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A LOWER BOUND FOR THE RADIUS OF THE SMALLEST BALL CONTAINING A MANIFOLD WITH CURVATURE BOUNDED FROM ABOVE

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1. Introduction

Let $\mathcal{F}$ be a certain family of triples (φ, N, M) where N and M are riemannian manifods and $\varphi : N \to M$ is an isometric immersion. We look for bounds for the function $\rho : \mathcal{F} \to \mathbb{R}^+$ which assigns to each (φ, N, M) the radius of the smallest M-ball containing $\varphi(N)$ (extrinsic radius).

Spruck ([Sp]) proved that if $\mathcal{F}$ is the family of triples $(\varphi, N, \mathbb{R}^n)$ such that N is a compact, connected riemannian manifold of dimension $n-1$ with sectional curvature $K \geq \beta > 0$, then the supremum of the function ρ is $\frac{\pi}{2\sqrt{\beta}}$.

Menninga in [Me] has generalized Spruck's result by considering a family of triples (φ, N, M) where N is a compact, connected riemannian manifold of dimension $s \geq 2$ with sectional curvature $K \geq \beta > 0$ and M is a riemannian manifold of dimension n with injectivity radius $i(M) \geq \frac{\pi}{\sqrt{\beta}}$ and sectional curvature $\overline{K} \leq \frac{\beta}{4}$.

For the infimum of the function ρ Jorge and Koutroufiotis have obtained in [JK] a result which implies that if $\mathcal{F}$ is the family of triples (φ, N, M) where N is a complete manifold of dimension s with scalar curvature bounded from below and sectional curvature $K \leq \beta$, and M is a complete connected riemannian manifold of dimension $n > 2s$ with sectional curvature $\lambda \geq \overline{K} \geq \alpha$ and the diameter of $\varphi(N)$ is less than $\frac{\pi}{2\sqrt{\lambda}}$ if $\lambda > 0$, then $\beta > \alpha$ and a lower bound for the function ρ is

$$\left[\frac{C_\lambda}{S_\lambda}\right]^{-1}\left(\sqrt{\beta-\alpha}\right)$$

where

$$S_\lambda(t) = \begin{cases} \frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}} & if \quad \lambda > 0 \\ t & if \quad \lambda = 0 \\ \frac{\sinh(\sqrt{|\lambda|}t)}{\sqrt{|\lambda|}} & if \quad \lambda < 0 \end{cases}$$

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and $C_\lambda(t) = S'_\lambda(t)$.

This bound is a minimum if $\lambda = \alpha$. In fact, it is attained at $(\varphi, N, M) \equiv (i, S^s_R, \mathbb{K}^n(\lambda))$, where $\mathbb{K}^n(\lambda)$ is the simply connected space of constant sectional curvature λ, S^s_R is the geodesic sphere of radius $R = \left[\frac{C_\lambda}{S_\lambda}\right]^{-1}\left(\sqrt{\beta - \lambda}\right)$ in $\mathbb{K}^{s+1}(\lambda)$ (where $\mathbb{K}^{s+1}(\lambda)$ is considered as a totally geodesic submanifold of $\mathbb{K}^n(\lambda)$) and $i : S^s_R \to \mathbb{K}^n(\lambda)$ is the canonical immersion. One has

$$\rho(i, S^s_R, \mathbb{K}^n(\lambda)) = R = \left[\frac{C_\lambda}{S_\lambda}\right]^{-1}\left(\sqrt{\beta - \lambda}\right)$$

The aim of this note is to obtain lower bounds for the function ρ defined on some families of triples $(\varphi, N, \mathbb{C}K^n(\lambda))$ where $\mathbb{C}K^n(\lambda)$ is the simply connected Kähler manifold with constant holomorphic sectional curvature 4λ and real dimension $2n$.

Now we state the main results of this paper. Let N be a complete riemannian manifold of dimension s, and let $\varphi : N \to \mathbb{C}K^n(\lambda)$ be an isometric immersion, and suppose that the scalar curvature of N is bounded from below. Let us denote by K and $\overline{K}$ the sectional curvatures of N and $\mathbb{C}K^n(\lambda)$, respectively. Let $B_r(p)$ be the geodesic ball of radius r about p in $\mathbb{C}K^n(\lambda)$. From now on, λ will denote a nonzero real number and $\beta \in \mathbb{R}$. In the next theorems, $d(\varphi(N))$ will denote the *extrinsic diameter* of $\varphi(N)$.

1.1 Theorem. *Let $\lambda > 0$. Let us suppose that $s > n+1$, $K \leq \beta$ and $d(\varphi(N)) < \frac{\pi}{2\sqrt{\lambda}}$. Then $\beta > \lambda$ and*

$$\rho(\varphi, N, M) \geq \left[\frac{C_\lambda}{S_\lambda}\right]^{-1}\left(\sqrt{\beta - \lambda}\right)$$

Let us recall that a holomorphic plane is that generated by two vectors of the form X, JX, and an antiholomorphic plane is that generated by two vectors X, Y such that Y is orthogonal to both X and JX.

1.2 Theorem. *Let us suppose that $s > n$, and that, for each plane σ in TN such that $\varphi_*(\sigma)$ is a holomorphic plane we have $K(\sigma) \leq \beta$. When $\lambda > 0$, let us assume, also, that $d(\varphi(N)) < \frac{\pi}{2\sqrt{\lambda}}$ if $s > n+1$, or $d(\varphi(N)) < \frac{\pi}{4\sqrt{\lambda}}$ if $s = n+1$. Then $\beta > 4\lambda$ and*
(a) for $\lambda < 0$, or $\lambda > 0$ and $s > n+1$, then

$$\rho(\varphi, N, M) \geq \left[\frac{C_\lambda}{S_\lambda}\right]^{-1}\left(\sqrt{\beta - 4\lambda}\right)$$

and
(b) for $\lambda > 0$ and $s = n+1$, then

$$\rho(\varphi, N, M) \geq \left[\frac{C_{4\lambda}}{S_{4\lambda}}\right]^{-1}\left(\sqrt{\beta - 4\lambda}\right).$$

1.3 Theorem. *Let us suppose that $s > n+1$, and that for each plane σ in TN such that $\varphi_*(\sigma)$ is an antiholomorphic plane, we have $K(\sigma) \leq \beta$. When $\lambda > 0$, let*

us assume, also, that $d(\varphi(N)) < \frac{\pi}{2\sqrt{\lambda}}$ *if* $s > n+2$, *or* $d(\varphi(N)) < \frac{\pi}{4\sqrt{\lambda}}$ *if* $s = n+2$. *Then* $\beta > \lambda$ *and*
(a) for $\lambda < 0$, *or* $\lambda > 0$ *and* $s > n+2$, *then*

$$\rho(\varphi, N, M) \geq \left[\frac{C_\lambda}{S_\lambda}\right]^{-1} \left(\sqrt{\beta - \lambda}\right)$$

and
(b) for $\lambda > 0$ *and* $s = n+2$, *then*

$$\rho(\varphi, N, M) \geq \left[\frac{C_{4\lambda}}{S_{4\lambda}}\right]^{-1} \left(\sqrt{\beta - \lambda}\right)$$

2. Proof of the main theorems

We shall use the following

Omori's lemma([Om]). *Let* M *be a complete Riemannian manifold whose sectional curvature is bounded from below. Let* $f : M \longrightarrow \mathbb{R}$ *be a smooth function bounded from above. Then, for every* $m \in M$ *and every* $\epsilon > 0$, *there is an* $m' \in M$ *with the properties*

$$f(m') \geq f(m), \quad \|grad \quad f\| (m') < \epsilon, \quad \nabla^2 f(X, X) < \epsilon \|X\|^2$$

for every $X \in T_{m'}M$, *where* ∇ *is the riemannian connection on* M.

In the computations below, we shall denote by the same symbol a tangent vector X in N and its image $\varphi_*(X)$. For any fixed $p \in \mathbb{C}K^n(\lambda)$ let $r : \mathbb{C}K^n(\lambda) \to \mathbb{R}$ be the distance to p. Let $\overline{\nabla}$ and ∇ be the riemannian connections on $\mathbb{C}K^n(\lambda)$ and N respectively, and let α be the second fundamental form of the immersion φ. The following formula is well known ([JK])

$$\left(\nabla^2 r \circ \varphi\right)(X, Y) \quad = \quad \left(\overline{\nabla}^2 r\right)(X, Y) + < \partial r, \alpha(X, Y) > \tag{1}$$

where ∂_r denotes the gradient of r in $\mathbb{C}K^n(\lambda)$. We shall also use the fact that $-\overline{\nabla}^2 r$ is the second fundamental form of the geodesic sphere of radius r in $\mathbb{C}K^n(\lambda)$ ([GW]) and ([Gr, p.138])

$$\begin{aligned}\left(\overline{\nabla}^2 r\right)(X, X) \quad &= \quad \frac{C_\lambda(r)}{S_\lambda(r)} \left(\|X\|^2 - < X, J\partial_r >^2\right) \\ &\quad + \frac{C_{4\lambda}(r)}{S_{4\lambda}(r)} < X, J\partial_r >^2\end{aligned} \tag{2}$$

2.1 Proof of theorem 1.1

We shall follow the ideas in [JK] and [Gi].

Given any $p \in \mathbb{C}K^n(\lambda)$, let R be any real number such that $\varphi(N) \subset B_R(p)$. Pick a point $m \in N - \varphi^{-1}(\{p\})$. Observe that, if $\varphi^{-1}(\{p\}) = \emptyset$, then $r \circ \varphi$ is smooth; if $\varphi^{-1}(\{p\}) \neq \emptyset$ let $\delta := \frac{1}{2} r(m)$ and redefine now r such that r is smooth in $\varphi(N)$ and coincide with the old $r_{|\varphi(N)}$ outside $\varphi(N) \cap B_{\frac{\delta}{2}}(p)$. Thus, we may apply Omori's lemma to $r \circ \varphi$, whence, for every $\epsilon' > 0$, there exists a point $m_{\epsilon'} \in N$ such that

$$0 < r \circ \varphi(m) \leq r \circ \varphi(m_{\epsilon'}) \leq R, \qquad \|\mathrm{grad}(r \circ \varphi)\|_{(m_{\epsilon'})} < \epsilon',$$

$$\nabla^2 (r \circ \varphi)(X, X) < \epsilon' \|X\|^2 \tag{3}$$

for every $X \in T_{m_{\epsilon'}} N$. Using Cauchy-Schwarz inequality and the fact that $\|\partial_r\| = 1$ outside $B_{\frac{\delta}{2}}(p)$ and $m_{\epsilon'} \notin B_{\frac{\delta}{2}}(p)$, we get

$$\begin{aligned} \nabla^2 (r \circ \varphi)(X, X) &= \overline{\nabla}^2 (r)(X, X) + < \partial_r, \alpha(X, X) > \\ &\geq \overline{\nabla}^2 (r)(X, X) - \|\alpha(X, X)\| \end{aligned} \tag{4}$$

for every $X \in T_{m_{\epsilon'}} N$.

From (4) and the third inequality of (3) we get that

$$\frac{\overline{\nabla}^2 (r)(X, X)}{\|X\|^2} - \epsilon' < \frac{\|\alpha(X, X)\|}{\|X\|^2} \tag{5}$$

for all $X \in T_{m_{\epsilon'}} N$, $X \neq 0$.

Using (2), the first inequality of (3), inequality (5) and the fact that $t \longmapsto \frac{C_\lambda(t)}{S_\lambda(t)}$ and $t \longmapsto \frac{C_{4\lambda}(t)}{S_{4\lambda}(t)}$ are decreasing , we obtain

$$\begin{aligned} \frac{\|\alpha(X, X)\|}{\|X\|^2} &\geq \left(\frac{C_\lambda(R)}{S_\lambda(R)}\right)\left(1 - \frac{< X, J\partial_r >^2}{\|X\|^2}\right) + \\ &\quad + \left(\frac{C_{4\lambda}(R)}{S_{4\lambda}(R)}\right)\frac{< X, J\partial_r >^2}{\|X\|^2} - \epsilon'. \end{aligned} \tag{6}$$

We claim that there are $X, Y \in \varphi_*(T_{m_{\epsilon'}} N)$ such that

$$\alpha(X, X) = \alpha(Y, Y) \neq 0 \text{ and } \alpha(X, Y) = 0, < X, J\partial_r >= 0. \tag{7}$$

To prove (7) we identify $W = \varphi_*(T_{m_{\epsilon'}} N)$ and $\varphi_*(T_{m_{\epsilon'}} N)^\perp$ with $\mathbb{R}^s$ and $\mathbb{R}^{2n-s}$ respectively. We may extend $\alpha_{|W \times W}$ to a symmetric bilinear map $\mathbb{C}^s \times \mathbb{C}^s \to \mathbb{C}^{2n-s}$. Since $s > 2n - s + 2$ (because $s > n + 1$ by hypothesis), the system of $2n - s + 2$ equations

$$\begin{aligned} \alpha(Z, Z) &= 0 \\ < \mathrm{Re}(Z), \mathrm{Im}(Z) > &= 0 \\ < \mathrm{Re}(Z), J\mathrm{Im}(Z) > &= 0 \end{aligned}$$

of s unknowns has a nontrivial solution $Z = X + iY$ which satisfies (7).

We have from Gauss equation that

$$\begin{aligned} K(X,Y)-\overline{K}(X,Y) &= \frac{<\alpha(X,X),\alpha(Y,Y)> - \|\alpha(Y,Y)\|^2}{\|X\|^2\|Y\|^2 - <X,Y>^2} \\ &\geq \frac{\|\alpha(X,X)\|}{\|X\|^2}\frac{\|\alpha(Y,Y)\|}{\|Y\|^2} \\ &\geq \left[\frac{C_\lambda(R)}{S_\lambda(R)} - \epsilon'\right]^2 \end{aligned}$$

Now, let $\epsilon > 0$, taking $\epsilon' > 0$ such that

$$\left[\frac{C_\lambda(R)}{S_\lambda(R)} - \epsilon'\right]^2 = \left(\frac{C_\lambda(R)}{S_\lambda(R)}\right)^2 - \epsilon,$$

we conclude that

$$K(X,Y)-\overline{K}(X,Y) \geq \left(\frac{C_\lambda(R)}{S_\lambda(R)}\right)^2 - \epsilon,$$

Since $K(X,Y)-\overline{K}(X,Y) \leq \beta - \alpha$ we have that

$$\left(\frac{C_\lambda(R)}{S_\lambda(R)}\right)^2 - \epsilon \leq \beta - \alpha.$$

Since the inequality holds for every $\epsilon > 0$ and the function $\left(\frac{C_\lambda}{S_\lambda}\right)^2$ is decreasing then

$$R \geq \left[\frac{C_\lambda}{S_\lambda}\right]^{-1}\left(\sqrt{\beta-\alpha}\right)$$

This finishes the proof of theorem 1.1.

2.2 Proof of theorems 1.2 and 1.3

It is similar to that of the theorem 1.1, but using the fact that $\frac{C_\lambda}{S_\lambda} \geq \frac{C_{4\lambda}}{S_{4\lambda}}$ if $\lambda > 0$ and $\frac{C_\lambda}{S_\lambda} \leq \frac{C_{4\lambda}}{S_{4\lambda}}$ if $\lambda < 0$.

3. Tightness of bounds

Here we shall prove that the bounds obtained in theorems 1.2 and 1.3 are the best possible when $s = dim(N)$ is odd. Let $\beta > 4\lambda$. If $s = 2s'-1$ we take $N = S^s_R$, the geodesic sphere of radius $R = \left[\frac{C_\lambda}{S_\lambda}\right]^{-1}(\sqrt{\beta - 4\lambda})$ on $\mathbb{C}K^{s'}(\lambda)$. Let $\varphi : N \to \mathbb{C}K^{s'}(\lambda)$ be the canonical immersion. It it obvious that $\rho(\varphi, S^s_R, \mathbb{C}K^n(\lambda)) = R$. Moreover $(\varphi, S^s_R, \mathbb{C}K^n(\lambda))$ satisfies the conditions of theorem 2.1. In fact, let $\sigma =< \{X,Y\} >$, $||X|| = ||Y|| = 1$, a plane in TN such that $\varphi_*(\sigma)$ is a holomorphic plane, then from Gauss equation,

$$\begin{aligned} K(\sigma) &= \overline{K}_H(\varphi_*(\sigma)) + \left[\frac{C_\lambda(R)}{S_\lambda(R)}\right]^2 (1- < X, J\partial_r >^2) \\ &\quad + \left[\frac{C_{4\lambda}(R)}{S_{4\lambda}(R)}\right]^2 < X, J\partial_r >^2 \\ &\leq 4\lambda + \left[\frac{C_\lambda(R)}{S_\lambda(R)}\right]^2 \\ &= 4\lambda + (\beta - 4\lambda) = \beta, \end{aligned}$$

where ∂_r is the gradient of the distance (in $\mathbb{C}K^{s'}(\lambda)$) to the center of the sphere S^s_R, considered as a vector in the tangent space to $\mathbb{C}K^n(\lambda)$. The second inequality follows from the fact that

$$\left[\frac{C_\lambda(R)}{S_\lambda(R)}\right]^2 \geq \left[\frac{C_{4\lambda}(R)}{S_{4\lambda}(R)}\right]^2 .$$

It is obvius that

$$\rho(\varphi, S^s_R, \mathbb{C}K^n(\lambda)) = R = \left[\frac{C_\lambda}{S_\lambda}\right]^{-1}\left(\sqrt{\beta - 4\lambda}\right)$$

thus, in this case, the bound obtained in the theorem 1.2 is a minimum.

For the theorem 1.3 a similar argument shows that there is a riemannian manifold N of odd dimension s and an isometric immersion $\varphi : N \rightarrow \mathbb{C}K^n(\lambda)$ satisfying the conditions of theorem 1.3 and such that the bound is just $\rho(\varphi, N, \mathbb{C}K^n(\lambda))$.

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