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2—CODIMENSIONAL LIGHTLIKE SUBMANIFOLDS OF ALMOST PARA—HERMITIAN MANIFOLDS

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0. Introduction. Let $(\bar{M}, \bar{g})$ be a semi–Riemannian (pseudo–Riemannian) manifold, [10] and let M be a submanifold of it. If the restriction $g = \bar{g}/M$ of $\bar{g}$ to M is still non–degenerate, then (M, g) becomes a semi–Riemannian manifold and it can be studied as the submanifolds of Riemannian manifolds. A different situation appears when g is degenerate and (M, g) is said to be a lightlike (degenerate) submanifold, [5], [6].

Among all semi–Riemannian manifolds, the class of the almost para–Hermitian manifolds is of special interest and it was considered by several authors, as in the list of references of [4].

Bejancu and Etayo initiated in [7] the study of degenerate hypersurfaces of almost para–Hermitian manifolds. Our purpose here is to step from the hypersurfaces to the submanifolds of arbitrary codimension. The present note deals with degenerate submanifolds of codimension two of almost para–Hermitian manifolds, since for higher codimension the results are similar but the calculus is much longer. We classify them, we prove their main properties and we also give some examples.

1. Preliminaries. Let $(\bar{M}, \bar{g})$ be a semi–Riemannian manifold endowed with an almost product structure, i.e., a tensor field $\bar{P}$ of type $(1,1)$ such that

$$\bar{P}^2 = Id; \ \bar{P} \neq \pm Id. \tag{1.1}$$

The pair $(\bar{P}, \bar{g})$ is said to be an almost para–Hermitian (or hyperbolic Hermitian) structure on $\bar{M}$ if

$$\bar{g}(\bar{P}X, \bar{P}Y) = -\bar{g}(X, Y), \ \forall X, Y \in \Gamma(T\bar{M}). \tag{1.2}$$

These manifolds appeared for the first time in [11].

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Remark 1.1 (i) $\bar{M}$ is of even dimension $2m,\ m = 1, 2, ...$
(ii) The eigendistributions $\bar{P}^+$ and $\bar{P}^-$ of $\bar{P}$ corresponding to the eigenvalues 1 and (-1) resp., are of rank m:

$$T\bar{M} = \bar{P}^+ \oplus \bar{P}^-. \tag{1.3}$$

(iii) $\bar{P}^+$ and $\bar{P}^-$ are isotropic, i.e. the restriction of $\bar{g}$ to either $\bar{P}^+$ or $\bar{P}^-$ is zero.

The notion of the almost para–Hermitian structure given on a differentiable manifold can be extended to an arbitrary vector bundle:

Definition 1. [2] Let $(E, \pi, \bar{M})$ be a vector bundle on the base manifold $\bar{M}$ endowed with a semi–Riemannian structure $\bar{g}$. If $\bar{P} : \Gamma(E) \longrightarrow \Gamma(E)$ satisfies (1.1) it is said to be a product structure of E and if moreover

$$\bar{g}(\bar{P}s, \bar{P}s') = -\bar{g}(s, s'),\ \forall s, s' \in \Gamma(E), \tag{1.4}$$

then $(\bar{P}, \bar{g})$ is called a para–Hermitian structure of E.

Notations. [9] Let M be a submanifold of codimension two of an almost para–Hermitian manifold $(\bar{M}, \bar{P}, \bar{g})$.

For any $q \in M$, the T_qM–perpendicular space [12] is defined by

$$T_qM^\perp = \{y \in T_q\bar{M} / \bar{g}(y, x) = 0,\ \forall x \in T_qM\} \tag{1.5}$$

and the radical of T_qM, cf. [1], is denoted by

$$\Delta_q = T_qM \cap T_qM^\perp. \text{ We suppose } \Delta_q \neq \{0\},\ \forall q \in M. \tag{1.6}$$

The submanifold M is called lightlike (degenerate) if the mapping $\Delta : q \in M \longrightarrow \Delta_q \subseteq T_qM$ is a non–trivial distribution. This notion is motivated by the restriction $g = \bar{g}/M$ of $\bar{g}$ to M which is degenerate, i.e. there is locally a non–zero vector field $\xi \in \Gamma(TM)$:

$$g(\xi, X) = 0,\ \forall X \in \Gamma(TM). \tag{1.7}$$

Remark 1.2. [9] $T_qM^\perp$ is a degenerate 2–dimensional subspace of $T_q\bar{M}$ and $(T_qM^\perp)^\perp = T_qM,\ \forall q \in M$.

Any orthogonal complement of Δ in TM is called a screen distribution SM and it is proved to be non–degenerate,

$$TM = SM \boxed{\perp} \Delta \text{ (orthogonal direct sum)}. \tag{1.8}$$

Moreover, there is a degenerate vector bundle NM associated to SM called the canonical affine normal bundle such that

$$T\bar{M} = SM \boxed{\perp} (TM^\perp \oplus NM) \tag{1.9}$$

where the first sum is orthogonal and the second one is only a direct Whitney sum.

The radical Δ being a non–trivial distribution with $\Delta_q \subseteq T_qM^\perp$, $q \in M$, then its dimension can be either one or two. The degenerate submanifold M is said to be half lightlike in the first case and totally lightlike in the second one. Remark that the ambient manifold $\bar{M}$ does not need to be almost para–Hermitian, but only semi–Riemannian, as in

Proposition 1.1 *Let M be a degenerate submanifold of codimension two of a semi–Riemannian manifold $(\bar{M}, \bar{g})$. Then the following conditions are equivalent:* (i) *M is totally lightlike;* (ii) *$\Delta = TM^\perp$;* (iii) *$TM^\perp \subseteq TM$;* (iv) *$\bar{g}$ restricted to $TM^\perp$ is zero.*

The proof follows from (1.5), (1.6) and Remark 1.2.

When the ambient manifold $\bar{M}$ is almost para–Hermitian, then we give an other classification of the degenerate submanifolds, according to the action of $\bar{P}$ to $TM^\perp$.

2. Classification of degenerate submanifolds. Everywhere in this section we take $(\bar{M}, \bar{P}, \bar{g})$ to be a $2m$–dimensional almost para–Hermitian manifold and let M be a degenerate submanifold of codimension two. Since all submanifolds of a 4–dimensional almost para–Hermitian manifold were classified by [10] then we consider $m > 2$.

Case 1. $\bar{P}$ is invariant on $TM^\perp$, i.e. $\bar{P}(T_qM^\perp) \subseteq T_qM^\perp$, $q \in M$.

From (1.1) we get

$$\bar{P}(T_qM^\perp) = T_qM^\perp, \; q \in M. \tag{2.1}$$

Theorem 2.1. *In the* Case 1 *we have:* (i) *$TM^\perp$ is locally spanned by two lightlike eigenvector fields of $\bar{P}$, tangent to M and orthogonal to each other;* (ii) *M is a totally lightlike submanifold;* (iii) *M is invariant with respect to $\bar{P}$, being an almost product manifold.*

Proof. First we prove that $TM^\perp$ is locally spanned by two lightlike vector fields of $\bar{P}$. We consider the vector bundle $TM^\perp$ on the base manifold M. From (2.1) the restriction of $\bar{P}$ to $TM^\perp$ is either $\pm id$ (and therefore we take any frame of $TM^\perp$) or a product structure of $TM^\perp$. In this last situation, as $TM^\perp$ is of rank two it follows that locally there are two eigenvector fields ξ_1, ξ_2 of $\bar{P}$ such that $\bar{P}\xi_i = (-1)^{i+1}\xi_i$, $i = 1, 2$.

From (1.4) we get $\bar{g}(\xi_i, \xi_i) = 0$, that is ξ_i, $i = 1, 2$, are lightlike in both situations. As the radical Δ is a non–trivial distribution and $\Delta \subseteq TM^\perp$, then there is locally a non–zero vector field $U = \alpha\xi_1 + \beta\xi_2 \in \Gamma(\Delta)$, with ξ_1, ξ_2 given above and $\alpha, \beta \in \mathcal{F}(M)$. Then $\bar{g}(U, Z) = 0$, $\forall Z \in \Gamma(TM^\perp)$. By taking consequently $Z = \xi_1$ and $Z = \xi_2$, we get $\bar{g}(\xi_1, \xi_2) = 0$. Since ξ_1, ξ_2 are lightlike from above, then $\bar{g}$ restricted to $TM^\perp$ is zero. Now (ii) follows from Proposition 1.1. As from (ii), we get that ξ_1, ξ_2 are tangent to M, then (i) is proved. (iii) From (1.2) we obtain

$$\bar{g}(\bar{P}X, \xi_i) = -\bar{g}(X, \bar{P}\xi_i) = \pm\bar{g}(X, \xi_i) = 0, \; i = 1, 2, \text{ for any } X \in \Gamma(TM),$$

where ξ_1, ξ_2 are given above. From Remark 1.2, $\bar{P}X \in \Gamma(TM)$, $\forall X \in \Gamma(TM)$, that is TM is invariant by $\bar{P}$. We denote by

$$P = \bar{P}/\Gamma(TM) : \Gamma(TM) \longrightarrow \Gamma(TM) \text{ the restriction of } \bar{P} \text{ to } M.$$

From Remark 1.1 (ii), TM can be contained neither in $\bar{P}^+$ nor in $\bar{P}^-$ since M is of $2(m-1)$–dimension, $m > 2$. Then $P \neq \pm Id$ and from (1.1) P is an almost product structure on M.

By Definition 1, we get

Theorem 2.2 *In the* Case 1, *there is in* TM *a screen distribution* SM *which inherits a para–Hermitian structure.*

Proof. From the proof of Theorem 2.1, P is an almost product structure on M whose eigendistributions P^+ and P^- split TM :

$$TM = P^+ \oplus P^-. \tag{2.2}$$

From the proof of Theorem 2.1, the restriction $\bar{P}/TM^\perp$ of $\bar{P}$ to $TM^\perp$ falls in one of the following three situations:

(a) $\bar{P}/TM^\perp = Id$. Then $\Delta \subset P^+$ and let D_0 be any supplementary distribution of Δ in P^+:

$$P^+ = D_0 \boxed{\perp} \Delta, \tag{2.3}$$

orthogonal direct sum, since Δ is orthogonal to any vector field tangent to M.
We take $SM = D_0 \oplus P^-$.

(b) $\bar{P}/TM^\perp = -Id$. (analogous).

(c) $\bar{P}/TM^\perp$ is a product structure, (1.1).

$P^+ + \Delta$ (resp. $P^- + \Delta$) is not a direct sum since from Theorem 2.1 (i), $\xi_1 \in \Gamma(P^+ \cap \Delta)$ and $\xi_2 \in \Gamma(P^- \cap \Delta)$. Let D_1 (resp. D_2) be any supplementary distribution of Δ in $P^+ + \Delta$ (resp. $P^- + \Delta$)

$$P^+ + \Delta = D_1 \boxed{\perp} \Delta \quad \text{and} \quad P^- + \Delta = D_2 \boxed{\perp} \Delta, \tag{2.4}$$

ortogonal direct sum since Δ is ortogonal to any vector field tangent to M. We take $SM = D_1 \oplus D_2$.

Therefore in all three situations (a), (b), (c), SM satisfies (1.8) and it is invariant by P. From Remark 1.1 (iii), SM being non–degenerate with respect to $\bar{g}$, it is contained neither in P^+ nor in P^-, that is $\bar{P}/SM \neq \pm Id$, i.e. $\bar{P}/SM$ is a product structure. Therefore, from Definition 1, SM inherits a para–Hermitian structure and we complete the proof.

Let $SM^\perp$ be the orthogonal supplement of SM in $T\bar{M}$ which is also non–degenerate with respect to $\bar{g}$ and it contains $TM^\perp$:

$$T\bar{M} = SM \boxed{\perp} SM^\perp. \tag{2.5}$$

From (1.4), (2.5) and Definition 1, we get

Corollary 2.3. *In the* Case 1*, let SM be the screen distribution given by Theorem* 2.2*. Then $SM^{\perp}$ inherits a para–Hermitian structure.*

Remark 2.1. (i) $rank\, SM = \dim \bar{M} - 4$; (ii) $index\, SM = index\, \bar{M} - 2$; (iii) $rank\, SM^{\perp} = 4$.

The following result will be used in §4 and it is an improvement of [9, Theorem 1].

Proposition 2.4. *In the* Case 1*, let SM be the screen distribution constructed in Theorem* 2.2*. Then the local frame $\{\xi_1, \xi_2\}$ of $TM^{\perp}$ given by Theorem* 2.1 (i)*, can be extended to a local frame $\{\xi_1, \xi_2, N_1, N_2\}$ of $SM^{\perp}$ such that*

(1) *N_1, N_2 are two orthogonal lightlike vector fields satisfying*

$$\bar{g}(\xi_i, N_j) = (-1)^{i+1}\delta_{ij}, \quad i, j = 1, 2 \text{ (no sum)}; \tag{2.6}$$

(2) *N_i is invariant (resp. anti–invariant) by $\bar{P}$ when ξ_i is anti–invariant (resp. invariant) by $\bar{P}$, $i = 1, 2$.*

Proof. From the proof of Theorem 2.1 (i) the restriction $\bar{P}/TM^{\perp}$ is either $\pm Id$ (when the existence of N_1, N_2 follows from [2, Proposition 1.4]) or a product structure (1.1). In this last situation we proceed as follows. As $SM^{\perp}$ is a para–Hermitian vector bundle of rank 4, let δ^{+} and δ^{-} be the eigenvector bundles of $\bar{P}/SM^{\perp}$ corresponding to the eigenvalues 1 and (-1), resp.

$$SM^{\perp} = \delta^{+} \oplus \delta^{-}. \tag{2.7}$$

From the proof of Theorem 2.1 we get $\xi_1 \in \Gamma(\delta^{+})$, $\xi_2 \in \Gamma(\delta^{-})$. Since δ^{-} is of rank two, let $V_1 \in \Gamma(\delta^{-})$ be a vector field lin.independent with ξ_2. Then $\bar{P}V_1 = -V_1$ and from (1.4) we get $\bar{g}(V_1, V_1) = \bar{g}(\xi_2, V_1) = 0$. Since $SM^{\perp}$ is non–degenerate with respect to $\bar{g}$ and ξ_1, ξ_2, V_1 are lin.independent, then $\bar{g}(\xi_1, V_1) \neq 0$. If we denote

$$N_1 = V_1/\bar{g}(\xi_1, V_1), \tag{2.8}$$

we get

$$\bar{P}N_1 = -N_1;\ \bar{g}(N_1, N_1) = \bar{g}(\xi_2, N_1) = 0 \ \text{ and } \ \bar{g}(\xi_1, N_1) = 1. \tag{2.9}$$

Since δ^{+} is of rank two there is $V_2 \in \Gamma(\delta^{+})$ lin.independent with ξ_1. From (1.4) we have

$$\bar{P}V_2 = V_2 \ \text{ and } \ \bar{g}(V_2, V_2) = \bar{g}(\xi_1, V_2) = 0. \tag{2.10}$$

If we denote $W_2 = -\bar{g}(V_2, N_1)\xi_1 + V_2$, then from (1.4) and (2.10) we get

$$\bar{P}W_2 = W_2 \ \text{ and } \ \bar{g}(W_2, W_2) = \bar{g}(W_2, N_1) = \bar{g}(\xi_1, W_2) = 0. \tag{2.11}$$

Since $SM^\perp$ is non–degenerate, with respect to $\bar{g}$ and $\{\xi_1, \xi_2, N_1, W_2\}$ is a local frame of $SM^\perp$ then from (2.11) we obtain $\bar{g}(\xi_2, W_2) \neq 0$. We take

$$N_2 = W_2/\bar{g}(\xi_2, W_2) \tag{2.12}$$

and we complete the proof.

Remark 2.2. There is a normal bundle NM satisfying (1.9) spanned locally by $\{N_1, N_2\}$.

In the Case 1, we assumed the invariance of $\bar{P}$ on $TM^\perp$ that is the intersection $\bar{P}(TM^\perp) \cap TM^\perp$ is a vector bundle of rank two on M. Now, we consider

Case 2. $E = \bigcup\limits_{q \in M} \bar{P}(T_q M^\perp) \cap T_q M^\perp$ is a vector bundle of rank one on M.

Proposition 2.5. *In the* Case 2*, the vector bundle E is a distribution on M contained in Δ and $E = \Delta$ if and only if M is a half lightlike submanifold of $\bar{M}$.*

Proof. As E is of rank one then either $E \subseteq \Delta$ or $E \cap \Delta = \{0\}$. In the last case $TM^\perp = E \oplus \Delta$, since $TM^\perp$ is of rank two. By (1.2), we have

$$\bar{g}(\sigma, \sigma) = 0, \quad \forall \sigma \in \Gamma(E), \tag{2.13}$$

since E is of rank one and invariant by $\bar{P}$. From (2.13) and the orthogonality of Δ to any section of $TM^\perp$ we get that $\bar{g}$ restricted to $TM^\perp$ is zero. From Proposition 1.1 it follows $\Delta = TM^\perp$ and therefore $E \subset \Delta$.

Case 3. $\bar{P}(T_q M^\perp) \cap T_q M^\perp = \{0\}$, $q \in M$.

Proposition 2.6. *Let M be a half lightlike submanifold of the almost para–Hermitian manifold $(\bar{M}, \bar{P}, \bar{g})$, falling in the* Case 3*. Then:* (i) *there is a screen distribution SM containing the tangent part of $\bar{P}(\Delta)$, i.e. the projection of $\bar{P}(\Delta)$ on TM;* (ii) *corresponding to SM, there is a unique lightlike normal bundle of TM in $T\bar{M}$ satisfying* (1.9) *which is orthogonal to $\bar{P}(\Delta)$.*

Proof. (i) As Δ is one dimensional it follows from [9] that $\bar{P}(\Delta)$ is contained in $TM + TM^\perp$ (no direct sum), but not in $TM^\perp$. Then we construct a screen distribution SM which contains the tangent part of $\bar{P}(\Delta)$. (ii) follows from (i) and [9, Section 2].

In the Case 1, by virtue of Theorem 2.1, the submanifold M is invariant with respect to $\bar{P}$, which is not true in all the other cases, as we show next section by examples. However, in the Case 2, E is invariant by $\bar{P}$ and therefore we give

Definition 2. Any degenerate submanifold of codimension two of an almost para–Hermitian manifold falling in the Case 1 (resp. Case 2, Case 3) is said to be invariant (resp. semi–invariant, non–invariant).

From the theory of degenerate hypersurfaces developed in [7], we get

Proposition 2.7. *Let $(\bar{M}, \bar{P}, \bar{g})$ be an almost para–Hermitian manifold and let $\mathcal{H}$ be an invariant* (*resp. non–invariant*) *hypersurface of it. If M is a degenerate* *submanifold of codimension two of* $(\bar{M}, \bar{P}, \bar{g})$ *embedded in* $\mathcal{H}$, *then* M *is either invariant or semi–invariant (resp. non–invariant).*

Corollary 2.8. *Let* (N, G) *be a Riemannian manifold and* TN *its tangent bundle endowed with the almost para–Hermitian structure introduced in [8]. The unit tangent bundle* $T'N$ *is considered as an invariant hypersurface of TN,* [10]. *Then no 2–codimensional non–invariant submanifold of* TN *lies in* $T'N$.

3. Examples. Let R^{2m}_m be the semi–Euclidean space of dimension $2m$ and of index m, endowed with the semi–Euclidean metric

$$\bar{g} = \sum_{i=1}^{m}[(dx^{m+1})^2 + (dx^i)^2] \text{ of signature } (m, m). \tag{3.1}$$

Let $\bar{P}$ be a product structure defined on R^{2m}_m by

$$\bar{P}\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^{m+i}} \text{ and } \bar{P}\left(\frac{\partial}{\partial x^{m+i}}\right) = \frac{\partial}{\partial x^i},\ i = \overline{1, m}. \tag{3.2}$$

Then $(R^{2m}_m, \bar{P}, \bar{g})$ is just a para–Hermitian space, as $\bar{P}$ is integrable. The next result given for codimension two only to fix the ideas, shows how [7, Proposition 1] can be generalized for any codimension.

Proposition 3.1. *Let* M *be a submanifoldof codimension two of* $(R^{2m}_m, \bar{P}, \bar{g})$ *given by* $x^{2m-1} = F(x^1, ..., x^{2m-2})$; $x^{2m} = G(x^1, ..., x^{2m-2})$, *where* F, G *are differentiable functions. Then:*

(i) *M is a degenerate submanifold if and only if*

$$AC = B^2, \quad \textit{with} \tag{3.3}$$

$$A = 1 - ((F, F));\ B = ((F, G));\ C = 1 - ((G, G)), \quad \textit{where} \tag{3.4}$$

$$((h, k)) = \sum_{i=1}^{m} \frac{\partial h}{\partial x^i}\,\frac{\partial k}{\partial x^i} - \sum_{a=1}^{2m-2} \frac{\partial h}{\partial x^a}\,\frac{\partial k}{\partial x^a} \quad \textit{for any functions } h, k; \tag{3.5}$$

(ii) *under the degeneration assumption,* M *is totally lightlike (resp. half lightlike) if and only if* A, B, C *are (resp. are not) all zero.*

Proof. (i) In the natural parametrization

$$\begin{gathered} x^1 = u^1; ...; x^{2m-2} = u^{2m-2}; \\ x^{2m-1} = F(u^1, ..., u^{2m-2});\ x^{2m} = G(u^1, ..., u^{2m-2}) \end{gathered} \tag{3.6}$$

the natural local frame on M is

$$\frac{\partial}{\partial u^\alpha} = \frac{\partial}{\partial x^\alpha} + \frac{\partial F}{\partial u^\alpha}\,\frac{\partial}{\partial x^{2m-1}} + \frac{\partial G}{\partial u^\alpha}\,\frac{\partial}{\partial x^{2m}},\ \alpha = \overline{1, 2m-2}. \tag{3.7}$$

From (1.7) and (3.1), M is degenerate if and only if there is a non–zero vector field $\xi = \sum_{\alpha=1}^{2m-2} \xi^{\alpha} \frac{\partial}{\partial u^{\alpha}}$ whose components satisfy the homogeneous system:

$$\begin{cases} \xi^i - \frac{\partial F}{\partial u^i}\xi(F) - \frac{\partial G}{\partial u^i}\xi(G) = 0, \ i = \overline{1, m} \\ \xi^a + \frac{\partial F}{\partial u^a}\xi(F) + \frac{\partial G}{\partial u^a}\xi(G) = 0, \ a = \overline{m+1, 2m-2} \end{cases} \tag{3.8}$$

ξ is a non–zero solution of (3.8) if and only if $\xi(F)$ and $\xi(G)$ are not simultaneously zero. Now, by multiplying the α equation of (3.8) with $\frac{\partial G}{\partial u^{\alpha}}$ (resp. $\frac{\partial F}{\partial u^{\alpha}}$), $\alpha = \overline{1, 2m-2}$, and then by adding altogether we get

$$C\xi(G) = B\xi(F) \quad (\text{resp. } A\xi(F) = C\xi(G)). \tag{3.9}$$

We denote by (S) the homogeneous linear system of both equations (3.9). Therefore $\xi(F)$ and $\xi(G)$ are not simultaneously zero if and only if the determinant of (S) is zero, i.e. (3.3) is satisfied.

(ii) Suppose M is degenerate. Then M is totally lightlike (resp. half lightlike) if and only if the solutions space of the system (3.8) is two (resp. one) dimensional, that is (S) is of rank zero (resp. one), i.e., A, B, C are (resp. are not) all zero.

Remark. Let M be a half lightlike submanifold of $(R^{2m}_m, \bar{P}, \bar{g})$. Since A, B, C are not all zero, we may suppose $C \neq 0$, otherwise is similar. From (3.7), (3.8), (3.3), (3.9) the vector field $\xi = \sum_{\alpha=1}^{2m} \frac{\partial}{\partial u^{\alpha}}$ from the proof of Propostion 3.1 is

$$\begin{aligned} \xi = \sum_{i=1}^{m} \left[C\frac{\partial F}{\partial x^i} + B\frac{\partial G}{\partial x^i}\right] \frac{\partial}{\partial x^i} - \sum_{a=m+1}^{2m-2} \left[C\frac{\partial F}{\partial x^a} + B\frac{\partial G}{\partial x^a}\right] \frac{\partial}{\partial x^a} \\ +C\frac{\partial}{\partial x^{2m-1}} + B\frac{\partial}{\partial x^{2m}}. \end{aligned} \tag{3.10}$$

The next three examples follows from Proposition 3.1 and (3.2).

Example 1. (Case 1) The submanifold M of $(R^{2m}_m, \bar{P}, \bar{g})$ given by $x^{2m-1} = x^{m-1}$; $x^{2m} = x^m$ is a degenerate submanifold falling in the invariant case whose $\Delta = TM^{\perp}$ is spanned by

$$\xi_1 = \frac{\partial}{\partial x^{m-1}} + \frac{\partial}{\partial x^{2m-1}}; \ \xi_2 = \frac{\partial}{\partial x^m} + \frac{\partial}{\partial x^{2m}}. \tag{3.11}$$

Example 2. (Case 2) The submanifold M of $(R^{2m}_m, \bar{P}, \bar{g})$ given by
(3.12)

$$x^{2m-1} = 2x^m + \sum_{k=1}^{m-2} (x^{m+k} - x^k)^2; \quad x^{2m} = 2x^{m-1} - 3x^m + \sum_{k+1}^{m-2} (x^{m+k} - x^k)^2.$$

Since from (3.4) $A = -3,\ B = -6;\ C = -12$, then M is half lightlike falling in the semi–invariant case whose radical distribution Δ is spanned by

$$\begin{aligned}\xi &= 36\sum_{k=1}^{m-2}(x^{m+k} - x^k)\left(\frac{\partial}{\partial x^k} + \frac{\partial}{\partial x^{m+k}}\right)\\ &-12\left(\frac{\partial}{\partial x^{m-1}} + \frac{\partial}{\partial x^{2m-1}}\right) - 6\left(\frac{\partial}{\partial x^m} + \frac{\partial}{\partial x^{2m}}\right)\end{aligned} \tag{3.13}$$

which verifies (3.10).

Example 3. (Case 3, totally lightlike) The submanifold M of $(R^{2m}_m, \bar{P}, \bar{g})$ given by $x^{2m-1} = x^1;\ x^{2m} = x^2$ is totally lightlike, falling in the non–invariant case whose radical distribution $\Delta = TM^{\perp}$ is spanned by

$$\sigma = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^{2m-1}};\ \zeta = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^{2m}}. \tag{3.14}$$

Besides all examples given above which are applications of Proposition 3.1, the next example is different, being a submanifold of an other para–Hermitian manifold.

Example 4. (Case 3, half lightlike) Let $T^3 = S^1 \times S^1 \times S^1$ be the 3–dimensional torus with the Riemannian metric G which is the product of the standard metrics on the factors S^1 and let $\{X_i\}_{i=\overline{1,3}}$ be an orthonormal basis giving a parallelization of T^3. $\bar{M} = T^3 \times T^3$ is endowed with the almost para–Hermitian structure

$$\bar{g} = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix};\ \bar{P} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}; \tag{3.15}$$

with respect to the frame $\{(X_i, 0),\ i = \overline{1,3};\ (0, X_b),\ b = \overline{1,3}\}$ of $\bar{M}$.

Then $\gamma : x \in S^1 \longrightarrow ((0,0,x),(x,0,0)) \in T^3 \times T^3 = \bar{M}$ is a lightlike curve on $\bar{M}$. By identifying γ with its image in $\bar{M}$, we take

$$M = T^2 \times \gamma \times S^1;\ \text{i.e. } M = T^2 \times \gamma \times (S^1 \times \{0\}) \subset T^2 \times (S^1 \times S^1) \times T^2 = \bar{M}. \tag{3.16}$$

Then $\{(X_1,0),(X_2,0),(X_3,X_1),(0,X_2)\}$ is a basis on TM and Δ is spanned by $\{(X_3, X_1)\}$. Therefore M is a half lightlike submanifold of $\bar{M}$ which provides an example of a degenerate non–invariant submanifold.

4. Invariant submanifolds of para–Kähler manifolds. Let $(\bar{M}, \bar{P}, \bar{g})$ be a $2m$–dimensional para–Kähler manifold $(m > 2)$, i.e. an almost para–Hermitian manifold such that $\bar{P}$ is parallel with respect to the Levi–Civita connection $\overline{\nabla}$ of $\bar{g}$:

$$\overline{\nabla}\bar{P} = 0 \tag{4.1}$$

Let M be a degenerate submanifold of codimension two falling in the invariant case. Now, we take the screen distribution SM given by Theorem 2.2 and the local frame $\{\xi_1, \xi_2, N_1, N_2\}$ of Proposition 2.4. The Gauss–Weingarten formulae for M are

$$\overline{\nabla}_X Y = \overline{\nabla}_X Y + B_1(X,Y)N_1 + B_2(X,Y)N_2 \quad \text{and} \tag{4.2}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \ \forall X, Y \in \Gamma(TM), \ N \in \Gamma(NM), \tag{4.3}$$

where $\overline{\nabla}_X Y$ and $A_N X$ are tangent to M and $\nabla^{\perp}$ denotes the linear normal connection of $\Gamma(NM)$. Similarly, the Gauss–Weingarten formulae for SM are

$$\nabla_X hY = \overset{*}{\nabla}_X hY + C_1(X,hY)\xi_1 + C_2(X,hY)\xi_2 \quad \text{and} \tag{4.4}$$

$$\nabla_\xi Y = -A_\xi X + \overset{*}{\nabla}{}^{\perp}_X \xi, \ \forall X, Y \in \Gamma(TM), \ \xi \in \Gamma(TM^{\perp}), \tag{4.5}$$

where h denotes the projection of TM on SM; $\overset{*}{\nabla}{}^{\perp}$ denotes the metric linear connection of $TM^{\perp}$ and $\overset{*}{\nabla}_X hY$, $A_\xi X \in \Gamma(SM)$, [9].

Now, we apply (4.1) to (4.2) and (4.4) and we take the components from $\Gamma(TM^{\perp})$ and $\Gamma(NM)$, respectively, to obtain:

$$C_i(X, PhY) = \pm C_i(X, hY); \ B_i(X, PhY) = \pm B_i(hX, hY); \tag{4.6}$$

$$B_i(PhX, PhY) = B_i(hX, hY), \ i = 1, 2, \ \forall X, Y \in \Gamma(TM). \tag{4.7}$$

Proposition 4.1. *The screen distribution SM is integrable if and only if*

$$C_i(PhX, PhY) = C_i(hX, hY), \ i = 1, 2, \ \forall X, Y \in \Gamma(TM). \tag{4.8}$$

Proof. SM is integrable if and only if the forms C_i, $i = 1, 2$, are symmetric, [9], which is equivalent to (4.8).

M is said to be totally umbilical if

$$B_i(X,Y) = f_i g(X,Y), \ i = 1, 2, \ \forall X, Y \in \Gamma(TM), \tag{4.9}$$

where $g = \bar{g}/M$ and f_1, f_2 are differentiable functions.

M is said to be totally geodesic when $f_1 = f_2 = 0$.

Since SM is a non–degenerate distribution, from Remark 2.1, there is locally an orthonormal basis $\{E_j\}$ of SM where $E_1, ..., E_{m-2}$ are timelike and $E_{m-1}, ..., E_{2m-2}$ are spacelike. The mean vector field

$$H = (1/2(m-2)) \sum_{i=1}^{2} \sum_{k=1}^{m-2} \{B_i(E_{m-2+k'} E_{m-2+k}) - B_i(E_k, E_k)\} N_i \tag{4.10}$$

does not depend on the basis.

M is said to be minimal if $H = 0$.

Theorem 4.2. *Any invariant submanifold M of a para–Kähler manifold is minimal.*

Proof. From Theorem 2.2 and [2, Proposition 1.4] there is a local frame $\{E_j\}$ of SM such that $E_{m-2+k} = PE_k$, $k = \overline{1, m-2}$. From (4.7) we get $H = 0$.

Theorem 4.3. *Any totally umbilical invariant submanifold of a para–Kähler manifold is totally geodesic.*

Proof. From (4.9) it follows $B_i(X, \xi) = 0$, $i = 1, 2$, $\forall X \in \Gamma(TM)$, $\xi \in \Gamma(TM^{\perp})$. By (4.7), (4.9) and (1.2), we get $f_i g(hX, hY) = 0$, $i = 1, 2$, $\forall X, Y \in \Gamma(TM)$. Since SM is non–degenerate then f_1, f_2 vanish identically.

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