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# ON THE SPACE OF 3–DIMENSIONAL HOMOGENEOUS RIEMANNIAN MANIFOLDS

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ABSTRACT. We answer the following question: Let $\lambda$, $\mu$, $\nu$ be arbitrary real numbers. Does there exist a 3–dimensional homogeneous Riemannian manifold whose eigenvalues of the Ricci tensor are just $\lambda$, $\mu$, and $\nu$ ?

It is essential to observe that inspite of the fact that the curvature of a 3–manifold is uniquely determined by the Ricci tensor, nevertheless, the constancy of all its three eigenvalues does not imply the manifold to be locally homogeneous, see Ref. 1 for examples. But the answer to the above question does not alter if we replace the word "homogeneous" by the notion "locally homogeneous". In principle, this question could have been answered since decades already. But the answer seems not to be published until this Conference. And, accidentally, the answer was found independently by Kowalski/Nikčević, Ref. 2, on the one hand, and by Rainer/Schmidt, Ref. 3, on the other hand.

Of course, both answers are equivalent. But the formulations are so different, that it is useful to compare them here. In Ref. 2 one defines $\rho_1 = max\{\lambda, \mu, \nu\}$, $\rho_3 = min\{\lambda, \mu, \nu\}$ and $\rho_2 = \lambda + \mu + \nu - \rho_1 - \rho_3$ and formulates the conditions as inequalities between the $\rho_i$.

In Ref. 3 the formulation uses curvature invariants; first step: one observes that the answer does not alter, if one multiplies $\lambda$, $\mu$, $\nu$ with the same positive constant. [Remark: this fails to be the case if we take a negative constant]; second step: we take curvature invariants which do not alter by this multiplication, i.e., we look for suitable homothetic invariants; third step: we formulate the answer by use of the homothetic invariants.

First step: this can be achieved by a homothetic transformation of the metric (i.e., by a conformal transformation with constant conformal factor). Then one excludes the trivial case $\lambda = \mu = \nu = 0$.

Second step: $R = \lambda + \mu + \nu$, $N = \lambda^2 + \mu^2 + \nu^2$ are two curvature invariants with $N > 0$, and $\hat{R} = R/\sqrt{N}$ is a homothetic invariant. For defining a second

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homothetic invariant we introduce the trace–free part of the Ricci tensor

$$S_{ij} = R_{ij} - \frac{R}{3} g_{ij}$$

and its invariant $S = S_i^j \; S_j^k \; S_k^i$. The desired second homothetic invariant reads $\hat{S} = S/\sqrt{N^3}$. Of course, one can also directly define $\hat{R}$ and $\hat{S}$ as functions of $\lambda$, $\mu$, and $\nu$, e.g., $S$ is a cubic polynomial. [Remark: the inequalities $S > 0$ in Ref. 3 and $\rho_2 < \frac{\rho_1+\rho_3}{2}$ in Ref. 2 are equivalent.]

Third step: The answer is "yes" if and only if one of the following conditions are fulfilled:

a) $N = 0$ [flat space]
b) $\hat{R}^2 = 3$ and $\hat{S} = 0$ [non–flat spaces of constant curvature]
c) $\hat{R}^2 = 2$ and $9\hat{R}\hat{S} = -1$ [real line times non–flat plane of constant curvature]
d) $\hat{R}^2 = 1$ and $9\hat{R}\hat{S} = 2$
e) $18\hat{S} > \hat{R}(9 - 5\hat{R}^2)$ [equivalent to $\lambda\mu\nu > 0$ ]
f) $-\sqrt{3} < \hat{R} < -\sqrt{2}$ and

$$0 < \hat{S} \leq (3 - \hat{R}^2)^2 \frac{5\hat{R}^2 - 6}{-18\hat{R}^3}$$

[Remarks: 1. In Ref. 2 this last inequality is more elegantly written as

$$\rho_2 \geq \frac{\rho_1^2 + \rho_3^2}{\rho_1 + \rho_3} \; .$$

2. There exists a one–parameter set (parameter $a$ with $0 < a < 1$ of Bianchi–type VI manifolds having $\rho_1 = -a - a^2$, $\rho_2 = -1 - a^2$, $\rho_3 = -1 - a$; they represent the case where in subsection f) the right inequalities are fulfilled with "=". ]

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