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ON A SPECIAL CLASS OF HERMITIAN MANIFOLDS

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Abstract

In this paper, we study a particular class of hermitian manifolds which we call special c'-hyperbolic hermitian manifolds with $c' \in \mathbb{R}$, $c' \neq 0$. As main result, we prove that the universal covering space of a special c'-hyperbolic hermitian manifold is the product of a c-sasakian manifold with a hyperbolic space of odd dimension and of constant sectional curvature $-(c')^2$.

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I Introduction

An almost hermitian manifold V is called locally conformal Kähler if its metric is conformally related to a Kähler metric in some neighbourhood of every point of V (see [V1]). Examples of locally conformal Kähler manifolds are provided by the generalized Hopf manifolds which are locally conformal Kähler manifolds with parallel Lee form (see [V2]). The main non-Kähler examples of such manifolds are the Hopf manifolds (see [KN] and [V1]), which are defined as the quotients

$$H_o^n = \frac{(\mathbb{C}^n - \{0\})}{\Delta_\lambda},$$

being Δ_λ a cyclic group of transformations, and the nilmanifold $N(r,1) \times S^1$, where $N(r,1) = \Gamma(r,1) \setminus H(r,1)$ is a compact quotient of the generalized Heisenberg group $H(r,1)$ by a discrete subgroup $\Gamma(r,1)$ (see [CFL]). The product of a c-sasakian manifold N with the $(2m+1)$-dimensional hyperbolic space H_c^{2m+1} of constant sectional curvature $-c^2$ also admits a locally conformal Kähler structure (J,g), where g is the product metric (see [MR]). In [MR] we have studied a particular class of locally conformal Kähler manifolds with similar properties to the locally conformal Kähler manifold $N \times H_c^{2m+1}$.

On the other hand if c and c' are arbitrary constants, c, $c' \neq 0$, then the almost hermitian structure (J,g) on the manifold $N \times H_{c'}^{2m+1}$ is hermitian but, in general, it is not locally conformal Kähler (see corollary III.1). In this paper we study a particular class of hermitian manifolds which we call special c'-hyperbolic hermitian manifolds,

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with $c' \in I\!R$, $c' \neq 0$. These manifolds have similar properties to the product $N \times H_{c'}^{2m+1}$. We generalize some facts obtained in [MR] and, as main result, we prove that the universal covering space of a special c'-hyperbolic hermitian manifold is almost complex isometric to the product of a c-sasakian manifold with a hyperbolic space of odd dimension and of constant sectional curvature $-(c')^2$ (see theorem VI.1).

II Preliminaries

Let V be a C^∞ almost hermitian manifold with metric g and almost complex structure J. Denote by $\mathfrak{X}(V)$ the Lie algebra of C^∞ vector fields on V. The almost hermitian manifold (V, J, g) is said to be: Hermitian if $N_J = 0$, where N_J denotes the Nijenhuis tensor of J; Locally conformal Kähler if it is hermitian, $d\omega = 0$ and $d\Omega = \omega \wedge \Omega$, where ω and Ω are the Lee 1-form and the Kähler 2-form respectively of V (see [KN] and [V1]).

On the other hand if M is an almost contact metric manifold with metric g and almost contact structure (φ, ξ, η) then it is said to be c'-kenmotsu, where $c' \in I\!R$, $c' \neq 0$, if $N_\varphi + 2d\eta \otimes \xi = 0$, $d\phi = -2c'\eta \wedge \phi$ and $d\eta = 0$ being N_φ the Nijenhuis tensor of φ and ϕ the fundamental 2-form of M (see [JV]).

Let $(H_{c'}^{2m+1}, (ds^2)_{c'})$ be the $(2m+1)$-dimensional hyperbolic space, i.e.,

$$H_{c'}^{2m+1} = \{(x_1, \ldots, x_{2m+1}) \in I\!R^{2m+1} \ / \ x_{2m+1} > 0\}$$

and $(ds^2)_{c'}$ is the Riemannian metric on $H_{c'}^{2m+1}$ given by

$$(ds^2)_{c'} = \frac{1}{(c'x_{2m+1})^2} \sum_{i=1}^{2m+1} (dx_i)^2, \quad (c' \neq 0).$$

$(H_{c'}^{2m+1}, (ds^2)_{c'})$ is a complete simply connected Riemannian manifold with constant sectional curvature $-(c')^2$. Moreover, if $(\varphi_{H_{c'}^{2m+1}}, \xi_{H_{c'}^{2m+1}}, \eta_{H_{c'}^{2m+1}}, (ds^2)_{c'})$ is the almost contact metric structure on $H_{c'}^{2m+1}$ defined by

$$\begin{aligned} \varphi_{H_{c'}^{2m+1}} &= (c'x_{2m+1}) \sum_{i=1}^{m} \Big(\frac{\partial}{\partial x_i} \otimes \alpha_{m+i} - \frac{\partial}{\partial x_{m+i}} \otimes \alpha_i \Big), \\ \xi_{H_{c'}^{2m+1}} &= (c'x_{2m+1}) \frac{\partial}{\partial x_{2m+1}}, \qquad \eta_{H_{c'}^{2m+1}} = \frac{dx_{2m+1}}{(c'x_{2m+1})}, \end{aligned} \tag{II.1}$$

where α_i, $i \in \{1, \ldots, 2m\}$, are the 1-forms on $H_{c'}^{2m+1}$ given by

$$\alpha_i = \frac{dx_i}{(c'x_{2m+1})}, \tag{II.2}$$

then $(\varphi_{H_{c'}^{2m+1}}, \xi_{H_{c'}^{2m+1}}, \eta_{H_{c'}^{2m+1}}, (ds^2)_{c'})$ is a c'-kenmotsu structure on $H_{c'}^{2m+1}$ (see [ChG]).

An almost contact metric manifold $(N, \varphi, \xi, \eta, g)$ is said to be c-sasakian (see [JV]), with $c \in I\!R$, $c \neq 0$ if $N_\varphi + 2d\eta \otimes \xi = 0$ and $d\eta = c\phi$. In [MR] we have proved that if $(N, \varphi, \xi, \eta, g)$ and $(N', \varphi', \xi', \eta', g')$ are $(2n-1)$-dimensional complete simply connected c-sasakian manifolds of constant φ-sectional curvature k then, N is almost contact isometric to N'. In fact, the unit sphere S^{2n-1} has a c-sasakian structure of constant φ-sectional curvature k, for all $k > -3c^2$; the $(2n-1)$-dimensional real number space

$\mathbb{R}^{2n-1}$ is a c-sasakian manifold of constant φ-sectional curvature $-3c^2$ and the product manifold $\mathbb{R}\times CD^{n-1}$, where CD^{n-1} is a simply connected bounded complex domain in C^{n-1} with negative constant holomorphic sectional curvature, has a c-sasakian structure of constant φ-sectional curvature k, for all $k < -3c^2$ (see [MR]). We shall denote by $S^{2n-1}(c,k)$ $(k > -3c^2)$, $\mathbb{R}^{2n-1}(c)$ and $(\mathbb{R}\times CD^{n-1})(c,k)$ $(k < -3c^2)$ respectively the c-sasakian manifolds with these structures.

III c'-Hyperbolic hermitian structures

In this section, we study a particular class of structures on an almost hermitian manifold which we call c'-hyperbolic hermitian structures.

Let $(N,\varphi_N,\xi_N,\eta_N,g_N)$ be a $(2n-1)$-dimensional c-sasakian manifold and $(M,\varphi_M,\xi_M,\ \eta_M,g_M)$ a $(2m+1)$-dimensional c'-kenmotsu manifold, with $c,c'\in\mathbb{R}$, $c\neq 0$, and $c'\neq 0$. Let us consider the product manifold $V = N\times M$ with the almost hermitian structure (J,g) defined by:

$$\left.\begin{aligned} J(X,X') &= \big(\varphi_N X - \eta_M(X')\,\xi_N,\ \varphi_M X' + \eta_N(X)\,\xi_M\big) \\ g((X,X'),(Y,Y')) &= g_N(X,Y) + g_M(X',Y') \end{aligned}\right\} \tag{III.1}$$

where $X,Y\in\mathfrak{X}(N)$ and $X',Y'\in\mathfrak{X}(M)$.

Proposition III.1 *The almost hermitian manifold (V,J,g) is a hermitian manifold with Lee 1-form*

$$\omega = -2\left(c' + \frac{n-1}{n+m-1}(c-c')\right)\pi_M^*\eta_M$$

where $\pi_M : N\times M \longrightarrow M$ is the canonical projection onto the second factor.

We remark that if $c = c'$ then (V,J,g) is a locally conformal Kähler manifold [MR].

Denote by $l = -2(c' + \frac{n-1}{n+m-1}(c-c'))$. Suppose that $l > 0$. Thus we deduce that $l = \|w\|$. Moreover, using theorem 2.1 of [M] (see also [MR]) we have

Proposition III.2 *Let (J,g) be the almost hermitian structure given by (III.1) on the product manifold $N\times M$. Then, for every point $(p,q)\in N\times M$ there exists an open neighbourhood U of q in M and $2m$ independent 1-forms $\alpha_1,\ldots,\alpha_{2m}$ on U, such that:*

$$\left.\begin{aligned} &\pi_U^*\alpha_j\circ J = \pi_U^*\alpha_{m+j}, \qquad \pi_U^*\alpha_{m+j}\circ J = -\pi_U^*\alpha_j, \quad j\in\{1,\ldots,m\} \\ &d(\pi_U^*\alpha_i) = \frac{c'}{l}\pi_U^*\alpha_i\wedge\omega, \quad (\pi_U^*\alpha_i)(B) = 0, \qquad i\in\{1,\ldots,2m\} \end{aligned}\right\} \tag{III.2}$$

where $\pi_U : N\times U \longrightarrow U$ is the projection onto the second factor and ω, B are the Lee 1-form and the Lee vector field respectively of $N\times M$.

The above result suggests us to consider the following particular class of hermitian structure:

Definition III.1 *Let (V,J,g) be a $(2n+2m)$-dimensional hermitian manifold with Lee 1-form $\omega\neq 0$ at every point and Lee vector field B and let $\alpha_1,..,\alpha_{2m}$ be independent*

1-*forms on* V, *with* $m \geq 0$. *We say that* $(J, g, \alpha_1, \ldots, \alpha_{2m})$ *is a* **c'-hyperbolic hermitian structure** *on* V *if*

$$\begin{aligned} &\alpha_j \circ J = \alpha_{m+j}, \quad \alpha_{m+j} \circ J = -\alpha_j, && j \in \{1, \ldots, m\}, \\ &d\alpha_i = c'\alpha_i \wedge u, && i \in \{1, 2, \ldots, 2m\} \\ &\alpha_i(B) = 0, && i \in \{1, 2, \ldots, 2m\} \end{aligned} \tag{III.3}$$

where c' *is a constant,* $c' \neq 0$ *and* u *is the unit Lee* 1-*form.*

Remark If $(N, \varphi_N, \xi_N, \eta_N, g_N)$ is a $(2n-1)$-dimensional c-sasakian manifold and $(M, \varphi_M, \xi_M, \eta_M, g_M)$ is a $(2m+1)$-dimensional c'-kenmotsu manifold, with $c, c' \in \mathbb{R}$, $c \neq 0, c' \neq 0$ and $c'm + c(n-1) < 0$, then, from proposition III.2, we deduce that for every point $(p, q) \in N \times M$, there exists an open neighbourhood U of q in M and $2m$ 1-forms $\alpha_1, \ldots, \alpha_{2m}$ on U, such that $(J, g, \pi_U^*\alpha_1, \ldots, \pi_U^*\alpha_{2m})$ is a c'-hyperbolic hermitian structure on $N \times U$, where (J, g) is the almost hermitian structure given by (III.1) on the manifold $N \times M$ and $\pi_U : N \times U \longrightarrow U$ is the projection onto the second factor.

Now, let $H_{c'}^{2m+1}$ be the $(2m+1)$-dimensional hyperbolic space. Denote by $\alpha_1, \ldots, \alpha_{2m}$ the 1-forms on $H_{c'}^{2m+1}$ given by (II.2) and by $(\varphi_{H_{c'}^{2m+1}}, \xi_{H_{c'}^{2m+1}}, \eta_{H_{c'}^{2m+1}}, (ds^2)_{c'})$ the c'-kenmotsu structure on $H_{c'}^{2m+1}$ given by (II.1). Then, if N is a c-sasakian manifold, $\pi_{H_{c'}^{2m+1}} : N \times H_{c'}^{2m+1} \longrightarrow H_{c'}^{2m+1}$ is the projection onto the second factor, (J, g) is the almost hermitian structure on $N \times H_{c'}^{2m+1}$ given by (III.1) and $mc' + (n-1)c < 0$, we obtain that

Corollary III.1 $(J, g, \pi_{H_{c'}^{2m+1}}^*\alpha_1, \ldots, \pi_{H_{c'}^{2m+1}}^*\alpha_{2m})$ *is a* c'-*hyperbolic hermitian structure on the hermitian manifold* $N \times H_{c'}^{2m+1}$ *and we have that*

$$\begin{aligned} d\Omega &= -\frac{2c}{l}\omega \wedge \Omega - \frac{4(c'-c)}{l} \sum_{j=1}^{m} \omega \wedge \pi_{H_{c'}^{2m+1}}^*\alpha_j \wedge \pi_{H_{c'}^{2m+1}}^*\alpha_{m+j}, \\ \nabla\omega &= -l\, c' \sum_{j=1}^{2m} (\pi_{H_{c'}^{2m+1}}^*\alpha_j) \otimes (\pi_{H_{c'}^{2m+1}}^*\alpha_j), \\ \nabla\pi_{H_{c'}^{2m+1}}^*\alpha_i &= \frac{c'}{l}(\pi_{H_{c'}^{2m+1}}^*\alpha_i) \otimes \omega, \end{aligned} \tag{III.4}$$

for $i \in \{1, \ldots, 2m\}$, *where* ∇ *is the Levi-Civita connection of the Riemannian metric* g, Ω *is the Kähler* 2-*form of* $N \times H_{c'}^{2m+1}$ *and* ω *is the Lee* 1-*form.*

IV Special c'-hyperbolic hermitian manifolds

The results obtained in section III suggest us to introduce the following definition.

Definition IV.1 *Let* $(J, g, \alpha_1, \ldots, \alpha_{2m})$ *be a* c'-*hyperbolic hermitian structure on a manifold* V^{2n+2m} *of dimension* $(2n+2m)$, *such that* $\alpha_1, \ldots, \alpha_{2m}$ *are unit* 1-*forms. We say* *that* V^{2n+2m} *is a* special c'-hyperbolic hermitian manifold *if*

$$\left.\begin{aligned} d\Omega &= -\frac{2c}{l}\omega\wedge\Omega - \frac{4(c'-c)}{l}\sum_{i=1}^{m}\omega\wedge\alpha_i\wedge\alpha_{m+i} \\ \nabla\omega &= -l\,c'\sum_{j=1}^{2m}\alpha_j\otimes\alpha_j \\ \nabla\alpha_i &= \frac{c'}{l}\alpha_i\otimes\omega \end{aligned}\right\} \tag{IV.1}$$

for $i\in\{1,\ldots,2m\}$, *where* ω *is the Lee form of* V^{2n+2m}, ∇ *is the Levi-Civita connection of the metric* g, $l=\|\omega\|\neq 0$ *at every point,* c' *is a constant,* $c'\neq 0$ *and* $c=c'-\frac{1}{2}(\frac{n+m-1}{n-1})(l+2c')$.

Remark From (IV.1) we deduce that ω is a closed 1-form.

Let $(V^{2n+2m},J,g,\alpha_1,..,\alpha_{2m})$ be a special c'-hyperbolic hermitian manifold and denote by A_i, with $1\leq i\leq 2m$, the vector fields on V^{2n+2m} given by

$$\alpha_i(X)=g(X,A_i) \tag{IV.2}$$

for all $X\in\mathfrak{X}(V^{2n+2m})$. From (III.3) and (IV.2), we obtain that

$$JA_i=-A_{m+i},\;\; JA_{m+i}=A_i \tag{IV.3}$$

for $i\in\{1,\ldots,m\}$. Moreover, using (IV.1) we have that the vector fields A_i and A_j, with $i\neq j$, are orthogonal and that the Lee 1-form has constant norm.

We also remark that if $c'=-\frac{l}{2}$ then $(V^{2n+2m},J,g,\alpha_1,\ldots\alpha_{2m})$ is an m-hyperbolic locally conformal Kähler manifold (see [MR]).

Let $(V^{2n+2m},J,g,\alpha_1,\ldots,\alpha_{2m})$ be a special c'-hyperbolic hermitian manifold with Lee vector field B and Lee form ω. Then, in the rest of this paper, we shall use the following notation

$$l=\|\omega\|,\;\; u=\frac{\omega}{l},\;\; U=\frac{B}{l},\;\; v=-u\circ J,\;\; V=JU. \tag{IV.4}$$

From (III.3), (IV.3) and (IV.4) we obtain that

$$\begin{aligned} &u(V)=v(U)=u(A_i)=v(A_i)=0 \\ &\alpha_i(U)=\alpha_i(V)=0 \end{aligned} \tag{IV.5}$$

for $i\in\{1,\ldots,2m\}$. Moreover, if Ω is the Kähler 2-form of V^{2n+2m} then, using that Ω is non-degenerate and (IV.5), we deduce that

Proposition IV.1 *On a special c'-hyperbolic hermitian manifold* V^{2n+2m}

$$\Omega=\psi+2\Big(\sum_{j=1}^{m}(\alpha_j\wedge\alpha_{m+j})+v\wedge u\Big),$$

where ψ *is a 2-form of rank* $(2n-2)$ *such that:*

$$\psi^{n-1}\wedge u\wedge v\wedge\alpha_1\wedge\ldots\wedge\alpha_{2m}\neq 0,$$

$$\psi(X,A_i)=\psi(X,U)=\psi(X,V)=0$$

for $i\in\{1,\ldots,2m\}$.

From (IV.1) and by a well-known result (see [KN], pg. 148) we have

Proposition IV.2 *On a special c'-hyperbolic hermitian manifold V^{2n+2m}*

$$\begin{aligned}(\nabla_X J)Y \;=\; & -\Big(\frac{c}{l}(\omega(JY)X - \omega(Y)JX - g(X,JY)B + g(X,Y)JB) + \\ & +\Big(\frac{c'-c}{l}\Big)\sum_{j=1}^{m}\Big((\omega(JY)\alpha_{m+j}(X) + \omega(Y)\alpha_j(X))A_{m+j} + \\ & +(\omega(JY)\alpha_j(X) - \omega(Y)\alpha_{m+j}(X))A_j + \\ & +(\alpha_j(X)\alpha_j(Y) + \alpha_{m+j}(X)\alpha_{m+j}(Y))JB - 2(\alpha_j\wedge\alpha_{m+j})(X,Y)B\Big)\Big).\end{aligned} \tag{IV.6}$$

From (IV.1) and (IV.6) we obtain some characterizations of special c'-hyperbolic hermitian manifolds.

Proposition IV.3 *Let $(J, g, \alpha_1, .., \alpha_{2m})$ be a c'-hyperbolic hermitian structure on a $(2n+2m)$-dimensional manifold V^{2n+2m} such that $\alpha_1, \ldots, \alpha_{2m}$ are unit 1-forms. Then, $(V^{2n+2m}, J, g, \alpha_1, \ldots, \alpha_{2m})$ is a special c'-hyperbolic hermitian manifold if and only if $l = \|\omega\|$ is constant, $d\Omega$ is given as in (IV.1) and one of the four following relations holds*

1. $\nabla u = -c'\sum_{j=1}^{2m}\alpha_j\otimes\alpha_j, \qquad \nabla\alpha_i = c'\alpha_i\otimes u,$
2. $\nabla U = -c'\sum_{j=1}^{2m}\alpha_j\otimes A_j, \qquad \nabla A_i = c'\alpha_i\otimes U,$
3. $\nabla V = c\Big(J + v\otimes U - u\otimes V + \sum_{j=1}^{m}(\alpha_j\otimes A_{m+j} - \alpha_{m+j}\otimes A_j)\Big),$
   $\nabla A_i = c'\alpha_i\otimes U,$
4. $\nabla v = -c\psi, \qquad \nabla\alpha_i = c'\alpha_i\otimes u$

for $i\in\{1,\ldots,2m\}$.

Next, using proposition IV.3, we deduce another results for a special c'-hyperbolic hermitian manifold V^{2n+2m}. Denote by $\mathcal{L}$ the Lie derivative on V^{2n+2m}.

Proposition IV.4 *Let $(V^{2n+2m}, J, g, \alpha_1, \ldots, \alpha_{2m})$ be a special c'-hyperbolic hermitian manifold. Then, V is a Killing vector field for the metric g. Moreover, the following relations hold*

$$[U,V]=0,\;\; [V,A_i]=0,\;\; [A_i,A_j]=0,\;\; [U,A_i]=c'A_i, \tag{IV.7}$$

$$\mathcal{L}_U J = 0,\;\; \mathcal{L}_V J = 0, \tag{IV.8}$$

$$\mathcal{L}_{A_k} J = c'(v\otimes A_k - u\otimes A_{m+k}), \tag{IV.9}$$

$$\mathcal{L}_{A_{m+k}} J = c'(v\otimes A_{m+k} + u\otimes A_k), \tag{IV.10}$$

$$\mathcal{L}_U v = 0,\;\; \mathcal{L}_{A_i} v = 0,\;\; dv = -c\,\psi \tag{IV.11}$$

for $i, j \in \{1, \ldots, 2m\}$ *and* $k \in \{1, \ldots, m\}$.

If $(V^{2n+2m}, J, g, \alpha_1 \ldots \alpha_{2m})$ is a special c'-hyperbolic hermitian manifold with $m \geq 1$ then, from (III.3) and (IV.11), we obtain that the volume element γ on V^{2n+2m} given by $\gamma = \alpha_1 \wedge \ldots \wedge \alpha_{2m} \wedge u \wedge v \wedge \psi^{n-1}$ is exact. In fact, $\gamma = d((\frac{1}{-2c'm})\alpha_1 \wedge \ldots \wedge \alpha_{2m} \wedge v \wedge \psi^{n-1})$. Therefore,

Corollary IV.1 *A compact manifold cannot admit a special c'-hyperbolic hermitian structure $(J, g, \alpha_1, .., \alpha_{2m})$with $m \geq 1$.*

V The curvature tensor on a special c'-hyperbolic hermitian manifold

In this section, we shall obtain some results about the Riemann curvature tensor and the sectional curvature of a special c'-hyperbolic hermitian manifold.

Let $(V^{2n+2m}, J, g, \alpha_1, \ldots, \alpha_{2m})$ be a $(2n+2m)$-dimensional special c'-hyperbolic hermitian manifold and let A_i be as in (IV.2) and l, u, U, v and V as in (IV.4). Then, using (III.3) and proposition IV.3, we have

Proposition V.1 *On a special c'-hyperbolic hermitian manifold V^{2n+2m}*

$$R(X,Y)U = -2(c')^2 \sum_{i=1}^{2m} (\alpha_i \wedge u)(X,Y) A_i, \tag{V.1}$$

$$R(X,Y)A_i = 2(c')^2 \Big\{ (\alpha_i \wedge u)(X,Y)\, U + \sum_{j=1}^{2m} (\alpha_i \wedge \alpha_j)(X,Y)\, A_j \Big\}, \tag{V.2}$$

where R is the Riemann curvature tensor of V^{2n+2m}, $i \in \{1, \ldots, 2m\}$, $X, Y \in \mathfrak{X}(V^{2n+2m})$.

Let x be a point of V^{2n+2m}. Denote by K_{XY} the sectional curvature for the plane section in T_xM with orthonormal basis $\{X, Y\}$. Then, by using (V.1) and (V.2), we deduce

Corollary V.1 *On a special c'-hyperbolic hermitian manifold V^{2n+2m}*

$$\begin{aligned} K_{XU} &= -(c')^2 \sum_{i=1}^{2m} (\alpha_i(X))^2, \\ K_{XA_i} &= -(c')^2 \Big\{ (u(X))^2 + \sum_{j=1, j \neq i}^{2m} (\alpha_j(X))^2 \Big\}, \\ K_{UA_i} &= K_{A_i A_j} = -(c')^2 \end{aligned}$$

for $i, j \in \{1, \ldots, 2m\}$.

VI The universal covering space of a special c'-hyperbolic hermitian manifold

In this section we shall study the universal covering space of a special c'-hyperbolic hermitian manifold.

Let $(V^{2n+2m}, J, g, \alpha_1, \ldots, \alpha_{2m})$ be a special c'-hyperbolic hermitian manifold and let A_i be $(1 \leq i \leq 2m)$ as in (IV.2) and l, u, U, v, V as in (IV.4). Denote by $c = c' - \frac{1}{2}(\frac{n+m-1}{n-1})(l+2c')$ and by $\mathfrak{F}$ the foliation given by $u = 0$, $\alpha_i = 0$, $1 \leq i \leq 2m$. $\mathfrak{F}$ defines on V^{2n+2m} a foliation of dimension $(2n-1)$, which we call the canonical foliation of V^{2n+2m}. Using (IV.7), proposition IV.3 and corollary V.1, we deduce

Proposition VI.1 *The canonical foliation $\mathfrak{F}$ of a special c'-hyperbolic hermitian manifold is totally geodesic with integrable normal bundle. Moreover, if $\mathfrak{F}^\perp$ is the foliation determined by the normal bundle of $\mathfrak{F}$, then $\mathfrak{F}^\perp$ also is totally geodesic and its leaves are of constant sectional curvature $-(c')^2$.*

Let $i : N \longrightarrow V^{2n+2m}$ be the immersion of a generic leaf N of the canonical foliation $\mathfrak{F}$. We define an almost contact metric structure $(\varphi_N, \xi_N, \eta_N, g_N)$ on N by

$$\varphi_N X = JX + (i^*v)(X)U\,|_N, \quad \xi_N = -V\,|_N, \quad \eta_N = -(i^*v), \quad g_N = i^*g \qquad \text{(VI.1)}$$

for all $X \in \mathfrak{X}(N)$. Then, from (IV.8), (IV.11) and since $N_J = 0$, we have

Proposition VI.2 *If $c \neq 0$ the almost contact metric structure $(\varphi_N, \xi_N, \eta_N, g_N)$ on N is c-sasakian.*

Now, we consider the immersion $j : M \longrightarrow V^{2n+2m}$ of a generic leaf M of the foliation $\mathfrak{F}^\perp$ on V^{2n+2m}. We define an almost contact metric structure $(\varphi_M, \xi_M, \eta_M, g_M)$ on M by

$$\varphi_M(Y) = JY + (j^*u)(Y)V\,|_M, \quad \xi_M = U\,|_M, \quad \eta_M = (j^*u), \quad g_M = j^*g, \qquad \text{(VI.2)}$$

for all $Y \in \mathfrak{X}(M)$. Then, using (IV.1), (IV.8) and since $N_J = 0$, we obtain

Proposition VI.3 *The almost contact metric structure $(\varphi_M, \xi_M, \eta_M, g_M)$ on M is c'-kenmotsu.*

Next, we give the following definition

Definition VI.1 *A special c'-hyperbolic hermitian manifold $(V^{2n+2m}, J, g, \alpha_1, \ldots, \alpha_{2m})$ of dimension $(2n+2m)$, with $c = c' - \frac{1}{2}(\frac{n+m-1}{n-1})(l+2c') \neq 0$ is called* k-special c'-hyperbolic hermitian manifold *$(k \in \mathbb{R})$ if every leaf N of the canonical foliation $\mathfrak{F}$ is of constant φ_N-sectional curvature k, where $(\varphi_N, \xi_N, \eta_N, g_N)$ is the induced c-sasakian structure on N given by (VI.1).*

If $(V^{2n+2m}, J, g, \alpha_1, \ldots, \alpha_{2m})$ is a k-special c'-hyperbolic hermitian manifold then V^{2n+2m} is said to have a k-special c'-hyperbolic hermitian structure $(J, g, \alpha_1, \ldots, \alpha_{2m})$.

Now, we introduce a definition which will be useful in the sequel. Let N, k be a $(2n-1)$-dimensional manifold and a real number respectively and let $(H^{2m+1}_{c'}, (ds^2)_{c'})$ be the $(2m+1)$-dimensional hyperbolic space.

Definition VI.2 *A* distinguished special c'-hyperbolic hermitian *(respectively* distinguished k-special c'-hyperbolic hermitian*) structure on $V^{2n+2m} = N \times H^{2m+1}_{c'}$ is a special c'-hyperbolic hermitian (respectively k-special c'-hyperbolic hermitian) structure $(J, g, \alpha_1, \ldots, \alpha_{2m})$ on V^{2n+2m} with $c = c' - \frac{1}{2}(\frac{n+m-1}{n-1})(l + 2c') \neq 0$ and such that:*

1. *The metric g is of the form $g = d\sigma^2 + (ds^2)_{c'}$ where $d\sigma^2$ is a Riemann metric on N and*
2. *The unit Lee 1-form u and the 1-forms α_i, $1 \leq i \leq 2m$, are given by*

$$u = \frac{dx_{2m+1}}{c'x_{2m+1}}, \quad \alpha_i = \frac{dx_i}{c'x_{2m+1}}$$

where $(x_1, \ldots, x_{2m+1})$ are the usual coordinates on $H^{2m+1}_{c'}$.

From (IV.5) and propositions IV.4 and VI.2 we deduce

Proposition VI.4 *If $(J, g, \alpha_1, \ldots, \alpha_{2m})$ is a distinguished special c'-hyperbolic hermitian structure on $V^{2n+2m} = N \times H^{2m+1}_{c'}$, then the manifold N carries an induced c-sasakian structure $(\varphi_N, \xi_N, \eta_N, g_N)$, where $c = c' - \frac{1}{2}(\frac{n+m-1}{n-1})(l + 2c')$, and the almost hermitian structure (J, g) on V^{2n+2m} is given by (III.1). Moreover, if $(J, g, \alpha_1, \ldots, \alpha_{2m})$ is a distinguished k-special c'-hyperbolic hermitian structure on V^{2n+2m}, then N is of constant φ_N-sectional curvature k.*

Next, we shall describe the universal covering space of a special c'-hyperbolic hermitian manifold.

Theorem VI.1 *The universal covering space of a complete special c'-hyperbolic hermitian manifold $(V^{2n+2m}, J, g, \alpha_1, \ldots, \alpha_{2m})$ of dimension $2n+2m$ with Lee form ω and such that $c = c' - \frac{1}{2}(\frac{n+m-1}{n-1})(l + 2c') \neq 0$, is a product space $\overline{V}^{2n+2m} = N \times H^{2m+1}_{c'}$, where N is the universal covering space of an arbitrary leaf of the canonical foliation of V^{2n+2m} and $H^{2m+1}_{c'}$ is the $(2m+1)$-dimensional hyperbolic space. The lift of the special c'-hyperbolic hermitian structure to $\overline{V}^{2n+2m}$ gives a distinguished special c'-hyperbolic hermitian structure on $\overline{V}^{2n+2m}$. Moreover, if the structure of V^{2n+2m} is a k-special c'-hyperbolic hermitian structure we have:*

1. *If $k > -3c^2$, then $\overline{V}^{2n+2m}$ is almost complex isometric to $S^{2n-1}(c, k) \times H^{2m+1}_{c'}$,*
2. *If $k = -3c^2$, then $\overline{V}^{2n+2m}$ is almost complex isometric to $\mathbb{R}^{2n-1}(c) \times H^{2m+1}_{c'}$ and,*
3. *If $k < -3c^2$, then $\overline{V}^{2n+2m}$ is almost complex isometric to $(\mathbb{R} \times CD^{n-1})(c, k) \times H^{2m+1}_{c'}$.*

Proof This result follows using propositions VI.1 and VI.4, theorem A of [BH] and the following lemma VI.1.

Lemma VI.1 *Let M be a $(2m+1)$-dimensional complete, simply connected, Riemannian manifold of constant negative curvature $-(c')^2$ ($c' \neq 0$) and U, A_i vector fields on M such that $\{U, A_1, \ldots, A_{2m}\}$ form an orthonormal basis for M and $[U, A_i] = c'A_i$,*

$[A_i, A_j] = 0$ *for* $i, j \in \{1, \ldots, 2m\}$. *Then, there is an isometry* F *of* M *to the* $(2m+1)$-*dimensional hyperbolic space* $H^{2m+1}_{c'}$, *satisfying*

$$F_*U = (c'x_{2m+1})\frac{\partial}{\partial x_{2m+1}}, \quad F_*A_i = (c'x_{2m+1})\frac{\partial}{\partial x_i},$$

for $i \in \{1, \ldots, 2m\}$, *where* $(x_1, \ldots, x_{2m+1})$ *are the usual coordinates on* $H^{2m+1}_{c'}$.

References

[BH] R.A. BLUMENTHAL, J.J. HEBDA, *De Rham decomposition theorems for foliated manifolds*, Ann. Inst. Fourier, Grenoble, 33 (1983), 183–198.

[ChG] D. CHINEA, J.C. GONZALEZ, *A classification of almost contact metric manifolds*, Annali di Mat. Pura ed Appl. (IV) 156 (1990), 15–36.

[CFL] L.A. CORDERO, M. FERNANDEZ, M. DE LEON, *Compact locally conformal Kähler nilmanifolds*, Geometriae Dedicata 21 (1986), 187–192.

[JV] D. JANSSENS, L. VANHECKE, *Almost contact structure and curvature tensors*, Kodai Math. J. 4 (1981), 1–27.

[KN] S. KOBAYASHI, K.NOMIZU, *Foundations of Differential Geometry*, Vol. II, Intersc. Publ., New York (1969).

[M] J.C. MARRERO, *The local structure of trans-sasakian manifolds*, to appear in Ann. di Mat. Pura ed Appl. (IV), CLXI (1992).

[MR] J.C. MARRERO, J. ROCHA, *Sasakian m-hyperbolic locally conformal Kähler manifolds*, to appear in Rend. Mat. di Roma.

[V1] I. VAISMAN, *On locally conformal almost Kähler manifolds*, Israel J. Math. 24 (1976), 338–351.

[V2] I. VAISMAN, *Generalized Hopf manifolds*, Geometriae Dedicata 13 (1982), 231–255.

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