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LAGRANGIAN DENSITIES INVARIANT UNDER THE FULL LIE ALGEBRA OF THE INFINITESIMAL AUTOMORPHISMS OF A PRINCIPAL BUNDLE

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Abstract. First order Lagrangian densities on the bundle of connections of a principal bundle invariant under the full Lie algebra of the infinitesimal automorphisms of the principal bundle are classified.

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0. Introduction

Let $\pi : P \longrightarrow M$ be a principal bundle and let $K = K(P) \longrightarrow M$ be the bundle of connections of P. In gauge theories a fundamental result (now known as Utiyama's theorem, see [U] and [G]) is the determination of the Lagrangian densities on $J^1(K)$ which are invariant under the gauge algebra of P.

The aim of this work is to determine all the Lagrangian densities on $J^1(K)$ not only gauge invariant but also invariant under the *full* Lie algebra of the infinitesimal automorphisms of P. Let us briefly describe our goal from a more technical point of view.

Let us denote by $X \longmapsto X_K$ the natural representation of G-invariant vector fields of P on K, and let $Y \longmapsto \overline{Y}$ stand for the 1-jet prolongation of vector fields (see below for more details concerning the notations). Let us fix a volume form v on the base manifold M. A Lagrangian density $\mathcal{L}v$, where $\mathcal{L}$ is a differentiable function on $J^1(K)$, is said to be *gauge invariant* iff $L_{\overline{X}_K}(\mathcal{L}v) = 0$ for every G-invariant π-*vertical* vector field X on P. As X is vertical, $L_{\overline{X}_K}(v) = 0$ and the above condition thus means $\overline{X}_K(\mathcal{L}) = 0$. We are presently interested in determining all the Lagrangian densities $\mathcal{L}v$ such that $L_{\overline{X}_K}(\mathcal{L}v) = 0$ for every G-invariant vector field X on P, not necessarily π-vertical. As is well-known ([K], §2.9, Prop. 6), a G-invariant vector field X on P is projectable onto M, and denoting by X' its projection, the above equation is equivalent to saying $\overline{X}_K(\mathcal{L}) + (\operatorname{div} X')\mathcal{L} = 0$, where $\operatorname{div} X'$ stands for the divergence of X' with respect to v.

As a consequence of our results, it can be proved that the condition "$\overline{X}_K(\mathcal{L}) = 0$ for every G-invariant vector field X on P" implies that $\mathcal{L}$ must be a constant. Nevertheless,

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we shall prove that non-trivial invariant Lagrangian densities do exist whenever $n = dim\, M$ is even, thus showing the importance of considering densities in dealing with the invariance under the full Lie algebra of infinitesimal automorphisms. Actually, each of such invariant Lagrangian densities is associated with a *Weil polynomial* p of degree $\frac{n}{2}$; let us call it $\mathcal{L}_p v$. The action functional defined by $\mathcal{L}_p v$ is not interesting from the variational point of view, in the sense that every section s_Γ is critical, $s_\Gamma : M \longrightarrow K$ being the section induced by a connection Γ on P, but because of this one can obtain theorems of Gauss-Bonnet type ([MS], Appendix C, p.311) for such functionals; for instance, if M is compact, a constant $c_p \in I\!R$ exists such that $\int_M (j^1 s_\Gamma)^*(\mathcal{L}_p v) = c_p$ for every connection Γ on P.

In the abelian case (i.e., for $U(1)$-bundles), up to scalar factors, a unique Weil polynomial of degree $\frac{n}{2}$ exists, $p(t) = t^{\frac{n}{2}}$, and we have $\mathcal{L}_p v = Pf(R_{hi})\, dx_1 \wedge \cdots \wedge dx_n$, where $Pf(R_{hi})$ stands for the Pfaffian of the curvature mapping coordinates, R_{hi}. Hence, in the general case the functions $\mathcal{L}_p$ can be understood as "generalized Pfaffians in the non-abelian setting".

Let us denote by $aut_v(P)$ the Lie algebra of G-invariant vector fields on P having constant divergence with respect to v (i.e., div $X' \in I\!R$) and let $aut_v^0(P)$ be the ideal of divergence-free vector fields in $aut_v(P)$. Also note that $[aut_v(P)\, ,\, aut_v(P)] \subset aut_v^0(P)$. Of course, the gauge algebra of P is contained in each $aut_v^0(P)$, and from our previous discussion it follows that if a Lagrangian density $\mathcal{L}v$ is gauge invariant, then it is also invariant under the larger algebra $aut_v^0(P)$. Moreover, as a consequence of our results one can conclude that a Lagrangian density $\mathcal{L}v$ is invariant under the full Lie algebra of infinitesimal automorphisms if and only if it is invariant under $aut_v(P)$. (We should more properly speak of invariance under the sheaf of germs $\mathcal{S}(U) = aut_v(P_U)$, because unlike the gauge algebra and the full algebra of infinitesimal automorphisms —which are the global sections of suitable vector bundles— the sheaf $\mathcal{S}$ may have no global section with non-vanishing divergence.) This basic fact measures the difference between gauge invariance and invariance under the full Lie algebra of infinitesimal automorphisms.

1. Preliminaries and notations

Most of notations have been taken from [M], where we send the reader for further information.

1.1. ad P and aut P. Let G be the structure group of $\pi : P \longrightarrow M$ and let be $\mathfrak{g}$ the Lie algebra of G. Let us denote by $\pi_G : Ad\, P \longrightarrow M$ the bundle associated to P by the *conjugation action* of G on itself, and by $\pi_{\mathfrak{g}} : ad\, P \longrightarrow M$ the bundle associated to P by the *adjoint representation* of G on $\mathfrak{g}$. We set $|u, \sigma|$ = orbit of $(u, \sigma) \in P \times G$ in $Ad\, P = (P \times G)/G$, and similarly $|u, A|$ = orbit of $(u, A) \in P \times \mathfrak{g}$ in $ad\, P = (P \times \mathfrak{g})/G$. The rule $|u, \sigma| \cdot |u, \tau| = |u,\, \sigma \cdot \tau|$ induces a Lie group structure on the fibres of $Ad\, P$, and similarly the rule $[|u, A|, |u, B|] = |u,\, [A, B]|$ induces a Lie algebra structure on the fibres of $ad\, P$. Note that for every $x \in M$, the fibre $\pi_{\mathfrak{g}}^{-1}(x) = (ad\, P)_x$ is the Lie algebra of the Lie group $\pi_G^{-1}(x) = (Ad\, P)_x$.

Moreover, $\Gamma(M, ad\, P)$ endowed with the corresponding Lie algebra structure, can be identified with the gauge algebra of P. Actually, we can associate to each section $\xi : M \longrightarrow ad\, P$ a G-invariant vertical vector field X_ξ on P by setting $X_\xi(u) = A_u^*$, where A^* stands for the *fundamental vector field* associated to $A \in \mathfrak{g}$ and $\xi(\pi(u)) = |u, A|$.

An *automorphism* of P is a diffeomorphism $\phi : P \longrightarrow P$ such that $\phi(u\sigma) = \phi(u)\sigma$, for every $u \in P, \sigma \in G$. Each ϕ induces a diffeomorphism $\phi' : M \longrightarrow M$ such that

$\pi \circ \phi = \phi' \circ \pi$. Assume a vector field X on P generates the 1-parameter group ϕ_t. Then, X is G-invariant if and only if every ϕ_t is an automorphism of P. Accordingly, we denote by $aut\,P$ the Lie algebra of G-invariant vector fields of P, and we think of $aut\,P$ as being the Lie algebra of the infinite Lie group $Aut\,P$ (cf. [GS]). There is a unique vector bundle $T_G(P) \longrightarrow M$ such that $aut\,P_U = \Gamma(U, T_G(P))$ for every open subset $U \subset M$, and the following exact sequence of vector bundles on M holds ([A]): $0 \longrightarrow ad\,P \longrightarrow T_G(P) \longrightarrow T(M) \longrightarrow 0$ (the so-called, *Atiyah sequence*).

1.2. Local expression of G-invariant vector fields. Assume $n = dim\,M$, $m = dim\,G$ and let $(U; x_1, ..., x_n)$, $(A_1, ..., A_m)$ be an open coordinate domain of M trivializing P (i.e., $P_U = \pi^{-1}(U) \approx U \times G$), and a basis of $\mathfrak{g}$, respectively. We define a 1-parameter group of gauge transformations by $\phi_t^j(x, \sigma) = (x, exp(tA_j)\sigma)$, $x \in U, \sigma \in G, 1 \leq j \leq m$. The corresponding vector field will be denoted by $\tilde{A}_j$. We have, $[\tilde{A}_j, \tilde{A}_k] = -\sum_{i=1}^m c_{jk}^i \tilde{A}_i$, where c_{jk}^i are the structure constants of $\mathfrak{g}$ in the basis $A_1, ..., A_m$. More generally, for every differentiable mapping $g : U \longrightarrow \mathfrak{g}$ and every 1-parameter group ϕ'_t of U we can define a 1-parameter family of automorphisms of P_U by setting $\phi_t(x, \sigma) = (\phi'_t(x), exp(t\,g(x))\sigma)$, and the vector field X that ϕ_t induces is

$$X = \sum_{i=1}^n f_i(\partial/\partial x_i) + \sum_{j=1}^m g_j \tilde{A}_j \quad \left(f_i(x) = (\partial \phi'_i/\partial t)(0, x)\,, g(x) = \sum_{j=1}^m g_j(x) A_j,\, x \in U \right).$$

This is the local expression for an arbitrary G-invariant vector field. Also note that if $\xi : U \longrightarrow ad\,P$ is the section defined by $g : U \longrightarrow \mathfrak{g}$ (i.e., $\xi(x) = |(x,1), g(x)|$), the corresponding gauge field X_ξ is given by $X_\xi = \sum_{j=1}^m g_j \tilde{A}_j$. In particular, if $\xi_j : U \longrightarrow ad\,P$ is the section defined by the constant map $A_j : U \longrightarrow \mathfrak{g}$, then $X_{\xi_j} = \tilde{A}_j$.

2. The representation of $aut\,P$ on $J^1(K)$

2.1. Representing aut P in $\mathfrak{X}$(K). The group $Aut\,P$ acts on the connections of P by pulling back connection forms: $\phi \cdot \omega = (\phi^{-1})^* \omega$ ([KN], Prop.6.2.(b)). It is checked that $(\phi^{-1})^* \omega_{|\pi^{-1}(\phi'(x))}$ only depends on $\omega_{|\pi^{-1}(x)}$. Hence, $Aut\,P$ also acts on $K = K(P)$. Assume ϕ_t is the 1-parameter group of a G-invariant vector field X. We denote by ϕ_t^K the 1-parameter group induced by ϕ_t acting on K, and by X_K the corresponding vector field. In this way, we obtain a representation of $aut\,P$ on K, $X \longmapsto X_K$.

2.2. Local expression of $\mathbf{X} \longmapsto \mathbf{X_K}$. We recall that the sections of K can be identified with the splittings of the Atiyah sequence as follows: $s_\Gamma : T(M) \longrightarrow T_G(P)$, $s_\Gamma(X) = X^*$, the horizontal lift of X with respect to Γ. With the same notations as above, we can introduce coordinates on K, $(K_U\,; x_i, y_{ij})$, $1 \leq i \leq n$, $1 \leq j \leq m$, by setting $s_\Gamma(\partial/\partial x_i) = \partial/\partial x_i + \sum_{j=1}^m (y_{ij} \circ s_\Gamma)\tilde{A}_j$. Locally, $\phi_t = (\phi'_t, L_{exp(tg)})$ (here, L stands for *left translation*), and according to our definitions,

$$\phi_t^K(s_\Gamma(x))(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_i} + \sum_{j=1}^m y_{ij}\left(\phi_t^K(s_\Gamma(x))\right)\tilde{A}_j,$$

$$\begin{aligned} 0 &= (\phi_{-t}^* \omega_\Gamma)\left(\phi_t^K(s_\Gamma(x)(\partial/\partial x_i)_{\phi'_t(x)}\right) \\ &= \omega_\Gamma \left((\phi_{-t})_*(\partial/\partial x_i)_{(\phi'_t(x),\sigma)} + \sum_{j=1}^m y_{ij}\left(\phi_t^K(s_\Gamma(x))\right)(\phi_{-t})_*(\tilde{A}_j)_{(\phi'_t(x),\sigma)} \right). \end{aligned}$$

Taking derivatives with respect to t and letting $t = 0$, the above equation yields:

$$0 = \left([X, \partial/\partial x_i]_{(x,\sigma)} + (X_K y_{ij})(s_\Gamma(x))(\tilde{A}_j)_{(x,\sigma)} + y_{ij}(s_\Gamma(x))[X, \tilde{A}_j]_{(x,\sigma)} \right).$$

Taking into account that $\omega_\Gamma \left(\partial/\partial x_i + \sum_{j=1}^m (y_{ij} \circ s_\Gamma)\tilde{A}_j \right) = 0$, and developing the Lie brackets in the previous equation we obtain,

$$0 = \omega_\Gamma \left(\sum_{j=1}^m \left(\sum_{h=1}^n \frac{\partial f_h}{\partial x_i} y_{hj} - \frac{\partial g_j}{\partial x_i} + X_K(y_{ij}) - \sum_{k,l=1}^m c_{kl}^j g_k y_{il} \right) \tilde{A}_j \right),$$

thus proving the vector field in brackets is horizontal. Since it is also vertical, it must vanish. Hence,

$$X_K = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i} + \sum_{i=1}^n \sum_{j=1}^m \left(\frac{\partial g_j}{\partial x_i} + \sum_{k,l=1}^m c_{kl}^j g_k y_{il} - \sum_{h=1}^n \frac{\partial f_h}{\partial x_i} y_{hj} \right) \frac{\partial}{\partial y_{ij}}.$$

2.3. Jet prolongation. Let us denote by $(x_i, y_{ij}, y_{ij,h})$, $1 \leq h, i \leq n$, $1 \leq j \leq m$, the natural coordinate system induced by (x_i, y_{ij}) on $J^1(K)$; i.e., $y_{ij,h}(j_x^1 s_\Gamma) = (\partial(y_{ij} \circ s_\Gamma)/\partial x_h)(x)$. The general formulas for vector field jet prolongation lead us to the following expression for the prolongation of X_K to $J^1(K)$:

$$\begin{aligned} \overline{X}_K = & \sum_{i=1}^n f_i \frac{\partial}{\partial x_i} + \sum_{i=1}^n \sum_{j=1}^m \left(\frac{\partial g_j}{\partial x_i} + \sum_{k,l=1}^m c_{kl}^j g_k y_{il} - \sum_{h=1}^n \frac{\partial f_h}{\partial x_i} y_{hj} \right) \frac{\partial}{\partial y_{ij}} \\ & + \sum_{i,u=1}^n \sum_{j=1}^m \left(\frac{\partial^2 g_j}{\partial x_i \partial x_u} - \sum_{h=1}^n \left(\frac{\partial f_h}{\partial x_i} y_{hj,u} + \frac{\partial f_h}{\partial x_u} y_{ij,h} + \frac{\partial^2 f_h}{\partial x_i \partial x_u} y_{hj} \right) \right. \\ & \left. \qquad + \sum_{k,l=1}^m c_{kl}^j \left(g_k y_{il,u} + \frac{\partial g_k}{\partial x_u} y_{il} \right) \right) \frac{\partial}{\partial y_{ij,u}}. \end{aligned}$$

Remark. If in the above formulas we consider the particular case $f_i = 0$, we obtain the usual representation of the gauge algebra on the bundle of connections and its corresponding jet prolongation (cf. [G]).

2.4. The curvature mapping. The *curvature form* Ω_Γ of a connection Γ on P is a tensorial 2-form of type $ad\, G$ and hence it can be viewed as a 2-form on M with values in $ad\, P$ ([KN], II, *Example* 5.2.); i.e., $\Omega_\Gamma \in \Gamma(M, \bigwedge^2 T^*(M) \otimes ad\, P)$. Accordingly, we can define a map of fibred manifolds over M, called *the curvature mapping*,

$$\Omega : J^1(K) \longrightarrow \bigwedge^2 T^*(M) \otimes ad\, P,\ \Omega\left(j_x^1(s_\Gamma)\right) = \Omega_\Gamma(x).$$

Let $(U; x_1, ..., x_n)$, $(A_1, ..., A_m)$, $\xi_j : U \longrightarrow ad\, P$ be as in §1.2. We can define a coordinate system (x_i, R_{hij}), $1 \leq h < i \leq n$, $1 \leq j \leq m$, on $\bigwedge^2 T^*(U) \otimes (ad\, P)_U$ as follows: $\eta_2 = \sum_{1 \leq h < i \leq n} \sum_{j=1}^m R_{hij}(\eta_2) dx_h \wedge dx_i \otimes \xi_j$, $\eta_2 \in \bigwedge^2 T^*(U) \otimes (ad\, P)_U$. For $h \geq i$ we shall also set: $R_{hij} = -R_{ihj}$. By considering in $J^1(K)$ the coordinate system introduced in §2.3, it is checked that the equations for Ω are:

$$R_{ilj} \circ \Omega = y_{lj,i} - y_{ij,l} + \sum_{1 \leq h < k \leq m} c^j_{hk}(y_{lh}y_{ik} - y_{ih}y_{lk}).$$

Utiyama's theorem can be stated geometrically by saying (cf. [G]) that: *a Lagrangian function $\mathcal{L} : J^1(K) \longrightarrow I\!R$ is gauge invariant if and only if $\mathcal{L}$ factors through Ω by means of a function $\overline{\mathcal{L}} : \bigwedge^2 T^*(M) \otimes ad\, P \longrightarrow I\!R$ which is invariant under the natural representation of the gauge algebra on $\bigwedge^2 T^*(M) \otimes ad\, P$.* Let us recall how this representation is defined ([M], §5). For every $\sigma \in G, u \in P, A \in \mathfrak{g}, \omega_2 \in \bigwedge^2 T^*_x(M), x = \pi(u)$ we have: $|u, \sigma| \cdot (\omega_2 \otimes |u, A|) = \omega_2 \otimes |u, Ad_\sigma(A)|$. Let us denote by $V(N)$ the *vertical bundle* of a fibred manifold $N \longrightarrow M$. As $(ad\, P)_x$ is the Lie algebra of $(Ad\, P)_x$ (cf. §1.1), we can define a homomorphism of vector bundles over $\bigwedge^2 T^*(M) \otimes ad\, P$,

$$\rho : \left(\bigwedge^2 T^*(M) \otimes ad\, P \right) \times_M ad\, P \longrightarrow V \left(\bigwedge^2 T^*(M) \otimes ad\, P \right)$$

by setting: $\rho(\eta_2, A_x) = ({}^*A_x)_{\eta_2}$, where $\eta_2 \in \bigwedge^2 T^*_x(M) \otimes (ad\, P)_x, A_x \in (ad\, P)_x$ and ${}^*A_x$ is the vector field tangent to the fibre of $\bigwedge^2 T^*(M) \otimes ad\, P$ over x generating the 1-parameter group $L_{exp(t\, A_x)}$.

The image of ρ is an involutive differential system $\mathcal{M}$ with singularities; actually, $\rho(\xi_k) = \sum c^j_{kl} R_{hil}(\partial / \partial R_{hij})$ (cf.[G], formula (5.1)), and the determination of all gauge invariant Lagrangians is reduced to calculating the ring $\mathcal{R}$ of first integrals of $\mathcal{M}$. In spite of the singularities, it is proved in [M] that $\mathcal{R}$ is "differentiably finitely-generated".

3. *aut* P-invariant Lagrangian density associated to a Weil polynomial

Definition. Assume $n = dim\, M$ is even, and let p be an invariant polynomial of degree $\frac{n}{2}$ with respect to the coadjoint representation of G on $\mathfrak{g}^*$; i.e., $p \in S^{n/2}(\mathfrak{g})^G$. For each $k \in I\!N$, the polynomial p gives rise to a function $\overline{p}_k : \bigoplus^k ad\, P \longrightarrow I\!R$ by setting $\overline{p}_k(|u, A_1|, ..., |u, A_k|) = p(A_1, ..., A_k)$. Let $\overline{\lambda}_p : \bigwedge^2 T^*(M) \otimes ad\, P \longrightarrow \bigwedge^n T^*(M)$ be the mapping defined by:

$$\overline{\lambda}_p(\eta_2)(X_1, ..., X_n) = \sum_{\sigma \in S_n} \epsilon_\sigma \overline{p}_{n/2} \left(\eta_2 \big(X_{\sigma(1)}, X_{\sigma(2)} \big), ..., \eta_2 \big(X_{\sigma(n-1)}, X_{\sigma(n)} \big) \right),$$

where $\eta_2 \in \bigwedge^2 T^*_x(M) \otimes (ad\, P)_x$, $X_i \in T_x(M)$, $1 \leq i \leq n$. Once a volume form v on M has been fixed, $\overline{\lambda}_p = \overline{\mathcal{L}}_p v$ for a certain differentiable function $\overline{\mathcal{L}}_p : \bigwedge^2 T^*(M) \otimes ad\, P \longrightarrow I\!R$. Let $\Omega : J^1(K) \longrightarrow \bigwedge^2 T^*(M) \otimes ad\, P$ be the curvature mapping (cf. §2.4). The composition $\mathcal{L}_p v = (\overline{\mathcal{L}}_p \circ \Omega) v$ is said to be the *Lagrangian density associated with the Weil polynomial p*.

Theorem 3.1. *The Lagrangian density associated with a Weil polynomial of degree $\frac{n}{2}$ is aut P-invariant.*

Proof. As $\mathcal{L}_p$ factors through Ω by its very definition, from Utiyama's theorem it follows that $\mathcal{L}_p v$ is gauge invariant. Let $\mathcal{C}_U$ be the set of G-invariant vector fields X whose projection X' onto M has a compact support contained in the open subset $U \subset M$. Taking into account that $aut\,P = \Gamma(M, T_G(P))$ (see §1.1), the following properties can be easily verified:

(1) $\mathcal{C}_M$ is dense in $aut\,P$.
(2) Every $X \in \mathcal{C}_M$ is a finite sum of vector fields $X_i \in \mathcal{C}_{U_i}$ so that U_i is an open coordinate domain trivializing P.
(3) If P is trivial on U and X^h stands for the horizontal lift of a vector field $X \in \mathfrak{X}(U)$ with respect to the flat connection on P_U, then for every $X \in aut\,P$ the difference $X - X'^h$ belongs to the gauge algebra; i.e., $(X - X'^h) \in \Gamma(U, ad\,P)$.

Since $\mathcal{L}_p v$ is gauge invariant, by applying (1), (2) and (3) successively it is clear that it will be sufficient to prove that

(A) For every $X \in \mathfrak{X}(U)$ with compact support, $L_{\overline{X}^h_K}(\mathcal{L}_p v) = 0$.

Note that the 1-parameter group generated by the vector field X^h is $\Phi_t(x,\sigma) = (\phi_t(x), \sigma)$, $x \in U$, $\sigma \in G$, where ϕ_t is the 1-parameter group generated by X.

As $ad\,P$ is a bundle associated to P, $\bigwedge^2 T^*(U) \otimes ad\,P_U = \bigwedge^2 T^*(U) \otimes \mathfrak{g}$. Hence, the sections of $\bigwedge^2 T^*(U) \otimes ad\,P_U$ can be viewed as $\mathfrak{g}$-valued 2-forms on U. For a differentiable mapping $\phi : M' \longrightarrow U$ and $\eta_2 \in \bigwedge^2 T^*(U) \otimes \mathfrak{g}$, $x = \phi(x')$, we can thus define $\phi^*\eta_2 \in \bigwedge^2 T^*_{x'}(M') \otimes \mathfrak{g}$ by: $(\phi^*\eta_2)(X'_1, X'_2) = \eta_2(\phi_* X'_1, \phi_* X'_2)$. Moreover, we have:

(B) The group of diffeomorphisms of U acts on the set of fibred mappings over U, $\overline{\lambda} : \bigwedge^2 T^*(U) \otimes ad\,P_U \longrightarrow \bigwedge^n T^*(U)$, $\overline{\lambda} = \overline{\mathcal{L}} v$, as follows: $(\phi \cdot \overline{\lambda})(\eta_2) = \overline{\mathcal{L}}(\phi^*\eta_2)(\phi^{-1})^* \left(v_{\phi^{-1}(x)}\right)$. Then, $\overline{\lambda}_p$ is invariant under this action.

Proof of (B). Assume $p = \sum c_{j_1,\dots,j_{n/2}} A^*_{j_1} \cdots A^*_{j_{n/2}}$, where $A^*_1, \dots, A^*_m$ is the dual basis of $A_1, \dots, A_m$ and the dots stand for the symmetric product. For every $\eta_2 = \sum_{j=1}^m \omega_2^j \otimes \xi_j(x)$ (recall: $\xi_j(x) = |(x,1), A_j|$, §1.2), $\omega_2^j \in \bigwedge^2 T^*_x(M)$, we have, $\overline{\lambda}_p(\eta_2) = \sum c_{j_1,\dots,j_{n/2}} \omega_2^{j_1} \wedge \dots \wedge \omega_2^{j_{n/2}} = \overline{\mathcal{L}}_p(\eta_2) v$. Hence, $\overline{\lambda}_p(\phi^*\eta_2) = \sum c_{j_1,\dots,j_{n/2}} \left(\phi^*\omega_2^{j_1}\right) \wedge \dots \wedge \left(\phi^*\omega_2^{j_{n/2}}\right) = \overline{\mathcal{L}}_p(\phi^*\eta_2) v_{\phi^{-1}(x)}$, and thus, $(\phi^{-1})^*\left(\overline{\lambda}_p(\phi^*\eta_2)\right) = \overline{\lambda}_p(\eta_2) = \overline{\mathcal{L}}_p(\phi^*\eta_2)$
$(\phi^{-1})^* v_{\phi^{-1}(x)} = (\phi \cdot \overline{\lambda}_p)(\eta_2)$.

According to (B), in order to prove (A) we only need to state the following

Lemma 3.2. *Assume U is an open coordinate domain of M on which P is trivial. A mapping of fibred manifolds over U, $\overline{\lambda} : \bigwedge^2 T^*(U) \otimes ad\,P_U \longrightarrow \bigwedge^n T^*(U)$, $\overline{\lambda} = \overline{\mathcal{L}} v$, is invariant under every 1-parameter group of diffeomorphisms of U (i.e., $\phi_t \cdot \overline{\lambda} = \overline{\lambda}$) if and only if for every $X \in \mathfrak{X}(U)$ we have, $L_{\overline{X}^h_K}\left((\overline{\mathcal{L}} \circ \Omega) v\right) = 0$.*

Proof of Lemma 3.2. Let us consider a coordinate system $(x_1, \dots, x_n)$ on U such that $v = dx_1 \wedge \dots \wedge dx_n$.We have $X^h = \sum_{i=1}^n f_i(\partial/\partial x_i)$, and in this case the formula for $\overline{X}_K$

in §2.3 reads as:

$$\overline{X}_K^h = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} - \sum_{h,i=1}^{n} \sum_{j=1}^{m} \left(\frac{\partial f_h}{\partial x_i} y_{hj} \right) \frac{\partial}{\partial y_{ij}} - \sum_{h,i,u=1}^{n} \sum_{j=1}^{m} \left(\frac{\partial f_h}{\partial x_i} y_{hj,u} + \frac{\partial f_h}{\partial x_u} y_{ij,h} + \frac{\partial^2 f_h}{\partial x_i \partial x_u} y_{hj} \right) \frac{\partial}{\partial y_{ij,u}}.$$

Hence, $L_{\overline{X}_K^h}\left((\overline{\mathcal{L}} \circ \Omega)v\right) = 0$ for every $X \in \mathfrak{X}(U)$ if and only if,

$$\frac{\partial \mathcal{L}}{\partial x_i} = 0 \;\; ; \;\; \sum_{j=1}^{m} y_{hj} \left(\frac{\partial \mathcal{L}}{\partial y_{ij,u}} + \frac{\partial \mathcal{L}}{\partial y_{uj,i}} \right) = 0,\ 1 \leq h, i, u \leq n,$$

$$\sum_{u=1}^{n} \sum_{j=1}^{m} \left(y_{hj} \frac{\partial \mathcal{L}}{\partial y_{ij}} + y_{hj,u} \frac{\partial \mathcal{L}}{\partial y_{ij,u}} + y_{uj,h} \frac{\partial \mathcal{L}}{\partial y_{uj,i}} \right) = 0,\ 1 \leq h \neq i \leq n,$$

$$\mathcal{L} = \sum_{j=1}^{m} \left(y_{ij} \frac{\partial \mathcal{L}}{\partial y_{ij}} + \sum_{u=1}^{n} \left(y_{ij,u} \frac{\partial \mathcal{L}}{\partial y_{ij,u}} + y_{uj,i} \frac{\partial \mathcal{L}}{\partial y_{uj,i}} \right) \right),\ 1 \leq i \leq n,$$

where we have set: $\mathcal{L} = \overline{\mathcal{L}} \circ \Omega$.

Moreover, as a computation shows, from the equations of the curvature mapping (cf. §2.4) we can conclude that the above four equations for $\mathcal{L}$ are equivalent to the following three equations for $\overline{\mathcal{L}}$,

$$\text{(I)} \qquad \frac{\partial \overline{\mathcal{L}}}{\partial x_i} = 0$$

$$\text{(II)} \qquad \sum_{a<i} \sum_{b=1}^{m} R_{ahb} \frac{\partial \overline{\mathcal{L}}}{\partial R_{aib}} = \sum_{i<a} \sum_{b=1}^{m} R_{ahb} \frac{\partial \overline{\mathcal{L}}}{\partial R_{iab}},\ 1 \leq h \neq i \leq n$$

$$\text{(III)} \qquad \overline{\mathcal{L}} = \sum_{a<i} \sum_{b=1}^{m} R_{aib} \frac{\partial \overline{\mathcal{L}}}{\partial R_{aib}} + \sum_{i<a} \sum_{b=1}^{m} R_{iab} \frac{\partial \overline{\mathcal{L}}}{\partial R_{iab}},\ 1 \leq i \leq n$$

Furthermore, by taking derivatives in the invariance condition $\phi_t \cdot (\overline{\mathcal{L}} v) = \overline{\mathcal{L}} v$ and letting $t = 0$, we obtain: $-\tilde{X}_2(\overline{\mathcal{L}})v - \overline{\mathcal{L}} \left(\sum_{i=1}^{n} (\partial f_i / \partial x_i) \right) v = 0$, where $\tilde{X}_2$ stands for the vector field induced by ϕ_t on $\bigwedge^2 T^*(M) \otimes \mathfrak{g}$, whose local expression is

$$\tilde{X}_2 = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} - \sum_{h,i=1}^{n} \sum_{j=1}^{m} \left(\sum_{a=1}^{n} R_{aij} \frac{\partial f_a}{\partial x_h} + \sum_{b=1}^{n} R_{hbj} \frac{\partial f_b}{\partial x_i} \right) \frac{\partial}{\partial R_{hij}},$$

thus finishing the proof of the lemma and completing that of the theorem.

4. Determining the $aut\,P$-invariant Lagrangian densities

Theorem 4.1. *Let $\pi : P \longrightarrow M$ be a principal fibre bundle with structure group G. Assume both G and M are connected, and M is oriented by a volume form v. We have:*
(i) *If $n = dim\,M$ is odd, the the unique $aut\,P$-invariant Lagrangian density is the zero density.*
(ii) *If $n = dim\,M$ is even, then the only $aut\,P$-invariant Lagrangian densities are those associated with Weil polynomials of degree $\frac{n}{2}$.*
(iii) *If M is compact, a constant $c_p \in I\!R$ exists only depending on $p \in S^{n/2}(\mathfrak{g}^*)^G$ such that for every connection Γ on P,*

$$\int_M \left(j^1 s_\Gamma\right)^* \left(\overline{\mathcal{L}}_p \circ \Omega\right) v = c_p.$$

Proof. Let $\mathcal{L}v$ be an $aut\,P$-invariant Lagrangian density. Since $\mathcal{L}v$ is gauge invariant, from Utiyama's theorem we obtain $\mathcal{L} = \overline{\mathcal{L}} \circ \Omega$, where $\overline{\mathcal{L}} : \bigwedge^2 T^*(M) \otimes ad\,P \longrightarrow I\!R$ is a differentiable function invariant under the natural representation of the gauge algebra. As G is connected, $\overline{\mathcal{L}}$ is also $Ad\,P$-invariant. Moreover, as $\mathcal{L}v$ is $aut\,P$-invariant, $\overline{\mathcal{L}}$ satisfies equations (I), (II), (III) in the proof of Lemma 3.2. Equation (I) thus shows that $\overline{\mathcal{L}}$ only depends on the variables (R_{hij}), $1 \le h < i \le n$, $1 \le j \le m$. First, let us consider the following elementary result:

Lemma 4.2. *Let N be a differentiable manifold. If $f \in \mathcal{C}^\infty(I\!R^n \times N)$ verifies $f = \sum_{i=1}^n x_i(\partial f/\partial x_i)$, then $\partial f/\partial x_i \in \mathcal{C}^\infty(N)$, $1 \le i \le n$.*

Proof of Lemma 4.2. As in the usual case we have: $f(tx, y) = t\,f(x, y)$ for every $t \in I\!R$, $x \in I\!R^n$, $y \in N$. Hence, $f(tx, y) = t\,f(x, y) = \sum_{i=1}^n tx_i(\partial f/\partial x_i)(tx, y)$. Thus, $f(x, y) = \sum_{i=1}^n x_i(\partial f/\partial x_i)(tx, y)$, and letting $t = 0$, we obtain $f = \sum_{i=1}^n x_i f_i$ for some $f_i \in \mathcal{C}^\infty(N)$. Since $\partial f_h/\partial x_i = 0$, we have $\partial f/\partial x_i = f_i$.

Taking this lemma into account, equation (III) proves that $\overline{\mathcal{L}}$ depends linearly on each of the n systems of variables (R_{aib}, R_{ihb}), $1 \le a \le i-1$, $i+1 \le h \le n$, $1 \le b \le m$, the index i being fixed. Consequently, $\overline{\mathcal{L}}$ is a finite sum of monomials

$$\lambda_{hij}\, R_{h_1 i_1 j_1} \cdots R_{h_k i_k j_k},\ h_a < i_a,\ 1 \le a \le k,$$

such that,
1) $\lambda_{hij} \neq 0$; 2) the indices $h_1, i_1, ..., h_k, i_k$ are pairwise different; 3) $\{1, ..., n\} = \{h_1, i_1, ..., h_k, i_k\}$. Hence, $n = 2k$, thus forcing n to be even.

Moreover, equation (II) implies that the coefficients λ_{hij} should be skew-symmetric in the indices h, i; i.e., $\lambda_{hij} = \epsilon_{h,i} c_j$, where $\epsilon_{h,i}$ stands for the sign of the permutation $(1, 2, ..., n-1, n) \longmapsto (h_1, i_1, ..., h_k, i_k)$. With the coefficients c_j we can construct a polynomial p as in the proof of Theorem 3.1. Since $\overline{\mathcal{L}}$ is $Ad\,P$-invariant, it is clear that p is $ad\,G$-invariant and by virtue of the above construction we have $\overline{\mathcal{L}} = \overline{\mathcal{L}}_p$.

This proves the first two parts of the statement. As for the third, by using the same notations as in [KN], XII, §1, it is easy to see that for every connection Γ on P we have: $(j^1 s_\Gamma)^* \left(\overline{\mathcal{L}}_p \circ \Omega\right) v = \overline{p}(\Omega_\Gamma)$, where $\overline{p}(\Omega_\Gamma)$ is the unique n-form on M such that, $\pi^*(\overline{p}(\Omega_\Gamma)) = p(\Omega_\Gamma)$ (cf. [KN], XII, Th.1.1). Hence, $\int_M (j^1 s_\Gamma)^* \left(\overline{\mathcal{L}}_p \circ \Omega\right) v = \int_M \overline{p}(\Omega_\Gamma)$, and it is known ([KN], XII, Th.1.1.(2)) that $\overline{p}(\Omega_\Gamma)$ is a closed n-form whose cohomology class in $H^n(M; I\!R)$ does not depend on the connection chosen.

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