Differential Geometry and Its Applications
Proc. Conf. Opava (Czechoslovakia), August 24–28, 1992
Silesian University, Opava, 1993, 369–376

ON RADON'S THEOREM FOR AFFINE KAEHLER IMMERSIONS *

Stefan Ivanov †

Abstract. We give a new geometric formulation of the fundamental existence theorem for affine Kaehler immersions in dimension not less than 6.

Keywords. Affine Kaehler immersions.

MS classification. 53A15.

1. Introduction

In the real affine differential geometry, the famous theorem of J. Radon gives a necessary and sufficient conditions for an affine 2-manifold to be affinely immersed in $\mathbf{R}^3$. This theorem received a new geometric formulation for all dimensions in [DNV], [K] and in more general situation in [I1] (see also [Si]). In [I2] we give a geometric formulation of the fundamental theorem for holomorphic affine immersions with complex transversal connection form.

The fundamental theorem for affine Kaehler immersions is proved in [NPP].

In the present paper, using the new approach of [DNV], we give a new geometric formulation (for dimension greater or equal to 6) of Radon's theorem for affine Kaehler immersions by means of conjugate connections.

2. Affine Kaehler immersions

In this section, we recall several definitions and preliminary facts in the geometry of affine Kaehler immersions. For more details, see [NPP], [NP], [O]. We shall use the real approach and therefore all the objects are assumed to be real and differentiable.

Let $\mathbf{M}^n$ be a complex manifold of complex dimension n with a complex structure J and let $f : \mathbf{M}^n \to \mathbf{C}^{n+1}$ be a holomorphic immersion of $\mathbf{M}^n$ into the standard complex affine space $\mathbf{C}^{n+1}$ equipped with the usual flat complex affine connection D. A transversal vector field ξ along f is said to be anti-holomorphic if $D_{JX}\xi = -JD_X\xi$ for every vector field X on $\mathbf{M}$.

*This paper is in final form and no version of it will be submitted for publication elsewhere.

†The author is supported by Contract MM 18/1991 with the Ministry of Science and Education of Bulgaria.

Let ξ be an arbitrary local field of transversal vectors to $f(\mathbf{M}^n)$. For any vector fields X and Y on $\mathbf{M}^n$, we write

$$D_X f_*(Y) = f_*(\nabla_X Y) + h(X,Y)\xi - h(X,JY)J\xi, \tag{2.1}$$

thus defining an affine torsion-free connection ∇ and symmetric tensor h of type (0,2) on $\mathbf{M}$. Since f is holomorphic and J is D-parallel, we get that ∇ is a complex affine connection (i.e. $\nabla J = 0$) and

$$h(JX,Y) = h(X,JY). \tag{2.2}$$

If h is non-degenerate (2.2) implies that h is a complex Riemannian metric; then it is called complex second fundamental form or complex affine metric. For the geometry of complex Riemannian manifolds see [GI].

We can also write

$$D_X\xi = -f_*(SX) + \tau(X)\xi + \kappa(X)J\xi \tag{2.3}$$

where S is the affine shape operator and τ and κ are 1-forms defining the transversal connection for the immersion (f,ξ). A transversal vector field ξ is anti-holomorphic iff

$$SJ = -JS \quad \text{and} \quad \kappa = \tau_o J \tag{2.4}$$

In this case (f,ξ) is called an affine Kaehler immersion [NPP].

Further, in the paper we consider only non-degenerate affine Kaehler immersions.

A torsion-free complex affine connection ∇ is said to be an affine Kaehler connection if its curvature tensor R has the property:

$$R(JX,JY) = R(X,Y). \tag{2.5}$$

Let R denote the curvature tensor of ∇. We have the following fundamental equations for affine Kaehler immersion:

Gauss:

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY - h(Y,JZ)JSX + h(X,JZ)JSY, \tag{2.6}$$

Codazzi I:

$$\begin{aligned}\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) = &- \tau(X)h(Y,Z) + \tau(Y)h(X,Z) - \\ &- \tau(JX)h(Y,JZ) + \tau(JY)h(X,JZ),\end{aligned} \tag{2.7}$$

Codazzi II:

$$\begin{aligned}\nabla_X S)Y - (\nabla_Y S)X = &\tau(X)SY - \tau(Y)SX + \\ &+ \tau(JX)JSY - \tau(JY)JSX,\end{aligned} \tag{2.8}$$

Ricci I:

$$h(X,SY) - h(Y,SX) = d\tau(X,Y), \tag{2.9}$$

Ricci II:

$$h(JX,SY) - h(JY,SX) = -d(\tau_o J)(X,Y). \tag{2.10}$$

From (2.6) and (2.4) it follows that the affine connection ∇ induced by an affine Kaehler immersion is affine Kaehler connection.

From (2.9), (2.10) and (2.4) it follows that

$$d\tau(JX, JY) = d\tau(X,Y); \quad d(\tau_o J)(JX, JY) = d(\tau_o J)(X,Y). \tag{2.11}$$

We need the following fundamental theorem for affine Kaehler immersions, proved in [NPP].

Theorem A [NPP]. *Let $(\mathbf{M}^n, J)$ be an n-dimensional complex manifold with a complex structure J endowed with an affine Kaehler connection ∇, a symmetric bilinear form h with the property (2.2), a $(1,1)$-tensor S with the property (2.4) and a 1-form τ satisfying equations (2.6)–(2.10). Then there exists an affine Kaehler immersion (f,ξ) of $(\mathbf{M}^n, J)$ into $\mathbf{C}^{n+1}$ such that h, ∇, S and τ are the induced objects of (f,ξ). Such an immersion is unique up to a holomorphic affine transformation of $\mathbf{C}^{n+1}$.*

If $\mathbf{M}^n$ is assumed to be connected, all the results above hold in a global setting.

From (2.7) it follows that the following (0,3)-tensor

$$C(X,Y,Z) = (\nabla_X h)(Y,Z) + \tau(X)h(Y,Z) + \tau(JX)h(Y,JZ), \tag{2.12}$$

is totally symmetric. The tensor C is called the cubic form for the affine Kaehler immersion.

3. H-Projective flatness

In his section we give a short review on H-projective structures and refer for the proofs to [T], [Y].

Let $(\mathbf{M}, J)$ be a $2n$-dimensional complex manifold with complex structure J. Two torsion-free complex affine connections ∇ and ∇' on $\mathbf{M}$ are said to be holomorphically-projective (H-projective) equivalent, if there exists a 1-form α such that

$$\nabla'_X Y = \nabla_X Y + \alpha(X)Y + \alpha(Y)X - \alpha(JY)JX - \alpha(JX)JY, \tag{3.1}$$

for all $X, Y \in \mathbf{XM}$.

One may also speak of a H-projective change from ∇ to ∇' or of a H-projective equivalence of ∇ and ∇' when (3.1) holds.

More geometrically two torsion-free complex affine connections are H-projectively equivalent iff their holomorphically planar curves coincide.

Given an affine connection ∇ on $\mathbf{M}$, the H-projective curvature tensor HW of ∇ is defined by

$$\begin{aligned} HW(X,Y)Z = {} & R(X,Y)Z - N(X,Z)Y + N(Y,Z)X - (N(X,Y) - N(Y,X))Z + \\ & + N(X,JZ)JY - N(Y,JZ)JX + (N(X,JY) - N(Y,JX))JZ. \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} N(X,Y) = -\frac{1}{2n+2}\{ & \mathrm{Ric}(X,Y) + \frac{1}{2n-2}[\mathrm{Ric}(X,Y) + \mathrm{Ric}(Y,X) - \\ & - \mathrm{Ric}(JX,JY) - \mathrm{Ric}(JY,JX)]\} \end{aligned}$$

If the Ricci tensor has the Kaehler property $\mathrm{Ric}(JX,JY)=\mathrm{Ric}(X,Y)$ the tensor N is given by:

$$N(X,Y)=-\frac{1}{2n+2}\,\mathrm{Ric}(X,Y) \tag{3.3}$$

We also need to consider the Codazzi equation for N:

$$(\nabla_X N)(Y,Z)=(\nabla_Y N)(X,Z). \tag{3.4}$$

An affine connection ∇ is said to be H-projectively flat if around each point there is a H-projective change of ∇ to a flat affine connection. Now the following result is well known (Y. Tashiro), (see [T], [Y]):

Theorem B. $(\mathbf{M},J)$ *be a* $2n$*-dimensional complex manifold with a complex torsion-free affine connection* ∇*. If* $\dim\mathbf{M}\geq 6$*, then* ∇ *is* H*-projectively flat iff* $HW=0$*. In this case (3.4) also holds.*

4. τ-Conjugate connections

Let $(\mathbf{M},J,h,\nabla)$ be a complex Riemannian manifold and ∇ be a torsion-free complex affine connection on $\mathbf{M}$. Let τ be a 1-form on $\mathbf{M}$.

A torsion-free complex affine connection ∇^* on $\mathbf{M}$ is said to be *τ-conjugate to ∇ relative to the metric h* if the following identity holds (see [NP]):

$$Xh(Y,Z)=h(\nabla^*_X Y,Z)+h(Y,\nabla_X Z)-\tau(X)h(Y,Z)-\tau(JX)h(JY,Z). \tag{4.1}$$

for arbitrary vector fields X,Y,Z on $\mathbf{M}$.

Conjugate connections are crucial in our investigations.

Proposition 4.1. *Let* ∇^* *be* τ*-conjugate to* ∇ *affine connection on* $(\mathbf{M},J,h)$*. Then the curvature tensors* R^* *and* R*, for* ∇^* *and* ∇ *respectively, are related by the equality:*

$$h(R^*(X,Y)Z,U)+h(R(X,Y)U,Z)=d\tau(X,Y)h(Z,U)+d(\tau_o J)(X,Y)h(JZ,U). \tag{4.2}$$

Proof: From (4.1) we obtain:

$$\begin{aligned}XYh(Z,U)&=h(\nabla^*_X\nabla^*_Y Z,U)+h(\nabla^*_Y Z,\nabla_X U)-\tau(X)h(\nabla^*_Y Z,U)-\\&-\tau(JX)h(J\nabla^*_Y Z,U)+h(\nabla_X\nabla_Y U,Z)+h(\nabla^*_X Z,\nabla_Y U)-\\&-\tau(X)h(Z,\nabla_Y U)-\tau(JX)h(JZ,\nabla_Y U)-X\tau(Y)h(Z,U)-\\&-\tau(Y)\{h(\nabla^*_X Z,U)+h(Z,\nabla_X U)-\tau(X)h(Z,U)-\tau(JX)h(JZ,U)\}-\\&-X\tau(JY)h(JZ,U)-\\&-\tau(JY)\{h(\nabla^*_X JZ,U)+h(JZ,\nabla_X U)+\tau(JX)h(Z,U)-\tau(X)h(JZ,U)\}.\end{aligned}$$

Subtracting from this equation the one obtained by interchanging X and Y and the equation

$$\begin{aligned}[X,Y]h(Z,U)&=h(\nabla^*_{[X,Y]}Z,U)+h(Z,\nabla_{[X,Y]}U)-\\&-\tau([X,Y])h(Z,U)-\tau(J[X,Y])h(JZ,U)\end{aligned}$$

gives the desired result. ∎

Corollary 4.2. *Let ∇ is affine Kaehler connection and ∇^* is τ-conjugate connection to ∇. Then ∇^* is affine Kaehler connection iff the 1-form τ satisfies the equations:*

$$d\tau(JX, JY) = d\tau(X, Y); \quad d(\tau_o J)(JX, JY) = d(\tau_o J)(X, Y).$$

Let K be the difference (1,2) tensor between ∇ and ∇^* i.e. $K(X,Y) = \nabla^*_X Y - \nabla_X Y$ for all vector fields on $\mathbf{M}$. Since ∇ and ∇^* are complex and torsion-free the tensor K satisfies the following equations:

$$K(X, Y) = K(Y, X); \quad K(X, JY) = JK(X, Y). \tag{4.3}$$

Proposition 4.3. *Let the affine connection ∇^* be τ-conjugate to ∇ for some 1-form τ. Then the following formulae are valid:*

$$h(K(X, Z), Y) = h(K(X, Y), Z) \tag{4.4}$$

$$\begin{aligned}(\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) = &- \tau(X)h(Y, Z) + \tau(Y)h(X, Z) - \\ &- \tau(JX)h(JY, Z) + \tau(JY)h(JX, Z);\end{aligned} \tag{4.5}$$

$$(\nabla^*_X h)(Y, Z) - (\nabla^*_Y h)(X, Z) = (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) \tag{4.6}$$

Proof: From (4.1) we get (4.4) immediately. Using (4.1) we easily obtain:

$$(\nabla_X h)(Y, Z) = h(K(X, Y), Z) - \tau(X)h(Y, Z) - \tau(JX)h(JY, Z); \tag{4.7}$$

$$(\nabla^*_X h)(Y, Z) = -h(K(X, Z), Y) - \tau(X)h(Y, Z) - \tau(JX)h(JY, Z); \tag{4.8}$$

Using (4.3) and (4.4) we get (4.5) and (4.6) from (4.7) and (4.8). ∎

A torsion-free complex affine connection ∇ is said to be τ-*compatible* with the complex Riemannian metric h if the tensor ∇h satisfies the equation (4.5).

Thus, Proposition 4.3 implies that the affine connections ∇ and ∇^* are τ-compatible with h.

From (4.5) and (4.7) we get that the cubic form C defined by (2.12) is totally symmetric and $C(X, Y, Z) = h(K(X, Y), Z)$. Then (4.3) and (4.4) imply that the cubic form C is totally pure i.e. $C(JX, Y, Z) = C(X, JY, Z) = C(X, Y, JZ)$. Thus, if ∇^* is τ-conjugate to ∇ there arises the cubic form C.

Proposition 4.4. *Let $(\mathbf{M}, J, h)$ be a complex Riemannian manifold, C a totally symmetric and totally pure $(0,3)$-tensor on $\mathbf{M}$ and τ is a 1-form on $\mathbf{M}$. Then*

a) there exist a unique affine connection ∇, τ-compatible with h, with cubic form equal to C, and a unique affine connection ∇^*, τ-compatible with h, such that ∇^* is τ-conjugate to ∇.

b) The complex Riemannian metric h is locally conformally equivalent to a complex analytic Riemannian metric.

Proof: If the connections ∇ and ∇^* with the required properties exist from (4.7) and (4.8) we have

$$(\nabla_X h)(Y, Z) = C(X, Y, Z) - \tau(X)h(Y, Z) - \tau(JX)h(JY, Z); \tag{4.9}$$

$$(\nabla^*_X h)(Y, Z) = -C^*(X, Y, Z) - \tau(X)h(Y, Z) - \tau(JX)h(JY, Z); \tag{4.10}$$

By the well known theorem (see [Sch, p. 132]) there exist unique torsion-free affine connections ∇ and ∇^* with the properties (4.9) and (4.10) respectively. Let $\hat{\nabla}$ denote the Levi-Civita connection of h. Then the connections ∇ and ∇^* are determined by:

$$\begin{aligned} h(\nabla_X Y, Z) = {} & h(\widehat{\nabla}_X Y, Z) - \tfrac{1}{2}\{C(X,Y,Z) + \tau(Z)h(X,Y) + \tau(JZ)h(JX,Y) - \\ & -\tau(X)h(Z,Y) - \tau(JX)h(JZ,Y) - \tau(Y)h(X,Z) - \tau(JY)h(JX,Z)\} \end{aligned}$$

$$\begin{aligned} h(\nabla^*_X Y, Z) = {} & h(\widehat{\nabla}_X Y, Z) - \tfrac{1}{2}\{-C(X,Y,Z) + \tau(Z)h(X,Y) + \tau(JZ)h(JX,Y) - \\ & -\tau(X)h(Z,Y) - \tau(JX)h(JZ,Y) - \tau(Y)h(X,Z) - \tau(JY)h(JX,Z)\} \end{aligned}$$

It is easy to check that ∇ and ∇^* are the required connections.

Since C is totally pure we get

$$h(\widehat{\nabla}_X J)Y, Z) = \tau(Z)h(X,JY) - \tau(JZ)h(X,Y) + \tau(Y)h(X,JZ) - \tau(JY)h(X,Z)$$

This condition implies that h is locally conformally equivalent to a complex analytic Riemannian metric (see [GI]). ∎

5. Radon's theorem and its reformulations. Main result

In this section we formulate and prove our main result. We have

Theorem 5.1. *Let* $(\mathbf{M}, h, J, \nabla)$ *be a* $2n$*-dimensional* $(2n \geq 6)$ *connected and simply connected complex Riemannian manifold with complex metric* h *and affine Kaehler connection* ∇. *Let* R *be the curvature tensor of* ∇ *and* τ *be a 1-form on* $\mathbf{M}$. *Then the following conditions are equivalent:*

i) There exists a non-degenerate affine Kaehler immersion $f : \mathbf{M} \to \mathbf{C}^{n+1}$ *with transversal connection form* τ *such that the induced fundamental form and the induced connection of the immersion are* h *and* ∇, *respectively. Such an immersion is unique up to holomorphic affine transformation on* $\mathbf{C}^{n+1}$.

ii) The affine Kaehler connection ∇ *is* τ*-compatible with* h *and the unique complex affine connection* ∇^*, τ*-conjugate to* ∇, *is a* H*-projectively flat affine Kaehler connection.*

Proof: Let the condition i) of Theorem 5.1 hold. From the Codazzi equation (2.7) we have that ∇ is τ-compatible with h. Let ∇^* denote the unique affine connection τ-conjugate to ∇ according to Proposition 4.4. From the Gauss equation (2.6) using (4.2) for the curvature tensor R^* of ∇^* we get:

$$\begin{aligned} R^*(X,Y)Z = {} & -h(SX,Z)Y + h(SY,Z)X + h(SX,JZ)JY - h(SY,JZ)JX + \\ & + d\tau(X,Y)Z + d(\tau_o J)(X,Y)JZ. \end{aligned} \tag{5.1}$$

Using (2.4), (2.11) and Corollary 4.2 we get that ∇^* is an affine Kaehler connection. From (5.1), the Ricci equations (2.9), (2.10) and Theorem B we get that ∇^* is a H-projectively flat connection.

Conversely, let the unique connection ∇^*, τ-compatible with ∇ be a H-projectively flat affine Kaehler connection. Then the Ricci tensor has the Kaehler property. From Theorem B for the curvature tensor R^* of ∇^* we have:

$$\begin{aligned} R^*(X,Y)Z = {} & N(X,Z)Y - N(Y,Z)X + (N(X,Y) - N(Y,X))Z - \\ & - N(X,JZ)JY + N(Y,JZ)JX - (N(X,JY) - N(Y,JX))JZ, \end{aligned} \tag{5.2}$$

where N is the normalized Ricci tensor for ∇^* , given by (3.3).

Using (4.2) and (5.2) for the curvature tensor R of ∇ we obtain:

$$\begin{aligned} h(R(X,Y)U,Z) = &-N(X,Z)h(Y,U) + N(Y,Z)h(X,U) - \\ &-(N(X,Y) - N(Y,X) - d\tau(X,Y))h(Z,U) + \\ &+ N(X,JZ)h(JY,U) - N(Y,JZ)h(JX,U) + \\ &+ (N(X,JY) - N(Y,JX) + d(\tau_o J)(X,Y))h(JZ,U). \end{aligned} \tag{5.3}$$

Since ∇ is torsion-free and $2n \geq 6$, applying the first Bianchi identity, from (5.3) we obtain:

$$N(X,Y) - N(Y,X) = d\tau(X,Y). \tag{5.4}$$

$$N(X,JY) - N(Y,JX) = -d(\tau_o J)(X,Y). \tag{5.5}$$

Since h is non-degenerate we can define the operator S by the equality:

$$h(SX,Y) = -N(X,Y). \tag{5.6}$$

Since the Ricci tensor has the Kaehler identity we have $N(JX,JY) = N(X,Y)$. Using this equality, from (5.6) we get

$$SJ = -JS \tag{5.7}$$

Using (5.4), (5.5) and (5.6), from (5.3) we obtain:

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY - h(Y,JZ)JSX + h(X,JZ)JSY,$$

which implies that the Gauss equation (2.6) is fulfilled.

Using (5.6), from (5.4) and (5.5) we get the Ricci equations (2.9) and (2.10). The Codazzi I equation (2.7) is valid since ∇ is τ-compatible with h.

Further we show that the Codazzi II equation (2.8) is valid.

From (5.6), using (4.1), we obtain:

$$-(\nabla^*_X h)(Y,Z) = h(\nabla_X SY, Z) - h(S\nabla^*_X Y, Z) - \tau(X)h(SY,Z) - \tau(JX)h(SY,JZ).$$

Applying the condition (3.4) to the connection ∇^* and taking into account that ∇ and ∇^* are torsion-free we get:

$$(\nabla_X S)Y - (\nabla_Y S)X = \tau(X)SY - \tau(Y)SX + \tau(JX)JSY - \tau(JY)JSX,$$

which is the Codazzi II equation (2.8). Together with (5.7) all the conditions of Theorem A are fulfilled. Applying Theorem A we complete the proof of the Theorem 5.1. ∎

References

[DNV] Dillen, F., Nomizu, K., Vranken, L., *Conjugate connections and Radon's theorem in Affine Differential Geometry,* Monatshefte für Math. **109** (1990), 221–231.

[GI] Ganchev, G., Ivanov, S., *Connections and curvatures on complex Riemannian manifolds,* Internal Report $IC/91/41$, I.C.T.P. Trieste 1991.

[I1] Ivanov, S., *On the Radon's theorem in affine differential geometry,* to appear.

[I2] Ivanov, S., *On the Radon's and Norden's theorems in complex affine differential geometry,* to appear.

[K] Kurose T., *On the realization of Statistical manifolds in Affine space,* Preprint 1991.

[NP] Nomizu, K., Podesta, F., *On the Cartan-Norden theorem for affine Kaehler immersions,* Nagoya Math. J. **121** (1991), 127–135.

[NPP] Nomizu, K., Pinkall, U., Podesta, F., *On the geometry of affine Kaehler immersions,* Nagoya Math. J. **120** (1990), 205–222.

[O] Opozda, B., *Equivalence theorem for complex affine hypersufaces.*, Preprint MPI Bonn / 91–81.

[Scho] Schouten, J.A., *Ricci calculus*, 2nd ed., Springer 1954.

[Si] Simon, U., *The fundamental theorem in affine hypersurface theory*, Geom. Ded. **26** (1988), 125–137.

[T] Tashiro, Y., *On holomorphically projective correspondence in an almost complex space,* Math J. Okayama Univ. **6** (1957), 147–152.

[Y] Yano, K., *Differential geometry on complex and almost complex spaces,* Pergamon Press 1965.

Author's address:
Stefan Ivanov
University of Sofia,
Faculty of Math. and Inf.,
Department of Geometry,
bul. James Bauchier 5,
1126 Sofia, BULGARIA.

Received 28 August 1992