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Odd variational sequences on finite order jet spaces*

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Abstract: For a fibered manifold $\pi : Y \to X$ with possibly non-orientable base we introduce an "odd variational sequence" of sheaves over the (finite) r-jet prolongation J^rY. We give a description of cohomology spaces of this odd variational sequence by means of cohomology spaces of the odd de Rham sequence over Y. With help of double cover of X we describe the relation between the spaces of odd base differential forms and ordinary differential forms over Y, which allows us to express the cohomology of the odd variational sequence in terms of ordinary de Rham cohomology.

Key words: Jet prolongation of a fibered manifold, differential form, odd base differential form, double cover of a manifold, sheaf.

MS classification: 58E99, 57N65, 57R22.

1. Introduction

The fundamental motivation of this article is the investigation of the global inverse variational problem on a fibered manifold $\pi : Y \to X$ whose base manifold X is non-orientable. According to [6], in this case we can define the Lagrangian structure as a couple (Y, λ), where the Lagrangian λ is an odd base differential form on some jet prolongation $\pi_r : J^rY \to X$ of the fibered manifold $\pi : Y \to X$. (This allows us to define the action $\gamma \to \int_X (J^r\gamma)^*\lambda$ on sections γ of the fibered manifold $\pi : Y \to X$ even if the manifold X is non-orientable.) As in the classical case we can define the complex

$$\{\text{Lagrangians}\} \xrightarrow{\text{Euler-Lagrange}} \{\text{differential equations}\} \xrightarrow{\text{Helmholtz}} \tag{1}$$

and study the structure of the factor space Ker Helmholtz/ Im (Euler-Lagrange), which contains obstructions for a locally variational equation to be globally variational. (For more details see [1],[6].)

The standard way to study this factor space for ordinary forms is based on study of variational bicomplexes over infinite jet spaces (see [2]). Another way using only finite-order jet spaces is described in [7] where with the help of sheaves ${}^Y\Omega^k_r$ of k-forms on J^rY (strictly speaking, with the help of direct images of these sheaves by the projection $\pi_{r,0} : J^rY \to Y$) and their subsheaves ${}^Y\Omega^k_{r(c)}$ of contact (resp. strongly contact) forms, the sheaves ${}^Y\Theta^k_r = d\,{}^Y\Omega^{k-1}_{r(c)} + {}^Y\Omega^k_{r(c)}$ are defined and the sequence

$$0 \to {}^Y\mathbf{R} \to {}^Y\Omega^0_r \overset{E_0}{\to} {}^Y\Omega^1_r/{}^Y\Theta^1_r \overset{E_1}{\to} {}^Y\Omega^2_r/{}^Y\Theta^2_r \overset{E_2}{\to} \dots \tag{2}$$

*This paper is in final form and no version of it will be submitted for publication elsewhere.

is constructed. This sequence is called the variational sequence of order r over Y. In [7] it was shown that the mapping E_n from the sequence (2) is just the Euler-Lagrange mapping and that the mapping E_{n+1} (called Helmholtz-Sonin mapping) corresponds to the standard conditions of local variationality. Moreover, it was shown there that the variational sequence (2) is an acyclic resolution of the constant sheaf ${}^Y\mathbf{R}$ over Y, which makes possible to apply the de Rham theorem to the investigation of its cohomology spaces.

From the properties of odd base forms it follows that the construction of variational sequence can be done in the same way as in [7] if we replace the sheaves ${}^Y\Omega^k_r, {}^Y\Omega^k_{r(c)}, {}^Y\Theta^k_r = d^Y\Omega^{k-1}_{r(c)} + {}^Y\Omega^k_{r(c)}$ of ordinary forms by the corresponding sheaves ${}^Y_X\hat{\Omega}^k_r, {}^Y_X\hat{\Omega}^k_{r(c)}, {}^Y_X\hat{\Theta}^k_r = d^Y_X\hat{\Omega}^{k-1}_{r(c)} + {}^Y_X\hat{\Omega}^k_{r(c)}$ of odd base forms and consider the sheaf ${}^Y_X\hat{\mathbf{R}}$ of locally constant fields of odd scalars instead of the sheaf ${}^Y\mathbf{R}$ of locally constant real functions. In this way we get a resolution

$$0 \to {}^Y_X\hat{\mathbf{R}} \to {}^Y_X\hat{\Omega}^0_r \xrightarrow{\hat{E}_0} {}^Y_X\hat{\Omega}^1_r/{}^Y_X\hat{\Theta}^1_r \xrightarrow{\hat{E}_1} {}^Y_X\hat{\Omega}^2_r/{}^Y_X\hat{\Theta}^2_r \xrightarrow{\hat{E}_2} \dots \tag{3}$$

of the sheaf ${}^Y_X\hat{\mathbf{R}}$ over Y, which can be called the odd variational sequence of order r over Y.

Furthermore, from the properties of the variational sequence (2) derived in [7] and from the relation between the variational sequence (2) and the odd variational sequence (3) it follows that, like the case of ordinary variational sequence, the part ${}^Y_X\hat{\Omega}^n_r/{}^Y_X\hat{\Theta}^n_r \xrightarrow{\hat{E}_n} {}^Y_X\hat{\Omega}^{n+1}_r/{}^Y_X\hat{\Theta}^{n+1}_r \xrightarrow{\hat{E}_{n+1}}$ of the odd variational sequence (3) corresponds to the sequence (1), which means that the $(n+1)$th cohomology space of the odd variational sequence contains obstructions for a locally variational equation to be globally variational.

The aim of this article is the description of the cohomology spaces $\mathbf{H}(\Gamma(Y, {}^Y_X\widehat{\nu}^r))$ of the odd variational sequence ${}^Y_X\widehat{\nu}^r$ of the fibered manifold $\pi : Y \to X$. The main implement used to this description is the double cover $\sigma : OrX \to X$ of the base manifold X. (The construction of the double cover can be found for example in [5].)

It is shown in this article that the ordinary variational sequence ${}^{\sigma^*Y}\nu^r$ over the pull-back $\sigma^*\pi : \sigma^*Y \to OrX$ of the fibered manifold $\pi : Y \to X$ along the projection $\sigma : OrX \to X$ can be decomposed in a natural way into the direct sum ${}^{\sigma^*Y}\nu^r = {}^Y\nu^r \oplus {}^Y_X\widehat{\nu}^r$, where ${}^Y\nu^r$ (resp. ${}^Y_X\widehat{\nu}^r$) is the ordinary (resp. odd) variational sequence over the fibered manifold $\pi : Y \to X$. This decomposition allows us to express the cohomology spaces $\mathbf{H}(\Gamma(Y, {}^Y_X\widehat{\nu}^r))$ of the odd variational sequence ${}^Y_X\widehat{\nu}^r$ over Y with the help of cohomology spaces of ordinary variational sequences over the fibered manifolds $\sigma^*\pi : \sigma^*Y \to OrX, \pi : Y \to X$; we get $\mathbf{H}(\Gamma(Y, {}^Y_X\widehat{\nu}^r)) \simeq \mathbf{H}(\Gamma(\sigma^*Y, {}^{\sigma^*Y}\nu^r))/\mathbf{H}(\Gamma(Y, {}^Y\nu^r))$. To this relation we can apply the results derived in [7], this leads to the expression $\mathbf{H}(\Gamma(Y, {}^Y_X\widehat{\nu}^r)) \simeq {}_X\widehat{\mathbf{H}}(Y)$, where ${}_X\widehat{\mathbf{H}}(Y)$ are cohomology spaces of the odd de Rham complex $0 \to {}^Y_X\widehat{\mathbf{R}} \xrightarrow{d} {}^Y_X\hat{\Omega}^0 \xrightarrow{d} {}^Y_X\hat{\Omega}^1 \xrightarrow{d} \dots$ of the fibered manifold $\pi : Y \to X$.

Moreover, a description of the odd cohomology spaces ${}_X\widehat{\mathbf{H}}(Y)$ is given in this article, which allows us to find the cohomology spaces of the odd variational sequence over a fibered manifold $\pi : Y \to X$ in terms of the usual de Rham cohomology spaces of the manifolds Y, σ^*Y: $\mathbf{H}(\Gamma(Y, {}^Y_X\widehat{\nu}^r)) \simeq {}_X\widehat{\mathbf{H}}(Y) \simeq \mathbf{H}(\sigma^*Y)/\mathbf{H}(Y)$.

Particularly, in Chapter 3 we will describe the relation between the cohomology spaces of ordinary and odd de Rham sequences over the r-jet prolongation $\pi_r : J^rY \to$

X of the fibered manifold $\pi : Y \to X$ and over associated fibered manifolds $\sigma^*\pi_r$: $\sigma^*(J^rY) \to OrX$, $(\sigma^*\pi)_r : J^r(\sigma^*Y) \to OrX$.

In Chapter 4 we will define so-called Θ-spaces, which will be used in Chapter 5 to the construction of the odd variational sequence. The main result of the Chapter 5 is the above description of cohomology spaces of the odd variational sequence.

2. Notation

The notation used in this article follows the notation from [6] and [7]. Throughout this paper we denote by Y a fibered manifold with base X and projection $\pi : Y \to X$, and $n = \dim X, m = \dim Y - n$. J^rY, where $r \geq 0$, denotes the r-jet prolongation of Y, with the canonical jet projection $\pi_r : J^rY \to X, \pi_{r,s} : J^rY \to J^sY, 0 \leq s \leq r$. If $W \subseteq Y$ is a set, we write $W_r = \pi_{r,0}^{-1}(W)$.

By $\Omega(Y) = \Sigma_k \Omega^k(Y)$ (resp. ${}_X\widehat{\Omega}(Y) = \Sigma_k {}_X\widehat{\Omega}^k(Y)$) we denote the graded real vector space of smooth differential forms (resp. smooth odd base differential forms) on Y (resp. on fibered manifold $\pi : Y \to X$). The graded vector cohomology spaces of differential spaces $(\Omega(Y), d), ({}_X\widehat{\Omega}(Y), d)$ are denoted by $\mathbf{H}(Y) = \Sigma_i \mathbf{H}^i(Y)$, ${}_X\widehat{\mathbf{H}}(Y) = \Sigma_i \widehat{\mathbf{H}}^i(Y)$. We will write $\Omega_r(Y), {}_X\widehat{\Omega}_r(Y)$ rather than $\Omega(J^rY), {}_X\widehat{\Omega}(J^rY)$. By $\Omega^k_{r(c)}(Y)$ (resp. ${}_X\widehat{\Omega}^k_{r(c)}(Y)$), where $1 \leq k \leq n$, we denote the real vector spaces of contact k-forms on J^rY (resp. contact odd base k-forms on $\pi_r : J^rY \to X$). By $\Omega^k_{r(c)}(Y)$ (resp. ${}_X\widehat{\Omega}^k_{r(c)}(Y)$), where $n+1 \leq k$, we denote the real vector spaces of strongly contact k-forms on J^rY (resp. strongly contact odd base k-forms on $\pi_r : J^rY \to X$) (see Chapter 4).

The definition and the basic properties of the double cover of a manifold X, which will be denoted by $\sigma : OrX \to X$, can be found for example in [5]. The pull-back of the fibered manifold $\pi : Y \to X$ along the projection $\sigma : OrX \to X$ will be denoted by $\sigma^*\pi : \sigma^*Y \to OrX$, where $\sigma^*Y = \{(y, \xi) \in Y \times OrX | \pi(y) = \sigma(\xi)\}$. The pull-back of $\sigma : OrX \to X$ along the projection $\pi : Y \to X$ will be denoted by $\pi^*\sigma : \sigma^*Y \to Y$.

Finally, if $0 \to \mathbf{R} \to \mathcal{A}$ is a resolution of a sheaf $\mathbf{R}$ over Y defined by $0 \to \mathbf{R} \to A_1 \to A_2 \to \ldots$ then by $\Gamma(Y, \mathcal{A})$ we denote the cochain complex of global sections $0 \to \Gamma(Y, \mathbf{R}) \to \Gamma(Y, A_1) \to \Gamma(Y, A_2) \to \ldots$, and by $\mathbf{H}^k(\Gamma(Y, \mathcal{A}))$ we denote its cohomology spaces.

3. Structure of spaces of odd base forms

3.1. Definition. Let $\pi : Y \to X$ be a fibered manifold, let δ be a continuous field of unit odd scalars on X. By ${}_\delta Z(Y) : \Omega(Y) \to {}_X\widehat{\Omega}(Y)$ we denote the mapping defined by $\Omega(Y) \ni \omega \mapsto (\pi^*\delta) \otimes \omega \in {}_X\widehat{\Omega}(Y)$.

(It is known that the mapping ${}_\delta Z(Y)$ is an isomorphism of graded differential spaces whenever it exists—see [6], [8].)

3.2. Definition. An *involution of the fibered manifold* $\pi : Y \to X$ is an isomorphism $(F, f) : \pi \to \pi$ such that $F \circ F = \mathrm{id}_Y, f \circ f = \mathrm{id}_X$.

It is easy to see that the involution (F, f) of the fibered manifold $\pi : Y \to X$ induces the decomposition $\Omega(Y) = \Omega_+(Y) \oplus \Omega_-(Y)$ (resp. ${}_X\widehat{\Omega}(Y) = {}_X\widehat{\Omega}_+(Y) \oplus {}_X\widehat{\Omega}_-(Y)$), where the graded spaces $\Omega_+(Y) = \{\omega \in \Omega(Y) | F^*\omega = \omega\}, \Omega_-(Y) = \{\omega \in \Omega(Y) | F^*\omega = -\omega\}$ (resp. ${}_X\widehat{\Omega}_+(Y) = \{\omega \in {}_X\widehat{\Omega}(Y) | F^*\omega = \omega\}, {}_X\widehat{\Omega}_-(Y) = \{\omega \in {}_X\widehat{\Omega}(Y) | F^*\omega = -\omega\}$) with the exterior derivative may be regarded as graded differential spaces. The relation of the mapping ${}_\delta Z(Y)$ to these decompositions shows the following lemma:

3.3. Lemma. *Let $\pi : Y \to X$ be a fibered manifold, let $(F, f) : \pi \to \pi$ be its involution, let δ be a continuous field of unit odd scalars on X.*

(a) If $f^*\delta = \delta$, then the mappings ${}_\delta Z(Y)|_{\Omega_+(Y)} : \Omega_+(Y) \to {}_X\widehat{\Omega}_+(Y)$, ${}_\delta Z(Y)|_{\Omega_-(Y)} : \Omega_-(Y) \to {}_X\widehat{\Omega}_-(Y)$ are isomorphisms of graded differential spaces.

(b) If $f^*\delta = -\delta$, then the mappings ${}_\delta Z(Y)|_{\Omega_+(Y)} : \Omega_+(Y) \to {}_X\widehat{\Omega}_-(Y)$, ${}_\delta Z(Y)|_{\Omega_-(Y)} : \Omega_-(Y) \to {}_X\widehat{\Omega}_+(Y)$ are isomorphisms of graded differential spaces.

Proof. ad(a) Let $f^*\delta = \delta$. Considering that the mapping ${}_\delta Z(Y) : \Omega(Y) = \Omega_+(Y) \oplus \Omega_-(Y) \to {}_X\widehat{\Omega}(Y) = {}_X\widehat{\Omega}_+(Y) \oplus {}_X\widehat{\Omega}_-(Y)$ is an isomorphism of graded differential spaces it is sufficient to show that ${}_\delta Z(Y)(\Omega_+(Y)) \subseteq {}_X\widehat{\Omega}_+(Y)$, ${}_\delta Z(Y)(\Omega_-(Y)) \subseteq {}_X\widehat{\Omega}_-(Y)$. Let $\omega \in \Omega_+(Y)$. Then $F^*({}_\delta Z(Y)(\omega)) = F^*(\pi^*\delta \otimes \omega) = F^*\pi^*\delta \otimes F^*\omega = \pi^* f^*\delta \otimes \omega = \pi^*\delta \otimes \omega = {}_\delta Z(Y)(\omega)$, so that ${}_\delta Z(Y)(\omega) \in {}_X\widehat{\Omega}_+(Y)$. The inclusion ${}_\delta Z(Y)(\Omega_-(Y)) \subseteq {}_X\widehat{\Omega}_-(Y)$ can be proved in the same way.

ad(b) The assertion (b) can be proved in a similar way as the assertion (a).

Now, let us consider a fibered manifold $\pi : Y \to X$. The double cover $\sigma : OrX \to X$ allows us to construct the commutative diagram

$$\begin{tikzcd} Y & \sigma^*Y \arrow[l, "\pi^*\sigma"'] \arrow[d, "\sigma^*\pi"] \\ X \arrow[from=u, "\pi"] & OrX \arrow[l, "\sigma"'] \end{tikzcd} \tag{D1}$$

where $\sigma^*\pi : \sigma^*Y \to OrX$ is the pull-back of the fibered manifold $\pi : Y \to X$ along the projection $\sigma : OrX \to X$.

The diagram (D1) induces the commutative diagram

$$\begin{tikzcd} \sigma^*J^rY \arrow[rr, leftrightarrow, "\simeq"] \arrow[dr, "\pi_r^*\sigma"'] \arrow[dddd, "\sigma^*\pi_r"] & & J^r\sigma^*Y \arrow[dl, "J^r(\pi^*\sigma)"] \arrow[dd, "(\sigma^*\pi)_{r,0}"] \\ & J^rY \arrow[dd, "\pi_{r,0}"] & \\ & & \sigma^*Y \arrow[dl, "\pi^*\sigma"'] \arrow[dd, "\sigma^*\pi"] \\ & Y \arrow[dd, "\pi"] & \\ OrX \arrow[rr, leftrightarrow, "\simeq"] \arrow[dr, "\sigma"'] & & OrX \arrow[dl, "\sigma"] \\ & X & \end{tikzcd} \tag{D2}$$

where $\sigma^*\pi_r : \sigma^*J^rY \to OrX$ is the pull-back of the fibered manifold $\pi_r : J^rY \to X$ along the projection $\sigma : OrX \to X$, and this diagram generates the two following commutative diagrams of morphisms of differential spaces

$$\begin{tikzcd}
 & {}_X\widehat{\Omega}_r(Y) \arrow[dl, "(\pi_r^*\sigma)^*"'] \arrow[dr, "(J^r(\pi^*\sigma))^*"] & \\
{}_{OrX}\widehat{\Omega}(\sigma^*J^rY) \arrow[rr, leftrightarrow, "\simeq"] & & {}_{OrX}\widehat{\Omega}_r(\sigma^*Y)
\end{tikzcd} \tag{D3}$$

$$\begin{tikzcd}
\Omega(\sigma^*J^rY) \arrow[rr, leftrightarrow, "\simeq"] & & \Omega_r(\sigma^*Y) \\
 & \Omega_r(Y) \arrow[ul, "(\pi_r^*\sigma)^*"] \arrow[ur, "(J^r(\pi^*\sigma))^*"'] &
\end{tikzcd} \tag{D4}$$

where $\Omega_r(Y) = \Omega(J^rY)$, ${}_X\widehat{\Omega}_r(Y) = {}_X\widehat{\Omega}(J^rY)$, $\Omega_r(\sigma^*Y) = \Omega(J^r\sigma^*Y)$, ${}_{OrX}\widehat{\Omega}_r(\sigma^*Y) = {}_{OrX}\widehat{\Omega}(J^r\sigma^*Y)$.

3.4. Definition. Let $\pi : Y \to X$ be a fibered manifold. The *canonical involution* of the fibered manifold $\sigma^*\pi : \sigma^*Y \to OrX$ is the isomorphism $(I, \iota) : \sigma^*\pi \to \sigma^*\pi$ defined by $\sigma^*Y \ni (y, \xi) \overset{I}{\mapsto} (y, \iota(\xi)) \in \sigma^*Y$, where $\iota : OrX \to OrX$ is the canonical involution of the double cover $\sigma : OrX \to X$ of X.

3.5. Lemma. *Let $\pi : Y \to X$ be a fibered manifold, let $(I, \iota) : \sigma^*\pi \to \sigma^*\pi$ (resp. $(I_r, \iota) : \sigma^*\pi_r \to \sigma^*\pi_r$) be the canonical involution of the fibered manifold $\sigma^*\pi : \sigma^*Y \to OrX$ (resp. $\sigma^*\pi_r : \sigma^*J^rY \to OrX$). Then the diagrams (D3), (D4) can be written in the form*

$$\begin{tikzcd}
 & {}_X\widehat{\Omega}_r(Y) \arrow[dl] \arrow[dr] & \\
{}_{OrX}\widehat{\Omega}_+(\sigma^*J^rY) \arrow[rr, leftrightarrow, "\simeq"] & & {}_{OrX}\widehat{\Omega}_{r+}(\sigma^*Y) \\
\oplus & & \oplus \\
{}_{OrX}\widehat{\Omega}_-(\sigma^*J^rY) \arrow[rr, leftrightarrow, "\simeq"] & & {}_{OrX}\widehat{\Omega}_{r-}(\sigma^*Y)
\end{tikzcd} \tag{D5}$$

$$\begin{tikzcd}
\Omega_-(\sigma^*J^rY) \arrow[rr, leftrightarrow, "\simeq"] & & \Omega_{r-}(\sigma^*Y) \\
\oplus & & \oplus \\
\Omega_+(\sigma^*J^rY) \arrow[rr, leftrightarrow, "\simeq"] & & \Omega_{r+}(\sigma^*Y) \\
 & \Omega_r(Y) \arrow[ul] \arrow[ur] &
\end{tikzcd} \tag{D6}$$

where ${}_{OrX}\widehat{\Omega}(\sigma^*J^rY) = {}_{OrX}\widehat{\Omega}_-(\sigma^*J^rY) \oplus {}_{OrX}\widehat{\Omega}_+(\sigma^*J^rY)$, $\Omega(\sigma^*J^rY) = \Omega_+(\sigma^*J^rY) \oplus \Omega_-(\sigma^*J^rY)$ are the decompositions generated by the canonical involution $(I_r, \iota) : \sigma^*\pi_r \to \sigma^*\pi_r$, and ${}_{OrX}\widehat{\Omega}_r(\sigma^*Y) = {}_{OrX}\widehat{\Omega}_{r-}(\sigma^*Y) \oplus {}_{OrX}\widehat{\Omega}_{r+}(\sigma^*Y)$, $\Omega_r(\sigma^*Y) = \Omega_{r+}(\sigma^*Y) \oplus \Omega_{r-}(\sigma^*Y)$ are the decompositions generated by the involution $(J^rI, \iota) : (\sigma^*\pi)_r \to (\sigma^*\pi)_r$. Moreover, all the mappings in the diagrams (D5), (D6) are isomorphisms of graded differential spaces.

Proof. First, let us remark that the mappings $(\pi_r^*\sigma)^*, (J^r(\pi^*\sigma))^*$ commute with the exterior derivative so that it is enough to prove that these mappings are bijections.

The projections $\pi_r^*\sigma : \sigma^*J^rY \to J^rY$, $J^r(\pi^*\sigma) : J^r\sigma^*Y \to J^rY$ are local diffeomorphisms so that the mappings $(\pi_r^*\sigma)^* : {}_X\widehat{\Omega}_r(Y) \to {}_{OrX}\widehat{\Omega}(\sigma^*J^rY)$, $(\pi_r^*\sigma)^* : \Omega_r(Y) \to \Omega(\sigma^*J^rY)$, $(J^r(\pi^*\sigma))^* : {}_X\widehat{\Omega}_r(Y) \to {}_{OrX}\widehat{\Omega}_r(\sigma^*Y)$, $(J^r(\pi^*\sigma))^* : \Omega_r(Y) \to \Omega_r(\sigma^*Y)$ are injections. Let us show the surjectivity of these mappings.

The equalities $I_r^*(\pi_r^*\sigma)^* = ((\pi_r^*\sigma) \circ I_r)^* = (\pi_r^*\sigma)^*$, $(J^rI)^*(J^r(\pi^*\sigma))^* = (J^r(\pi^*\sigma) \circ J^rI)^* = (J^r((\pi^*\sigma) \circ I))^* = (J^r(\pi^*\sigma))^*$ imply $(\pi_r^*\sigma)^*({}_X\widehat{\Omega}_r(Y)) \subseteq {}_{OrX}\widehat{\Omega}_+(\sigma^*J^rY)$, $(\pi_r^*\sigma)^*(\Omega_r(Y)) \subseteq \Omega_+(\sigma^*J^rY)$, $(J^r(\pi^*\sigma))^*({}_X\widehat{\Omega}_r(Y)) \subseteq {}_{OrX}\widehat{\Omega}_{r+}(\sigma^*Y)$, $(J^r(\pi^*\sigma))^*(\Omega_r(Y)) \subseteq \Omega_{r+}(\sigma^*Y)$, so that it is sufficient to prove the inclusions ${}_{OrX}\widehat{\Omega}_+(\sigma^*J^rY) \subseteq (\pi_r^*\sigma)^*({}_X\widehat{\Omega}_r(Y))$, $\Omega_+(\sigma^*J^rY) \subseteq (\pi_r^*\sigma)^*(\Omega_r(Y))$, ${}_{OrX}\widehat{\Omega}_{r+}(\sigma^*Y) \subseteq (J^r(\pi^*\sigma))^*({}_X\widehat{\Omega}_r(Y))$, $\Omega_{r+}(\sigma^*Y) \subseteq (J^r(\pi^*\sigma))^*(\Omega_r(Y))$.

Let $\varrho \in {}_{OrX}\widehat{\Omega}_+(\sigma^*J^rY)$ (resp. $\varrho \in \Omega_+(\sigma^*J^rY)$) be an odd differential form (resp. differential form) on σ^*J^rY. Let $\mathcal{A}_{J^rY} = \{(W_j, \tau_j) | j \in J\}$, $\mathcal{A}_X = \{(V_j, \psi_j) | j \in J\}$, $\mathcal{A}_{OrX} = \{(U_j, \varphi_j) | j \in J\} \cup \{(-U_j, \varphi_j^-) | j \in J\}$, $\mathcal{A}_{\sigma^*J^rY} = \{(\mathcal{W}_j, \phi_j) | j \in J\} \cup \{(-\mathcal{W}_j, \phi_j^-) | j \in J\}$ be atlases such that (W_j, τ_j) is a fiber chart on J^rY with respect to the projection $\pi_r : J^rY \to X$, (V_j, ψ_j) is the chart on X associated with the chart (W_j, τ_j), $\sigma|_{U_j} : U_j \to V_j$ is a diffeomorphism, $\varphi_j = \psi_j \circ \sigma|_{U_j}$, $-U_j = \iota(U_j)$, $\varphi_j^- = \varphi_j \circ \iota|{-U_j}$, $\pi_r^*\sigma|_{\mathcal{W}_j} : \mathcal{W}_j \to W_j$ is a diffeomorphism, $\phi_j = \tau_j \circ (\pi_r^*\sigma|_{\mathcal{W}_j})$, $-\mathcal{W}_j = I_r(\mathcal{W}_j)$, $\phi_j^- = \phi_j \circ I_r|_{-\mathcal{W}_j}$. (It is obvious that such atlases exist.) For each $j \in J$ define odd differential forms (resp. differential forms) $\omega_j = ((\pi_r^*\sigma|_{\mathcal{W}_j})^{-1})^*(\varrho|_{\mathcal{W}_j})$, $\bar{\omega}_j = ((\pi_r^*\sigma|_{-\mathcal{W}_j})^{-1})^*(\varrho|_{-\mathcal{W}_j})$ on W_j ($\subseteq J^rY$). The equality $(\pi_r^*\sigma) \circ I_r = \pi_r^*\sigma$ implies $\bar{\omega}_j = ((\pi_r^*\sigma|_{-\mathcal{W}_j})^{-1})^*(\varrho|_{-\mathcal{W}_j}) = ((\pi_r^*\sigma|_{\mathcal{W}_j} \circ I_r|_{-\mathcal{W}_j})^{-1})^*(\varrho|_{-\mathcal{W}_j}) = ((I_r|_{-\mathcal{W}_j})^{-1} \circ (\pi_r^*\sigma|_{\mathcal{W}_j})^{-1})^*(\varrho|_{-\mathcal{W}_j}) = ((I_r|_{\mathcal{W}_j}) \circ (\pi_r^*\sigma|_{\mathcal{W}_j})^{-1})^*(\varrho|_{-\mathcal{W}_j}) = ((\pi_r^*\sigma|_{\mathcal{W}_j})^{-1})^*(I|_{\mathcal{W}_j})^*(\varrho|_{-\mathcal{W}_j}) = ((\pi_r^*\sigma|_{\mathcal{W}_j})^{-1})^*(I^*\varrho)|_{\mathcal{W}_j} = ((\pi_r^*\sigma|_{\mathcal{W}_j})^{-1})^*(\varrho|_{\mathcal{W}_j}) = \omega_j$. Therefore, for each $j \in J$ we can define an odd differential form (resp. differential form) $\omega_j = ((\pi_r^*\sigma|_{\mathcal{W}_j})^{-1})^*(\varrho|_{\mathcal{W}_j}) = ((\pi_r^*\sigma|_{-\mathcal{W}_j})^{-1})^*(\varrho|_{-\mathcal{W}_j})$ on W_j.

Now, let $i, j \in J$ be indices such that $W_i \cap W_j \neq \emptyset$. Then $\mathcal{W}_i \cap \mathcal{W}_j \neq \emptyset$ or $\mathcal{W}_i \cap (-\mathcal{W}_j) \neq \emptyset$. If $\mathcal{W}_i \cap \mathcal{W}_j \neq \emptyset$ then $\omega_i|_{W_i \cap W_j} = ((\pi_r^*\sigma|_{\mathcal{W}_i \cap \mathcal{W}_j})^{-1})^*(\varrho|_{\mathcal{W}_i \cap \mathcal{W}_j}) = \omega_j|_{W_i \cap W_j}$, if $\mathcal{W}_i \cap (-\mathcal{W}_j) \neq \emptyset$ then $\omega_i|_{W_i \cap W_j} = ((\pi_r^*\sigma|_{\mathcal{W}_i \cap (-\mathcal{W}_j)})^{-1})^*(\varrho|_{\mathcal{W}_i \cap (-\mathcal{W}_j)}) = \omega_j|_{W_i \cap W_j}$, so that $\omega_i|_{W_i \cap W_j} = \omega_j|_{W_i \cap W_j}$ whenever $W_i \cap W_j \neq \emptyset$.

This allows us to define an odd differential form (resp. differential form) ω on J^rY by $\omega|_{W_i} = \omega_i$. It is easy to see that $(\pi_r^*\sigma)^*\omega = \varrho$ so that $\varrho \in (\pi_r^*\sigma)^*({}_X\widehat{\Omega}_r(Y))$ (resp. $\varrho \in (\pi_r^*\sigma)^*(\Omega_r(Y))$). This completes the proof of the inclusions ${}_{OrX}\widehat{\Omega}_+(\sigma^*J^rY) \subseteq (\pi_r^*\sigma)^*({}_X\widehat{\Omega}_r(Y))$, $\Omega_+(\sigma^*J^rY) \subseteq (\pi_r^*\sigma)^*(\Omega_r(Y))$.The rest inclusions can be proved in the similar way

3.6. Proposition. *Let the assumptions of the Lemma 3.5. be satisfied.*

(a) If the manifold X is orientable then the following commutative diagram of isomorphisms of graded real vector differential spaces can be constructed.

$$
\begin{tikzcd}
 & & {}_{X}\widehat{\Omega}_r(Y) \arrow[dl] \arrow[drrr] & & \\
 & {}_{OrX}\widehat{\Omega}_{+}(\sigma^* J^r Y) \arrow[rrr, leftrightarrow, "\simeq"] & & & {}_{OrX}\widehat{\Omega}_{r+}(\sigma^* Y) \\
 & \oplus & & & \oplus \\
{}_{OrX}\widehat{\Omega}_{-}(\sigma^* J^r Y) \arrow[rrr, leftrightarrow, "\simeq"] & & & {}_{OrX}\widehat{\Omega}_{r-}(\sigma^* Y) & \\
 & \Omega_{-}(\sigma^* J^r Y) \arrow[uuu] \arrow[ul] \arrow[rrr, leftrightarrow, "\simeq"] & & & \Omega_{r-}(\sigma^* Y) \arrow[uuu] \arrow[ul] \\
 & \oplus & & & \oplus \\
\Omega_{+}(\sigma^* J^r Y) \arrow[uuu] \arrow[uuuuur] \arrow[rrr, leftrightarrow, "\simeq"] & & & \Omega_{r+}(\sigma^* Y) \arrow[uuu] \arrow[uuuuur] & \\
 & & \Omega_r(Y) \arrow[ull] \arrow[ur] & &
\end{tikzcd}
\tag{D7}
$$

(b) If the manifold X is non-orientable then the following commutative diagram of isomorphisms of graded real vector differential spaces can be constructed.

$$
\begin{tikzcd}
 & & {}_{X}\widehat{\Omega}_r(Y) \arrow[dl] \arrow[drrr] & & \\
 & {}_{OrX}\widehat{\Omega}_{+}(\sigma^* J^r Y) \arrow[rrr, leftrightarrow, "\simeq"] & & & {}_{OrX}\widehat{\Omega}_{r+}(\sigma^* Y) \\
 & \oplus & & & \oplus \\
{}_{OrX}\widehat{\Omega}_{-}(\sigma^* J^r Y) \arrow[rrr, leftrightarrow, "\simeq"] & & & {}_{OrX}\widehat{\Omega}_{r-}(\sigma^* Y) & \\
 & \Omega_{-}(\sigma^* J^r Y) \arrow[uuu] \arrow[rrr, leftrightarrow, "\simeq"] & & & \Omega_{r-}(\sigma^* Y) \arrow[uuu] \\
 & \oplus & & & \oplus \\
\Omega_{+}(\sigma^* J^r Y) \arrow[uuu] \arrow[rrr, leftrightarrow, "\simeq"] & & & \Omega_{r+}(\sigma^* Y) \arrow[uuu] & \\
 & & \Omega_r(Y) \arrow[ull] \arrow[ur] & &
\end{tikzcd}
\tag{D8}
$$

Proof. If X is an orientable manifold then OrX is a union of two disjoint manifolds A, B which are diffeomorphic to the manifold X. In addition, for the canonical involution $\iota : OrX \to OrX$ the restrictions $\iota|_A : A \to B, \iota|_B : B \to A$ are diffeomorphisms. Hence we can choose two orientations on OrX so that the involution ι preserves one of these orientations and changes the other one. If δ^-, δ^+ are continuous unit fields of odd scalars on OrX according to these orientations then $\iota^*\delta^+ = \delta^+, \iota^*\delta^- = \delta^-$. Now, the existence of the diagram (D7) follows from Lemma 3.3. and Lemma 3.5.

If X is a connected non-orientable manifold then OrX is connected orientable manifold and the canonical involution $\iota : OrX \to OrX$ is an orientation reversing mapping (see [5], Proposition 5.14.). Hence on OrX there exists a continuous unit field of odd scalars δ and for this field holds $\iota^*\delta = -\delta$. So in this case the existence of the diagram (D8) follows from Lemma 3.3. and Lemma 3.5.

If X is a non-connected non-orientable manifold we can divide X in its components and to this components we can apply what we proved in the first part of the proof.

Now, let us consider the cohomology spaces ${}_X\widehat{\mathbf{H}}^k_r(Y)$, $\mathbf{H}^k_r(Y)$, ${}_{OrX}\widehat{\mathbf{H}}^k_r(\sigma^*Y)$, $\mathbf{H}^k_r(\sigma^*Y)$, ${}_{OrX}\widehat{\mathbf{H}}^k_{r+}(\sigma^*Y)$, $\mathbf{H}^k_{r+}(\sigma^*Y)$, ${}_{OrX}\widehat{\mathbf{H}}^k_{r-}(\sigma^*Y)$, $\mathbf{H}^k_{r-}(\sigma^*Y)$ of graded differential spaces ${}_X\widehat{\Omega}_r(Y)$, $\Omega_r(Y)$, ${}_{OrX}\widehat{\Omega}_r(\sigma^*Y)$, $\Omega_r(\sigma^*Y)$, ${}_{OrX}\widehat{\Omega}_{r+}(\sigma^*Y)$, $\Omega_{r+}(\sigma^*Y)$, ${}_{OrX}\widehat{\Omega}_{r-}(\sigma^*Y)$, $\Omega_{r-}(\sigma^*Y)$. As an immediate consequence of Proposition 3.6. we get the following assertions.

3.7. Proposition. *Let $\pi : Y \to X$ be a fibered manifold, let $\sigma : OrX \to X$ be double cover of X.*

(a) If the manifold X is orientable then the following diagram of isomorphisms of real vector spaces can be constructed

$$\begin{tikzcd}
 & & & & {}_X\widehat{\mathbf{H}}^k_r(Y) \\
{}_{OrX}\widehat{\mathbf{H}}^k_r(\sigma^*Y) & = & {}_{OrX}\widehat{\mathbf{H}}^k_{r-}(\sigma^*Y) & \oplus & {}_{OrX}\widehat{\mathbf{H}}^k_{r+}(\sigma^*Y) \arrow[u] \\
\mathbf{H}^k_r(\sigma^*Y) \arrow[u] & = & \mathbf{H}^k_{r+}(\sigma^*Y) \arrow[u] \arrow[urr] & \oplus & \mathbf{H}^k_{r-}(\sigma^*Y) \arrow[u] \arrow[ull] \\
 & & \mathbf{H}^k_r(Y) \arrow[u] & & (0 \le k \le \dim J^rY).
\end{tikzcd} \tag{D9}$$

(b) If the manifold X is non-orientable then the following diagram of isomorphisms of real vector spaces can be constructed

$$\begin{tikzcd}
 & & & & {}_X\widehat{\mathbf{H}}^k_r(Y) \\
{}_{OrX}\widehat{\mathbf{H}}^k_r(\sigma^*Y) & = & {}_{OrX}\widehat{\mathbf{H}}^k_{r-}(\sigma^*Y) & \oplus & {}_{OrX}\widehat{\mathbf{H}}^k_{r+}(\sigma^*Y) \arrow[u] \\
\mathbf{H}^k_r(\sigma^*Y) \arrow[u] & = & \mathbf{H}^k_{r+}(\sigma^*Y) \arrow[u] & \oplus & \mathbf{H}^k_{r-}(\sigma^*Y) \arrow[u] \\
 & & \mathbf{H}^k_r(Y) \arrow[u] & & (0 \le k \le \dim J^rY).
\end{tikzcd} \tag{D10}$$

3.8. Corollary. *Let $\pi : Y \to X$ be a fibered manifold, let $\sigma : OrX \to X$ be double cover of X.*

(a) If the manifold X is orientable then ${}_X\widehat{\mathbf{H}}^k_r(Y) \simeq \mathbf{H}^k_r(Y)$ $(0 \le k \le \dim J^rY)$.

(b) If the manifold X is non-orientable then $\mathbf{H}^k_r(\sigma^*Y) \simeq \mathbf{H}^k_r(Y) \oplus {}_X\widehat{\mathbf{H}}^k_r(Y)$ $(0 \le k \le \dim J^rY)$.

Particularly, for $r = 0$ we get

3.9. Corollary. *Let $\pi : Y \to X$ be a fibered manifold, let $\sigma : OrX \to X$ be double cover of X.*

(a) If the manifold X is orientable then ${}_X\widehat{\mathbf{H}}^k(Y) \simeq \mathbf{H}^k(Y)$ $(0 \leq k \leq \dim Y)$.

(b) If the manifold X is non-orientable then $\mathbf{H}^k(\sigma^* Y) \simeq \mathbf{H}^k(Y) \oplus {}_X\widehat{\mathbf{H}}^k(Y)$ $(0 \leq k \leq \dim Y)$.

Very simple is the situation when the fibered manifold $\pi : Y \to X$ admits the structure of a vector bundle.

3.10. Corollary. *Let $\pi : Y \to X$ be a vector bundle, let $\sigma : OrX \to X$ be double cover of X.*

(a) If the manifold X is orientable then ${}_X\widehat{\mathbf{H}}^k(Y) \simeq \mathbf{H}^k(X)$ $(0 \leq k \leq \dim Y)$.

(b) If the manifold X is non-orientable then ${}_X\widehat{\mathbf{H}}^k(Y) \simeq \mathbf{H}^k(OrX)/\mathbf{H}^k(X) \simeq \widehat{\mathbf{H}}^k(X)$ $(0 \leq k \leq \dim Y)$, where $\widehat{\mathbf{H}}^k(X)$ are cohomology spaces of the graded real vector differential space $(\widehat{\Omega}(X), d)$ of odd differential forms on X.

Proof. If $\pi : Y \to X$ is a vector bundle then the manifold X is a deformation retract of the total space Y so that $\mathbf{H}^k(Y) \simeq \mathbf{H}^k(X)$ for each k. Moreover, the pull-back $\sigma^*\pi : \sigma^* Y \to OrX$ has the structure of a vector bundle too, and so $\mathbf{H}^k(\sigma^* Y) \simeq \mathbf{H}^k(OrX)$ for each k. Now, the assertion (a) and the first part of the assertion (b) follow immediately from Corollary 3.9. For the relation $\mathbf{H}^k(OrX)/\mathbf{H}^k(X) \simeq \widehat{\mathbf{H}}^k(X)$ it is sufficient to apply Corollary 3.9. to the trivial fibered manifold $\mathrm{id}_X : X \to X$.

3.11. Example. Let $\pi : Y \to$ Möb be a vector bundle over the Möbius strip. Then $\mathbf{H}^k(\text{Möb}) \simeq \mathbf{H}^k(Or\text{Möb}) \simeq \mathbf{H}^k(S^1)$ for each k because the 1-sphere S^1 is a deformation retract of both manifolds Möb, OrMöb. Now, according to Corollary 3.10., ${}_{\text{Möb}}\widehat{\mathbf{H}}^k(Y) = 0$ for each k.

3.12. Example. Let $\pi : Y \to \mathbf{R}P^n$, where n is an even positive integer, be a vector bundle over the non-orientable projective space $\mathbf{R}P^n$. In this case $Or\mathbf{R}P^n = S^n$ so that $\mathbf{H}^0(Or\mathbf{R}P^n) = \mathbf{H}^n(Or\mathbf{R}P^n) = \mathbf{R}$, $\mathbf{H}^k(Or\mathbf{R}P^n) = 0$ for $1 \leq k < n$. Moreover $\mathbf{H}^0(\mathbf{R}P^n) = \mathbf{R}$, $\mathbf{H}^k(\mathbf{R}P^n) = 0$ for $1 \leq k$, and so, according to Corollary 3.10., ${}_{\mathbf{R}P^n}\widehat{\mathbf{H}}^k(Y) = 0$ for $k \neq n$, ${}_{\mathbf{R}P^n}\widehat{\mathbf{H}}^n(Y) \simeq \mathbf{R}$.

4. Θ-spaces

The aim of this chapter is to construct the Θ-spaces, which will be used in Chapter 5 to the construction of the variational sequence. Moreover, we will prove an analogue to the Proposition 3.6. for Θ-spaces.

In [6] were introduced the contact odd base differential forms. It is easily seen that these forms can be described in the following way.

4.1. Definition. Let $\pi : Y \to X$ be a fibered manifold, let $\{(W_i, \tau_i) | i \in I\}$ be an adapted atlas on Y. For each $i \in I$ denote by (V_i, ψ_i) the chart on X associated with (W_i, τ_i). Let $\varrho \in {}_X\widehat{\Omega}^k_r(Y)$, let $\varrho|_{(W_i)_r} = \pi^*_r \widehat{\psi}_i \otimes \varrho_{\psi_i}$ for each $i \in I$, where $\widehat{\psi}_i$ is the field of unit odd scalars on V_i associated with the chart (V_i, ψ_i), and where $\varrho_{\psi_i} \in \Omega^k_r(W_i)$. We say that the odd base form ϱ has the *order of contact* equal to an integer q, if all the forms ϱ_{ψ_i} have the order of contact equal to q.

4.2. Definition. Let $\pi : Y \to X$ be a fibered manifold, let $\varrho \in {}_X\widehat{\Omega}^k_r(Y)$. We say that ϱ is *contact* if the order of contact of ϱ is greater than 0. If $k > n = \dim X$ we say that ϱ is *strongly contact* if the order of contact of ϱ is greater than $k - n$. By ${}_X\widehat{\Omega}^k_{r(c)}(Y)$ we denote the space of all contact odd base k-forms on J^rY (if $1 \le k \le n$) or the space of all strongly contact odd base k-forms on J^rY (if $n < k$). By ${}_X\widehat{\Omega}_{r(c)}(Y)$ we denote the graded real vector space ${}_X\widehat{\Omega}_{r(c)}(Y) = \sum_{k\ge 1} {}_X\widehat{\Omega}^k_{r(c)}(Y)$.

4.3. Definition. Let $\pi : Y \to X$ be a fibered manifold. We set $\Theta^1_r(Y) = \Omega^1_{r(c)}(Y)$, $\Theta^k_r(Y) = \{\varrho \in \Omega^k_r(Y) \mid$ for each $y \in Y$ there is a neighbourhood $W \subseteq Y$ of the point y such that $\varrho|_{W_r} \in d\Omega^{k-1}_{r(c)}(W) + \Omega^k_{r(c)}(W)\}$, where $2 \le k$, $\Theta_r(Y) = \sum_{k\ge 1} \Theta^k_r(Y)$, ${}_X\widehat{\Theta}^1_r(Y) = {}_X\widehat{\Omega}^1_{r(c)}(Y)$, ${}_X\widehat{\Theta}^k_r(Y) = \{\varrho \in {}_X\widehat{\Omega}^k_r(Y) \mid$ for each $y \in Y$ there is a neighbourhood $W \subseteq Y$ of the point y such that $\varrho|_{W_r} \in d_X\widehat{\Omega}^{k-1}_{r(c)}(W) + {}_X\widehat{\Omega}^k_{r(c)}(W)\}$, where $2 \le k$, ${}_X\widehat{\Theta}_r(Y) = \sum_{k\ge 1} {}_X\widehat{\Theta}^k_r(Y)$.

Remark. It is apparent that $(\Theta_r(Y), d)$ (resp. $({}_X\widehat{\Theta}_r(Y), d)$) is a differential subspace of the differential space $(\sum_{k\ge 1} \Omega^k_r(Y), d)$ (resp. $(\sum_{k\ge 1} {}_X\widehat{\Omega}^k_r(Y), d)$).

If there is given an involution $(F, f) : \pi \to \pi$ of a fibered manifold $\pi : Y \to X$ we can decompose the spaces $\Theta_r(Y), {}_X\widehat{\Theta}_r(Y)$ in the same way as we decomposed the spaces $\Omega_r(Y), {}_X\widehat{\Omega}_r(Y)$. We get $\Theta_r(Y) = \Theta_{r+}(Y) \oplus \Theta_{r-}(Y)$, ${}_X\widehat{\Theta}_r(Y) = {}_X\widehat{\Theta}_{r-}(Y) \oplus {}_X\widehat{\Theta}_{r+}(Y)$, where $\Theta_{r+}(Y) = \Theta_r(Y) \cap \sum_{k\ge 1} \Omega^k_{r+}(Y)$, $\Theta_{r-}(Y) = \Theta_r(Y) \cap \sum_{k\ge 1} \Omega^k_{r-}(Y)$, ${}_X\widehat{\Theta}_{r+}(Y) = {}_X\widehat{\Theta}_r(Y) \cap \sum_{k\ge 1} {}_X\widehat{\Omega}^k_{r+}(Y)$, ${}_X\widehat{\Theta}_{r-}(Y) = {}_X\widehat{\Theta}_r(Y) \cap \sum_{k\ge 1} {}_X\widehat{\Omega}^k_{r-}(Y)$. Applying these constructions to the diagram (D2), we obtain the following modifications of the diagrams (D5), (D6).

$$\begin{array}{ccccc} & & & & {}_X\widehat{\Theta}_r(Y) \\ & & & & \Big\uparrow (J^r(\pi^*\sigma))^* \\ {}_{OrX}\widehat{\Theta}_r(\sigma^*Y) & = & {}_{OrX}\widehat{\Theta}_{r-}(\sigma^*Y) & \oplus & {}_{OrX}\widehat{\Theta}_{r+}(\sigma^*Y) \end{array} \tag{D11}$$

$$\begin{array}{ccccc} \Theta_r(\sigma^*Y) & = & \Theta_{r+}(\sigma^*Y) & \oplus & \Theta_{r-}(\sigma^*Y) \\ & & \Big\uparrow (J^r(\pi^*\sigma))^* & & \\ & & \Theta_r(Y) & & \end{array} \tag{D12}$$

where the mappings $(J^r(\pi^*\sigma))^* : {}_X\widehat{\Theta}_r(Y) \to {}_{OrX}\widehat{\Theta}_{r+}(\sigma^*Y)$, $(J^r(\pi^*\sigma))^* : \Theta_r(Y) \to \Theta_{r+}(\sigma^*Y)$ are isomorphisms of graded differential spaces.

4.4. Lemma. *Let $\pi : Y \to X$ be a fibered manifold, let δ be a continuous field of unit odd scalars on X. Then ${}_\delta Z(J^rY)(\Theta_r(Y)) = {}_X\widehat{\Theta}_r(Y)$.*

Proof. Let $\varrho \in \Omega_r(Y)$, $\omega \in {}_X\widehat{\Omega}_r(Y)$, let $\omega = {}_\delta Z(J^rY)(\varrho)$. Let $\mathcal{A} = \{(W_i,\tau_i) \mid i \in I\}$ be an adapted atlas on Y. For each $i \in I$ denote by (V_i,ψ_i) the chart on X associated with (W_i,τ_i); we can suppose that each V_i is connected. Then for each $i \in I$ there is $\xi_i \in \{-1,1\}$ such that $\widehat{\psi}_i = \xi_i\delta|_{V_i}$. Let $\omega|_{(W_i)_r} = \pi_r^*\widehat{\psi}_i \otimes \omega_{\psi_i}$. $\omega|_{(W_i)_r} = {}_\delta Z(J^rY)(\varrho)|_{(W_i)_r} = {}_{(\delta|_{V_i})}Z((W_i)_r)(\varrho|_{(W_i)_r}) = \pi_r^*(\delta|_{V_i}) \otimes \varrho|_{(W_i)_r} = \xi_i\pi_r^*\widehat{\psi}_i \otimes \varrho|_{(W_i)_r} = \pi_r^*\widehat{\psi}_i \otimes (\xi_i\varrho|_{(W_i)_r})$ so that $\omega_{\psi_i} = \xi_i\varrho|_{(W_i)_r}$.

Now, from the Definition 4.3. it follows $\varrho \in \Theta_r(Y)$ iff $\omega \in {}_X\widehat{\Theta}_r(Y)$. Moreover, the mapping ${}_\delta Z(J^rY) : \Omega_r(Y) \to {}_X\widehat{\Omega}_r(Y)$ is an isomorphism and so ${}_\delta Z(J^rY)(\Theta_r(Y)) = {}_X\widehat{\Theta}_r(Y)$.

4.5. Proposition. *Let $\pi : Y \to X$ be a fibered manifold.*

(a) If the manifold X is orientable then the following diagram of isomorphisms of graded real vector differential spaces can be constructed.

$$\begin{tikzcd}
 & & {}_X\widehat{\Theta}_r(Y) \\
{}_{OrX}\widehat{\Theta}_{r-}(\sigma^*Y) & \oplus & {}_{OrX}\widehat{\Theta}_{r+}(\sigma^*Y) \arrow[u] \\
\Theta_{r+}(\sigma^*Y) \arrow[u] \arrow[urr] & \oplus & \Theta_{r-}(\sigma^*Y) \arrow[u] \arrow[ull] \\
\Theta_r(Y) \arrow[u]
\end{tikzcd} \tag{D13}$$

(b) If the manifold X is non-orientable then the following diagram of isomorphisms of graded real vector differential spaces can be constructed.

$$\begin{tikzcd}
 & & {}_X\widehat{\Theta}_r(Y) \\
{}_{OrX}\widehat{\Theta}_{r-}(\sigma^*Y) & \oplus & {}_{OrX}\widehat{\Theta}_{r+}(\sigma^*Y) \arrow[u] \\
\Theta_{r+}(\sigma^*Y) \arrow[u] & \oplus & \Theta_{r-}(\sigma^*Y) \arrow[u] \\
\Theta_r(Y) \arrow[u]
\end{tikzcd} \tag{D14}$$

Proof. The assertion follows from Lemma 4.4. and Proposition 3.6.

5. Variational sequences

First, let us recall the construction of variational sequences of a fibered manifold $\pi : Y \to X$ (see [7]).

Let $\pi : Y \to X$ be a fibered manifold, let $\dim X = n$, $\dim Y = m+n$. For each r,k we can consider the sheaf ${}^Y\Omega_r^k$ over the manifold Y generated by the presheaf $Y \supseteq W \mapsto \Omega_r^k(W)$ which assigns to every open subset $W \subseteq Y$ the real vector space of differential k-forms defined on W_r. (The constant sheaf over Y we denote by ${}^Y\mathbf{R}$.) Further, we can define the sheaves ${}^Y\Omega^k_{r(c)}$ over Y by the presheaves $Y \supseteq W \mapsto \Omega^k_{r(c)}(W)$ of contact (resp. strongly contact) differential forms. The sheaves ${}^Y\Omega^k_{r(c)}$ are subsheaves of the sheaves ${}^Y\Omega^k_r$ so that we can define the sheaves ${}^Y\Theta^1_r = {}^Y\Omega^1_{r(c)}$, ${}^Y\Theta^k_r = d\,{}^Y\Omega^{k-1}_{r(c)} + {}^Y\Omega^k_{r(c)}$ $(k > 1)$. These sheaves can be involved in the following commutative diagram

$$\begin{array}{ccccccccccccccccc}
 & & & & 0 & & 0 & & & & 0 & & & & & & \\
 & & & & \downarrow & & \downarrow & & & & \downarrow & & & & & & \\
 & & & 0 \to & {}^Y\Theta^1_r & \xrightarrow{d} & {}^Y\Theta^2_r & \xrightarrow{d} & \dots & \xrightarrow{d} & {}^Y\Theta^P_r & \to 0 & & & & & \\
 & & & & \downarrow & & \downarrow & & & & \downarrow & & & & & & \\
0 \to & {}^Y\mathbf{R} \to & {}^Y\Omega^0_r & \xrightarrow{d} & {}^Y\Omega^1_r & \xrightarrow{d} & {}^Y\Omega^2_r & \xrightarrow{d} & \dots & \xrightarrow{d} & {}^Y\Omega^P_r & \xrightarrow{d} & {}^Y\Omega^{P+1}_r & \xrightarrow{d} & \dots & \xrightarrow{d} & {}^Y\Omega^N_r \to 0 \\
 & & & & \downarrow & & \downarrow & & & & \downarrow & & & & & & \\
 & & & & {}^Y\Omega^1_r/{}^Y\Theta^1_r & & {}^Y\Omega^2_r/{}^Y\Theta^2_r & & & & {}^Y\Omega^P_r/{}^Y\Theta^P_r & & & & & & \\
 & & & & \downarrow & & \downarrow & & & & \downarrow & & & & & & \\
 & & & & 0 & & 0 & & & & 0 & & & & & &
\end{array} \tag{D15}$$

where $N = \dim Y$, and where $P = m\binom{n+r-1}{n} + 2n - 1$ is an integer such that for each $k > P$ the sheaf ${}^Y\Theta^k_r$ is trivial (see [7]). The columns of diagram (D15) are short exact sequences so that the diagram (D15) can be uniquely completed to the commutative diagram

$$\begin{array}{ccccccccccccccccc}
 & & & & 0 & & 0 & & & & 0 & & & & & & \\
 & & & & \downarrow & & \downarrow & & & & \downarrow & & & & & & \\
 & & & 0 \to & {}^Y\Theta^1_r & \xrightarrow{d} & {}^Y\Theta^2_r & \xrightarrow{d} & \dots & \xrightarrow{d} & {}^Y\Theta^P_r & \to 0 & & & & & \\
 & & & & \downarrow & & \downarrow & & & & \downarrow & & & & & & \\
0 \to & {}^Y\mathbf{R} \to & {}^Y\Omega^0_r & \xrightarrow{d} & {}^Y\Omega^1_r & \xrightarrow{d} & {}^Y\Omega^2_r & \xrightarrow{d} & \dots & \xrightarrow{d} & {}^Y\Omega^P_r & \xrightarrow{d} & {}^Y\Omega^{P+1}_r & \xrightarrow{d} & \dots & \xrightarrow{d} & {}^Y\Omega^N_r \to 0 \\
 & & & \searrow^{E_0} & \downarrow & & \downarrow & & & & \downarrow & \nearrow^{E_P} & & & & & \\
 & & & & {}^Y\Omega^1_r/{}^Y\Theta^1_r & \xrightarrow{E_1} & {}^Y\Omega^2_r/{}^Y\Theta^2_r & \xrightarrow{E_2} & \dots & \to & {}^Y\Omega^P_r/{}^Y\Theta^P_r & & & & & & \\
 & & & & \downarrow & & \downarrow & & & & \downarrow & & & & & & \\
 & & & & 0 & & 0 & & & & 0 & & & & & &
\end{array} \tag{D16}$$

The sequence $0 \to {}^Y\mathbf{R} \to {}^Y\Omega^0_r \xrightarrow{E_0} {}^Y\Omega^1_r/{}^Y\Theta^1_r \xrightarrow{E_1} \dots \xrightarrow{E_{P-1}} {}^Y\Omega^P_r/{}^Y\Theta^P_r \xrightarrow{E_P} {}^Y\Omega^{P+1}_r \to \dots \to {}^Y\Omega^N_r \to 0$ defined by diagram (D16) is called the *variational sequence of order r over Y*; we denote this sequence by $0 \to {}^Y\mathbf{R} \to {}^Y\nu^r$.

The same construction can be done for the spaces of odd base forms. For each r, k let us denote by ${}^Y_X\widehat{\Omega}^k_r$ the sheaf over the manifold Y generated by the presheaf $Y \supseteq W \mapsto {}_{\pi(W)}\widehat{\Omega}^k_r(W)$ which assigns to every open subset $W \subseteq Y$ the real vector space ${}_{\pi(W)}\widehat{\Omega}^k_r(W)$ of odd base k-forms defined on W_r. (By ${}^Y_X\widehat{\mathbf{R}}_r$ we denote the sheaf over Y generated by the presheaf over Y which assigns to every open subset $W \subseteq Y$ the real vector space of all locally constant fields of odd base scalars on W_r.) Further, for each r, k by ${}^Y_X\widehat{\Omega}^k_{r(c)}$ we denote the sheaf over Y generated by the presheaf $Y \supseteq W \mapsto {}_{\pi(W)}\widehat{\Omega}^k_{r(c)}(W)$ of contact (resp. strongly contact) odd base forms. The sheaves ${}^Y_X\widehat{\Omega}^k_{r(c)}$ are subsheaves of the sheaves ${}^Y_X\widehat{\Omega}^k_r$ so that we can define the sheaves ${}^Y_X\widehat{\Theta}^1_r = {}^Y_X\widehat{\Omega}^1_{r(c)}$, ${}^Y_X\widehat{\Theta}^k_r = d\,{}^Y_X\widehat{\Omega}^{k-1}_{r(c)} + {}^Y_X\widehat{\Omega}^k_{r(c)}$ $(k > 1)$ of real vector spaces over Y. These sheaves can be involved in the following commutative diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] \\
 & & 0 \arrow[r] & {}^Y_X\widehat{\Theta}^1_r \arrow[r, "d"] \arrow[d] & {}^Y_X\widehat{\Theta}^2_r \arrow[r, "d"] \arrow[d] & \dots \arrow[r, "d"] & {}^Y_X\widehat{\Theta}^P_r \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & {}^Y_X\widehat{\mathbf{R}} \arrow[r] & {}^Y_X\widehat{\Omega}^0_r \arrow[r, "d"] & {}^Y_X\widehat{\Omega}^1_r \arrow[r, "d"] \arrow[d] & {}^Y_X\widehat{\Omega}^2_r \arrow[r, "d"] \arrow[d] & \dots \arrow[r, "d"] & {}^Y_X\widehat{\Omega}^P_r \arrow[r, "d"] \arrow[d] & {}^Y_X\widehat{\Omega}^{P+1}_r \arrow[r, "d"] & \dots \arrow[r, "d"] & {}^Y_X\widehat{\Omega}^N_r \arrow[r] & 0 \\
 & & & {}^Y_X\widehat{\Omega}^1_r/{}^Y_X\widehat{\Theta}^1_r \arrow[d] & {}^Y_X\widehat{\Omega}^2_r/{}^Y_X\widehat{\Theta}^2_r \arrow[d] & & {}^Y_X\widehat{\Omega}^P_r/{}^Y_X\widehat{\Theta}^P_r \arrow[d] \\
 & & & 0 & 0 & & 0
\end{tikzcd} \tag{D17}$$

which can be uniquely completed to the commutative diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] \\
 & & 0 \arrow[r] & {}^Y_X\widehat{\Theta}^1_r \arrow[r, "d"] \arrow[d] & {}^Y_X\widehat{\Theta}^2_r \arrow[r, "d"] \arrow[d] & \dots \arrow[r, "d"] & {}^Y_X\widehat{\Theta}^P_r \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & {}^Y_X\widehat{\mathbf{R}} \arrow[r] & {}^Y_X\widehat{\Omega}^0_r \arrow[r, "d"] \arrow[dr, "\widehat{E}_0"'] & {}^Y_X\widehat{\Omega}^1_r \arrow[r, "d"] \arrow[d] & {}^Y_X\widehat{\Omega}^2_r \arrow[r, "d"] \arrow[d] & \dots \arrow[r, "d"] & {}^Y_X\widehat{\Omega}^P_r \arrow[r, "d"] \arrow[d] & {}^Y_X\widehat{\Omega}^{P+1}_r \arrow[r, "d"] & \dots \arrow[r, "d"] & {}^Y_X\widehat{\Omega}^N_r \arrow[r] & 0 \\
 & & & {}^Y_X\widehat{\Omega}^1_r/{}^Y_X\widehat{\Theta}^1_r \arrow[r, "\widehat{E}_1"] \arrow[d] & {}^Y_X\widehat{\Omega}^2_r/{}^Y_X\widehat{\Theta}^2_r \arrow[r, "\widehat{E}_2"] \arrow[d] & \dots \arrow[r] & {}^Y_X\widehat{\Omega}^P_r/{}^Y_X\widehat{\Theta}^P_r \arrow[d] \arrow[ur, "\widehat{E}_P"'] \\
 & & & 0 & 0 & & 0
\end{tikzcd} \tag{D18}$$

since the columns of diagram (D17) are short exact sequences.

5.1. Definition. The sequence $0 \to {}^Y_X\widehat{\mathbf{R}}_r \to {}^Y_X\widehat{\Omega}^0_r \xrightarrow{\widehat{E}_0} {}^Y_X\widehat{\Omega}^1_r/{}^Y_X\widehat{\Theta}^1_r \xrightarrow{\widehat{E}_1} \dots \xrightarrow{\widehat{E}_{P-1}} {}^Y_X\widehat{\Omega}^P_r/{}^Y_X\widehat{\Theta}^P_r \xrightarrow{\widehat{E}_P} {}^Y_X\widehat{\Omega}^{P+1}_r \to \dots \to {}^Y_X\widehat{\Omega}^N_r \to 0$ of morphisms of sheaves of real vector spaces is called the *odd variational sequence of order r over Y*. We denote the sequence by $0 \to {}^Y_X\widehat{\mathbf{R}}_r \to {}^Y_X\widehat{\nu}^r$.

The odd variational sequence is obviously a complex. The aim of this chapter is the description of its cohomology spaces.

5.2. Lemma. *Let $\pi : Y \to X$ be a fibered manifold, let $\sigma : OrX \to X$ be double cover of X. Let $(\pi^*\sigma)({}^{\sigma^*Y}\Theta^k_r)$ be the direct image of the sheaf ${}^{\sigma^*Y}\Theta^k_r$ by the projection $\pi^*\sigma : \sigma^*Y \to Y$. Then $(\pi^*\sigma)({}^{\sigma^*Y}\Theta^k_r) \simeq {}^Y\Theta^k_r \oplus {}^Y_X\widehat{\Theta}^k_r$.*

Proof. According to the definition of direct image, the sheaf $(\pi^*\sigma)({}^{\sigma^*Y}\Theta^k_r)$ is generated by the presheaf $Y \supseteq W \mapsto \Gamma((\pi^*\sigma)^{-1}(W), {}^{\sigma^*Y}\Theta^k_r)$, which is isomorphic to the presheaf $Y \supseteq W \mapsto \Theta^k_r((\pi^*\sigma)^{-1}(W))$. (See Definition 4.3.)

The direct sum ${}^Y\Theta^k_r \oplus {}^Y_X\widehat{\Theta}^k_r$ is the sheaf generated by the presheaf $Y \supseteq W \mapsto \Gamma(W, {}^Y\Theta^k_r) \oplus \Gamma(W, {}^Y_X\widehat{\Theta}^k_r)$ which is isomorphic to the presheaf $Y \supseteq W \mapsto \Theta^k_r(W) \oplus {}_{\pi(W)}\widehat{\Theta}^k_r(W)$. (See Definition 4.3.) So it is sufficient to prove that there is an isomorphism between the presheaves $Y \supseteq W \mapsto \Theta^k_r((\pi^*\sigma)^{-1}(W))$ and $Y \supseteq W \mapsto \Theta^k_r(W) \oplus {}_{\pi(W)}\widehat{\Theta}^k_r(W)$.

Let δ be a continuous field of unit odd scalars on OrX such that $\iota^*\delta = -\delta$,where $\iota : OrX \to OrX$ is the canonical involution of the double cover $\sigma : OrX \to X$ of X, let

$W \subseteq Y$ be an open subset of Y. Let us denote $V = \pi(W)$. Then $\sigma|_{\sigma^{-1}(V)} : \sigma^{-1}(V) \to V$ is the double cover of V, $\iota|_{\sigma^{-1}(V)} : \sigma^{-1}(V) \to \sigma^{-1}(V)$ is the canonical involution of $\sigma|_{\sigma^{-1}(V)} : \sigma^{-1}(V) \to V$, and $\delta|_{\sigma^{-1}(V)}$ is a continuous field of unit odd scalars on $\sigma^{-1}(V)$ such that $(\iota|_{\sigma^{-1}(V)})^*(\delta|_{\sigma^{-1}(V)}) = -\delta|_{\sigma^{-1}(V)}$. Then, according to the Proposition 4.5., the mapping $\Phi_W = [J^r((\pi|_W)^*(\sigma|_{\sigma^{-1}(V)}))]^* \circ pr_1 + [_{\delta|_{\sigma^{-1}(V)}} Z(J^r((\sigma|_{\sigma^{-1}(V)})^* W))]^{-1} \circ [J^r((\pi|_W)^*(\sigma|_{\sigma^{-1}(V)}))]^* \circ pr_2 : \Theta^k_r(W) \oplus {}_V\widehat{\Theta}^k_r(W) \to \Theta^k_r((\pi^*\sigma)^{-1}(W))$ is a linear isomorphism.

In this way we get a system $\Phi = \{\Phi_W \mid W \text{ is an open subset of } Y\}$ which is obviously the requisite isomorphism of presheaves $Y \supseteq W \mapsto \Theta^k_r(W) \oplus {}_{\pi(W)}\widehat{\Theta}^k_r(W)$, $Y \supseteq W \mapsto \Theta^k_r((\pi^*\sigma)^{-1}(W))$.

5.3. Proposition. *The sheaves* ${}^Y_X\widehat{\Theta}^k_r$ *are soft.*

Proof. Let us consider the short exact sequence $0 \to {}^Y\Theta^k_r \to (\pi^*\sigma)({}^{\sigma^*Y}\Theta^k_r) \to {}^Y_X\widehat{\Theta}^k_r \to 0$ induced by the decomposition $(\pi^*\sigma)({}^{\sigma^*Y}\Theta^k_r) \simeq {}^Y\Theta^k_r \oplus {}^Y_X\widehat{\Theta}^k_r$. In [7], [9] it was proved that the sheaves ${}^Y\Theta^k_r$, ${}^{\sigma^*Y}\Theta^k_r$ are soft. Hence the direct image $(\pi^*\sigma)({}^{\sigma^*Y}\Theta^k_r)$ is soft too, and from the exactness of the sequence $0 \to {}^Y\Theta^k_r \to (\pi^*\sigma)({}^{\sigma^*Y}\Theta^k_r) \to {}^Y_X\widehat{\Theta}^k_r \to 0$ it follows that the sheaf ${}^Y_X\widehat{\Theta}^k_r$ is soft. (See [13].)

5.4. Corollary. *Let $\pi : Y \to X$ be a fibered manifold, let $\sigma : OrX \to X$ be double cover of X.*

(a) The complex $0 \to \Gamma(Y, {}^Y\Omega^0_r) \to \Gamma(Y, {}^Y\Omega^1_r/{}^Y\Theta^1_r) \to \ldots \to \Gamma(Y, {}^Y\Omega^P_r/{}^Y\Theta^P_r) \to \Gamma(Y, {}^Y\Omega^{P+1}_r) \to \ldots \to \Gamma(Y, {}^Y\Omega^N_r) \to 0$ of global sections of the variational sequence $0 \to {}^Y\mathbf{R} \to {}^Y\nu^r$ is isomorphic to the complex $0 \to \Omega^0_r(Y) \xrightarrow{d} \Omega^1_r(Y)/\Theta^1_r(Y) \xrightarrow{d} \ldots \xrightarrow{d} \Omega^P_r(Y)/\Theta^P_r(Y) \xrightarrow{d} \Omega^{P+1}_r(Y) \xrightarrow{d} \ldots \xrightarrow{d} \Omega^N_r(Y) \to 0$.

(b) The complex $0 \to \Gamma(Y, {}^Y_X\widehat{\Omega}^0_r) \to \Gamma(Y, {}^Y_X\widehat{\Omega}^1_r/{}^Y_X\widehat{\Theta}^1_r) \to \ldots \to \Gamma(Y, {}^Y_X\widehat{\Omega}^P_r/{}^Y_X\widehat{\Theta}^P_r) \to \Gamma(Y, {}^Y_X\widehat{\Omega}^{P+1}_r) \to \ldots \to \Gamma(Y, {}^Y_X\widehat{\Omega}^N_r) \to 0$ of global sections of the variational sequence $0 \to {}^Y_X\widehat{\mathbf{R}} \to {}^Y_X\widehat{\nu}^r$ is isomorphic to the complex $0 \to {}_X\widehat{\Omega}^0_r(Y) \xrightarrow{d} {}_X\widehat{\Omega}^1_r(Y)/{}_X\widehat{\Theta}^1_r(Y) \xrightarrow{d} \ldots \xrightarrow{d} {}_X\widehat{\Omega}^P_r(Y)/{}_X\widehat{\Theta}^P_r(Y) \xrightarrow{d} {}_X\widehat{\Omega}^{P+1}_r(Y) \xrightarrow{d} \ldots \xrightarrow{d} {}_X\widehat{\Omega}^N_r(Y) \to 0$.

(c) The complex $0 \to \Gamma(\sigma^*Y, {}^{\sigma^*Y}\Omega^0_r) \to \Gamma(\sigma^*Y, {}^{\sigma^*Y}\Omega^1_r/{}^{\sigma^*Y}\Theta^1_r) \to \ldots \to \Gamma(\sigma^*Y, {}^{\sigma^*Y}\Omega^P_r/{}^{\sigma^*Y}\Theta^P_r) \to \Gamma(\sigma^*Y, {}^{\sigma^*Y}\Omega^{P+1}_r) \to \ldots \to \Gamma(\sigma^*Y, {}^{\sigma^*Y}\Omega^N_r) \to 0$ of global sections of the variational sequence $0 \to {}^{\sigma^*Y}\mathbf{R} \to {}^{\sigma^*Y}\nu^r$ is isomorphic to the direct sum $0 \to \Omega^0_r(Y) \oplus {}_X\widehat{\Omega}^0_r(Y) \xrightarrow{d \oplus d} (\Omega^1_r(Y)/\Theta^1_r(Y)) \oplus ({}_X\widehat{\Omega}^1_r(Y)/{}_X\widehat{\Theta}^1_r(Y)) \xrightarrow{d\oplus d} \ldots \xrightarrow{d\oplus d} (\Omega^P_r(Y)/\Theta^P_r(Y)) \oplus ({}_X\widehat{\Omega}^P_r(Y)/{}_X\widehat{\Theta}^P_r(Y)) \xrightarrow{d\oplus d} \Omega^{P+1}_r(Y) \oplus {}_X\widehat{\Omega}^{P+1}_r(Y) \xrightarrow{d\oplus d} \ldots\ldots \xrightarrow{d\oplus d} \Omega^N_r(Y) \oplus {}_X\widehat{\Omega}^N_r(Y) \to 0$.

Proof. Ad (a) According to [7], [9], the sheaves ${}^Y\Theta^k_r$ are soft so that the diagram

$$
\begin{tikzcd}
 & & 0 \arrow[d] & & 0 \arrow[d] & & & & \\
 & 0 \arrow[r] & \Gamma(Y,{}^Y\Theta^1_r) \arrow[r] \arrow[d] & \ldots \arrow[r] & \Gamma(Y,{}^Y\Theta^P_r) \arrow[r] \arrow[d] & 0 & & & \\
0 \arrow[r] & \Gamma(Y,{}^Y\Omega^0_r) \arrow[r] \arrow[dr] & \Gamma(Y,{}^Y\Omega^1_r) \arrow[r] \arrow[d] & \ldots \arrow[r] & \Gamma(Y,{}^Y\Omega^P_r) \arrow[r] \arrow[d] & \Gamma(Y,{}^Y\Omega^{P+1}_r) \arrow[r] & \ldots \arrow[r] & \Gamma(Y,{}^Y\Omega^N_r) \arrow[r] & 0 \\
 & & \Gamma(Y,{}^Y\Omega^1_r/{}^Y\Theta^1_r) \arrow[r] \arrow[d] & \ldots \arrow[r] & \Gamma(Y,{}^Y\Omega^P_r/{}^Y\Theta^P_r) \arrow[ur] \arrow[d] & & & & \\
 & & 0 & & 0 & & & &
\end{tikzcd}
$$

of spaces of global sections generated by the diagram (D16) has exact columns. Consequently, the complex $0 \to \Gamma(Y, {}^Y\Omega_r^0) \to \Gamma(Y, {}^Y\Omega_r^1/{}^Y\Theta_r^1) \to \ldots \to \Gamma(Y, {}^Y\Omega_r^P/{}^Y\Theta_r^P) \to \Gamma(Y, {}^Y\Omega_r^{P+1}) \to \ldots \to \Gamma(Y, {}^Y\Omega_r^N) \to 0$ is isomorphic to the complex $0 \to \Gamma(Y, {}^Y\Omega_r^0) \to \Gamma(Y, {}^Y\Omega_r^1)/\Gamma(Y, {}^Y\Theta_r^1) \to \ldots \to \Gamma(Y, {}^Y\Omega_r^P)/\Gamma(Y, {}^Y\Theta_r^P) \to \Gamma(Y, {}^Y\Omega_r^{P+1}) \to \ldots \to \Gamma(Y, {}^Y\Omega_r^N) \to 0$.

The presheaves $Y \supseteq W \mapsto \Omega_r^k(W)$ satisfy the axioms of a sheaf so that $\Gamma(Y, {}^Y\Omega_r^k) \simeq \Omega_r^k(Y)$. Furthermore, from the Definition 4.3. and from the construction of the sheaves ${}^Y\Theta_r^k$ we obtain $\Gamma(Y, {}^Y\Theta_r^k) \simeq \Theta_r^k(Y)$, what completed the proof of the assertion (a).

Ad (b) The Proposition 5.3. shows that the sheaves ${}^Y_X\widehat{\Theta}_r^k$ are soft. Then, we can prove the assertion (b) in the same way as the assertion (a).

Ad(c) In the same way as we proved the assertion (a) we can prove that the complex $0 \to \Gamma(\sigma^*Y, {}^{\sigma^*Y}\Omega_r^0) \xrightarrow{d} \Gamma(\sigma^*Y, {}^{\sigma^*Y}\Omega_r^1/{}^{\sigma^*Y}\Theta_r^1) \xrightarrow{d} \ldots \xrightarrow{d} \Gamma(\sigma^*Y, {}^{\sigma^*Y}\Omega_r^P/{}^{\sigma^*Y}\Theta_r^P) \xrightarrow{d} \Gamma(\sigma^*Y, {}^{\sigma^*Y}\Omega_r^{P+1}) \xrightarrow{d} \ldots \xrightarrow{d} \Gamma(\sigma^*Y, {}^{\sigma^*Y}\Omega_r^N) \to 0$ is isomorphic to the complex $0 \to \Omega_r^0(\sigma^*Y) \xrightarrow{d} \Omega_r^1(\sigma^*Y)/\Theta_r^1(\sigma^*Y) \xrightarrow{d} \ldots \xrightarrow{d} \Omega_r^P(\sigma^*Y)/\Theta_r^P(\sigma^*Y) \xrightarrow{d} \Omega_r^{P+1}(\sigma^*Y) \xrightarrow{d} \ldots \xrightarrow{d} \Omega_r^N(\sigma^*Y) \to 0$. According to the Proposition 3.6. and Proposition 4.5., the last complex is isomorphic to the complex

$$0 \to \Omega_r^0(Y) \oplus {}_X\widehat{\Omega}_r^0(Y) \xrightarrow{d\oplus d} (\Omega_r^1(Y) \oplus {}_X\widehat{\Omega}_r^1(Y))/(\Theta_r^1(Y) \oplus {}_X\widehat{\Theta}_r^1(Y)) \xrightarrow{d\oplus d} \ldots$$
$$\ldots \xrightarrow{d\oplus d} (\Omega_r^P(Y) \oplus {}_X\widehat{\Omega}_r^P(Y))/(\Theta_r^P(Y) \oplus {}_X\widehat{\Theta}_r^P(Y)) \xrightarrow{d\oplus d} \Omega_r^{P+1}(Y) \oplus {}_X\widehat{\Omega}_r^{P+1}(Y) \xrightarrow{d\oplus d} \ldots$$
$$\ldots \xrightarrow{d\oplus d} \Omega_r^N(Y) \oplus {}_X\widehat{\Omega}_r^N(Y) \to 0,$$ which is obviously isomorphic to the direct sum
$$0 \to \Omega_r^0(Y) \oplus {}_X\widehat{\Omega}_r^0(Y) \xrightarrow{d\oplus d} (\Omega_r^1(Y)/\Theta_r^1(Y)) \oplus ({}_X\widehat{\Omega}_r^1(Y)/{}_X\widehat{\Theta}_r^1(Y)) \xrightarrow{d\oplus d} \ldots$$
$$\ldots \xrightarrow{d\oplus d} (\Omega_r^P(Y)/\Theta_r^P(Y)) \oplus ({}_X\widehat{\Omega}_r^P(Y)/{}_X\widehat{\Theta}_r^P(Y)) \xrightarrow{d\oplus d} \Omega_r^{P+1}(Y) \oplus {}_X\widehat{\Omega}_r^{P+1}(Y) \xrightarrow{d\oplus d} \ldots$$
$$\ldots \xrightarrow{d\oplus d} \Omega_r^N(Y) \oplus {}_X\widehat{\Omega}_r^N(Y) \to 0.$$

5.5. Proposition. *Let $\pi : Y \to X$ be a fibered manifold, let $\sigma : OrX \to X$ be double cover of X. Then $\mathbf{H}^k(\Gamma(\sigma^*Y, {}^{\sigma^*Y}\nu^r)) \simeq \mathbf{H}^k(\Gamma(Y, {}^Y\nu^r)) \oplus \mathbf{H}^k(\Gamma(Y, {}^Y_X\widehat{\nu}^r))$ for each $k \geq 0$.*

Proof. The assertion follows directly from the Corollary 5.4.

5.6. Proposition. *Let $\pi : Y \to X$ be a fibered manifold. Then $\mathbf{H}^k(\Gamma(Y, {}^Y_X\widehat{\nu}^r)) \simeq {}_X\widehat{\mathbf{H}}^k(Y)$ for each $k \geq 0$.*

Proof. Let $k \geq 0$.
From the Proposition 5.5. we get $\mathbf{H}^k(\Gamma(Y, {}^Y_X\widehat{\nu}^r)) \simeq \mathbf{H}^k(\Gamma(\sigma^*Y, {}^{\sigma^*Y}\nu^r))/\mathbf{H}^k(\Gamma(Y, {}^Y\nu^r))$. In [7] it was shown that $\mathbf{H}^k(\Gamma(\sigma^*Y, {}^{\sigma^*Y}\nu^r)) \simeq \mathbf{H}^k(\sigma^*Y)$, $\mathbf{H}^k(\Gamma(Y, {}^Y\nu^r)) \simeq \mathbf{H}^k(Y)$. Moreover, by Corollary 3.9., $\mathbf{H}^k(\sigma^*Y)/\mathbf{H}^k(Y) \simeq {}_X\widehat{\mathbf{H}}^k(Y)$ so that $\mathbf{H}^k(\Gamma(Y, {}^Y_X\widehat{\nu}^r)) \simeq \mathbf{H}^k(\Gamma(\sigma^*Y, {}^{\sigma^*Y}\nu^r))/\mathbf{H}^k(\Gamma(Y, {}^Y\nu^r)) \simeq \mathbf{H}^k(\sigma^*Y)/\mathbf{H}^k(Y) \simeq {}_X\widehat{\mathbf{H}}^k(Y)$.

5.7. Corollary. *Let $\pi : Y \to X$ be a fibered manifold, let $\sigma : OrX \to X$ be double cover of X. Then $\mathbf{H}^k(\Gamma(Y, {}^Y_X\widehat{\nu}^r)) \simeq \mathbf{H}^k(\sigma^*Y)/\mathbf{H}^k(Y)$ for each k.*

If the manifold X is orientable then $\mathbf{H}^k(\Gamma(Y, {}^Y_X\widehat{\nu}^r)) \simeq \mathbf{H}^k(\Gamma(Y, {}^Y\nu^r)) \simeq \mathbf{H}^k(Y)$ for each $k \geq 0$.

Proof. The assertion follows directly from the Proposition 5.6., Corollary 3.9., and [7].

If the fibered manifold $\pi : Y \to X$ admits the structure of a vector bundle, we get from Corollary 3.10.:

5.8. Corollary. *Let $\pi : Y \to X$ be a vector bundle, let $\sigma : OrX \to X$ be double cover of X.*

(a) If X is orientable then $\mathbf{H}^k(\Gamma(Y, {}^Y_X\widehat{\nu}^r)) \simeq \mathbf{H}^k(X)$ for each $k \geq 0$.

(b) If X is non-orientable then $\mathbf{H}^k(\Gamma(Y, {}^Y_X\widehat{\nu}^r)) \simeq \mathbf{H}^k(OrX)/\mathbf{H}^k(X)$ for each $k \geq 0$.

5.9. Example. (a) Let $\pi : Y \to$ Möb be a vector bundle over the Möbius strip. Then, by Example 3.11., $\mathbf{H}^k(\Gamma(Y, {}^Y_{\text{Möb}}\widehat{\nu}^r)) = 0$ for each k, r.

(b) Let $\pi : Y \to \mathbf{R}P^n$, where n is an even positive integer, be a vector bundle over the non-orientable projective space $\mathbf{R}P^n$. Then, by Example 3.12., for arbitrary r it holds $\mathbf{H}^k(\Gamma(Y, {}^Y_{\mathbf{R}P^n}\widehat{\nu}^r)) = 0$ if $k \neq n$, $\mathbf{H}^n(\Gamma(Y, {}^Y_{\mathbf{R}P^n}\widehat{\nu}^r)) \simeq \mathbf{R}$.

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