensional manifolds and local diffeomorphisms. Thus, if $1_M$ denotes the identity of $M$, we can construct $J^1(1_M,\alpha) : \check{J}^2(M,N) \to J^1(M,M)$ and $J^1(1_M,\beta) : \check{J}^2(M,N) \to J^1(M,N)$. Beside $\alpha_1$ we have another projection $\alpha_2 := \alpha \circ \beta_1 : \check{J}^2(M,N) \to M$ and a projection $\beta_0 = \beta \circ \beta_1 : \check{J}^2(M,N) \to N$. The set $\tilde{J}^2(M,N)$ of non holonomic 2–jets by Ehresmann, [Eh], is the subset of all $A \in \check{J}^2(M,N)$ satisfying $J^1(1_M,\alpha)(A) = j^1_x 1_M$, $x = \alpha_1 A = \alpha_2 A$. The subset $\bar{J}^2(M,N) \subset \tilde{J}^2(M,N)$ of semiholonomic 2–jets is further characterised by $\beta_1(A) = J^1(1_M,\beta)(A)$.

Let $x^i$ or $y^p$ be some local coordinates on $M$ or $N$. The additional coordinates on $J^1(M,N)$ are $y^p_i = \partial y^p/\partial x^i$, so that the corresponding coordinates on $J^1(M,N)$ are $(y^p, y^p_i, x^i)$. By iteration, the induced coordinates on $\check{J}^2(M,N)$ are

$$(y^p, y^p_i, x^i, y^p_{0i}, y^p_{ij}, x^i_j, u^i) \tag{13}$$

For an element $A$ with coordinates (13) we have $\alpha_1 A = (u^i)$, $\alpha_2 A = (x^i)$, $\beta_0 A = (y^p)$, $\beta_1 A = (y^p, y^p_i, x^i)$, $J^1(1_M,\beta)(A) = (y^p, y^p_{0i}, u^i)$, $J^1(1_M,\alpha)(A) = (x^i, x^i_j, u^i)$.

Let $[x]$ denote the constant map of $M$ into $x \in M$.

**Definition 3.1.** *An element $A \in \check{J}^2(M,N)$ will be called distinguished, if $J^1(1_M,\alpha)(A) = j^1_x[x]$ for some $x \in M$ and $\beta_1(A) = J^1(1_M,\beta)(A)$.*

The set of all distinguished iterated 2–jets will be denoted by $\hat{J}^2(M,N)$. Since $J^1(1_M,\alpha)(A) = j^1_x[x]$ implies $\alpha_1 A = \alpha_2 A$, we have only one projection $\alpha_0 : \hat{J}^2(M,N) \to M$. Analogously to the semiholonomic and nonholonomic jets, we shall denote by $\hat{J}^2_x(M,N)_y$ the subspace of all $A$'s satisfying $\alpha_0 A = x$, $\beta_0 A = y$.

## 4. The difference tensor

First of all, we recall how a semiholonomic 2–jet $A \in \bar{J}^2_x(M,N)_y$ defines the deviation $\Delta A \in T_y N \otimes \overset{2}{\wedge} T^*_x M$, [CaKo], [MaMo], [Pra], [Sau]. We have $A = j^1_x s$, where $s$ is a section of $\alpha : J^1(M,N) \to M$. Every $s(u) \in J^1_u(M,N)$ can be interpreted as a map $T_u M \to TN$, so that $s$ determines a map $TM \to TN$. Its tangent map $\lambda(A) : TT_x M \to TT_y N$ over $x \in M$ is determined by $A$. If $\kappa_{M,x}$ is the restriction of the canonical involution of the second tangent bundle over $x \in M$, then there exists a unique element $\kappa A \in \bar{J}^2_x(M,N)_y$ such that $\lambda(\kappa A) = \kappa_{N,y} \circ \lambda(A) \circ \kappa_{M,x}$. Since $\bar{J}^2(M,N) \to J^1(M,N)$ is an affine vector bundle with derived vector bundle $TN \otimes \overset{2}{\otimes} T^* M$, the vector $\Delta A = A - \kappa A$ belongs to $T_y N \otimes \overset{2}{\wedge} T^*_x M$. If $(x^i, y^p, y^p_i, y^p_{ij})$ are the essential coordinates of $A$, then $\Delta A = y^p_{[ij]}$.

This procedure can be modified to an element $A \in \hat{J}^2_x(M,N)_y$ as follows. We have $A = j^1_x s$, where $s(u) \in J^1_x(M,N)$. Hence $s$ defines a map $\sigma : M \times T_xM \to TN$,

$$\sigma(u,\xi) = s(u)(\xi), \qquad \xi \in T_xM \tag{14}$$

The tangent map is $T\sigma : TM \times T(T_xM) \to TTN$. If $s(u) = (y^p(u), y^p_i(u), x^i)$ and $\xi = (\xi^i)$, then the coordinate form of (14) is

$$y^p = y^p(u), \qquad \eta^p = y^p_i(u)\xi^i \tag{15}$$

provided $\eta^p$ are the additional coordinates on $TN$. Let $Y^p = dy^p$, $H^p = d\eta^p$ be the induced coordinates on $TTN$. On the other hand, since $T_xM$ is a vector space, we have $T(T_xM)) = T_xM \times T_xM$. Let $\xi^i, \Xi^i = d\xi^i$ be the corresponding coordinate on $T(T_xM)$ and $X^i = du^i$ be the induced coordinates on $TM$. Then the restriction of $T\sigma$ to $T_xM \times T_xM \times T_xM$ is

$$Y^p = y^p_i X^i, \qquad \eta^p = y^p_i \xi^i, \qquad H^p = y^p_{ij}\xi^i X^j + y^p_i \Xi^i \tag{16}$$

If we apply $\kappa_{N,y}$ and the echange map between the first and the second factor of $T_xM \times T_xM \times T_xM$, we obtain

$$Y^p = y^p_i X^i, \qquad \eta^p = y^p_i \xi^i, \qquad H^p = y^p_{ij} X^i \xi^j + y^p_i \Xi^i \tag{17}$$

For every $X \in T_xM$ and $\xi \in T_xM$ take any $\Xi \in T_xM$. Then the strong difference (see [KoMiSl, p.391]) of (16) and (17) does not depend on the choice of $\Xi$ and defines an antisymmetric map $T_xM \times T_xM \to T_yN$ of the form

$$(y^p_{ij} - y^p_{ji})\xi^i X^j \tag{18}$$

This will be denoted by $\delta(A) \in T_yN \otimes \overset{2}{\wedge} T^*_xM$.

**Definition 4.1.** *$\delta(A) \in T_yN \otimes \overset{2}{\wedge} T^*_xM$ will be called the difference tensor of $A \in \hat{J}^2_x(M,N)_y$.*

## 5. The case of a general connection

In order to show that the difference tensor can be applied to $\nabla^2_\wedge s$ from Section 1, we discuss a more general situation. Let $\gamma$ be any connection in $VY \otimes T^*M \to Y \to M$ over the connection $\Gamma$ on $Y$. Write $\nabla_\Gamma s$ for $\nabla s$, so that $\nabla_\gamma(\nabla_\Gamma s)$ means the absolute differential of $\nabla_\Gamma s$ with respect to $\gamma$. We have $V_yY \otimes T^*_xM = T_y(Y_x) \otimes T^*_xM = J^1_x(M, Y_x)_y$ for all $y \in Y$, $x = p(y)$, which implies $VY \otimes T^*M \subset J^1(M,Y)$. By iteration, $V(VY \otimes T^*M) \otimes T^*M \subset J^1(M, J^1(M,Y)) = \check{J}^2(M,Y)$.

**Proposition 5.1.** *For every section $s$ of $Y$ and every $x \in M$, we have $\nabla_\gamma(\nabla_\Gamma s)(x) \in \hat{J}^2(M,Y)$ and $\delta(\nabla_\gamma(\nabla_\Gamma s)(x)) \in VY \otimes \overset{2}{\wedge} T^*M \subset TY \otimes \overset{2}{\wedge} T^*M$.*

Proof. We shall use the coordinate expressions. If $(x^i, y^p, Y^p_i)$ are the coordinates on $VY \otimes T^*M$ from Section 1 and $(x^i, y^p, x^k_i, y^p_i, u^i)$ are the coordinates on $J^1(M,Y)$ from Section 3, then the inclusion $VY \otimes T^*M \subset J^1(M,Y)$ is expressed by

$$(x^i, y^p, Y^p_i) \mapsto (x^i, y^p, 0, Y^p_i, x^i) \tag{19}$$

By iteration, let

$$(x^i, y^p, Y_i^p, Y_{0i}^p, Y_{ij}^p) \tag{20}$$

be the analogous coordinates on $V(VY \otimes T^*M) \otimes T^*M$ and let

$$(x^i, y^p, x_i^k, y_i^p, u^i, x_j^i, y_{0j}^p, x_{ij}^k, y_{ij}^p, u_j^i, v^i) \tag{21}$$

be the corresponding coordinates on $\check{J}^2(M,Y)$. Then the image of (20) in $\check{J}^2(M,Y)$ has the form

$$(x^i, y^p, 0, Y_i^p, x^i, 0, Y_{0i}^p, 0, Y_{ij}^p, 0, x^i) \tag{22}$$

If $y^p = s^p(x)$ is the coordinate expression of $s$, then the coordinate form of $\nabla_\Gamma s$ is $y^p = s^p(x)$ and $Y_i^p = \frac{\partial s^p}{\partial x^i} - F_i^p(x, s(x)) =: s_i^p$. Since $\gamma$ is over $\Gamma$, the additional coordinate expression of $\nabla_\gamma(\nabla_\Gamma s)$ is $Y_{0i}^p = s_i^p$ and $Y_{ij}^p$ are some functions of $x^i$, which we do not need. Then (22) proves that each iterated 2–jet $\nabla_\gamma(\nabla_\Gamma s)(x)$ is distinguished. The zero at the eighth place in (22) indicates that the values of $\delta(\nabla_\gamma(\nabla_\Gamma s))$ lie in $VY \otimes \overset{2}{\wedge} T^*M$. $\boxed{QED}$

Consider now the case $\gamma = \mathcal{V}\Gamma \otimes \Lambda^*$, so that $\nabla_{\mathcal{V}\Gamma\otimes\Lambda^*}(\nabla_\Gamma s) = \nabla_\Lambda^2 s$. Then we have three sections $\delta(\nabla_\Lambda^2 s)$, $R \circ s$ and $\langle S, \nabla s\rangle$ of $VY \otimes \overset{2}{\wedge} T^*M$.

**Proposition 5.2.** *(The Ricci identity for general connections.) It holds*

$$\delta(\nabla_\Lambda^2 s) = -R \circ s + \langle S, \nabla s\rangle \tag{23}$$

Proof. By (18), the coordinate form of (23) is the same as in Proposition 2.1. $\boxed{QED}$

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