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On the Wheeler-deWitt equation for homogeneous cosmological models

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Abstract

We derive the Wheeler-deWitt equation for fourth order gravity, specialized to isotropic Friedmann models. Some special solutions are found.

Keywords: Wheeler-deWitt equation, Fourth order gravity.

PACS-Classification: 9880 Cosmology.

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1 Introduction

We consider[1] quantum cosmology as canonically quantized general relativistic cosmology. The Schrödinger equation replacing the Friedmann equation of General Relativity is called Wheeler-DeWitt equation [3] [6] [9]. We deduce this equation for a spatially flat Friedmann model and classify the solutions for different theories. Thereby we concentrate on mathematical rigour (e. g. in showing the possible ambiguities) and postpone the interpretation of the solutions to later work. We only mention that $|\psi|^2$ is a kind of probability distribution of the possible geometries if ψ is a suitably chosen solution of the Wheeler-Dewitt equation. Sct. 2 summarizes results which are already known to specialists [4] [5] [6]. We present them to have a larger audience but mainly to ease the comparison with the new calculations in sct. 3 on fourth order gravity [2] [11].

2 Quantization of the Friedmann equation

We deduce the classical Friedmann equation in 2.1., quantize it in 2.2. and add a scalar field as source in 2.3.

2.1 The classical equation

The metric for the spatially flat Friedmann model is

$$ds^2 = N(t)^2 dt^2 - a(t)^2(dx^2 + dy^2 + dz^2) \tag{2.1}$$

[1]This paper is in final form and no version of it will be submitted for publication elsewhere

where N and a are restricted to positive values. For $N = 1$, t is synchronized time: With $a = \exp\alpha$ we get

$$ds^2 = N(t)^2 dt^2 - e^{2\alpha}(dx^2 + dy^2 + dz^2) \tag{2.2}$$

The curvature scalar R for metric (2.2) reads

$$R = -6\ddot{\alpha}/N^2 + 6\dot{\alpha}\dot{N}/N^3 - 12(\dot{\alpha}/N)^2 \tag{2.3}$$

where the dot denotes d/dt. The Einstein–Hilbert Lagrangian density is

$$\mathbf{L}_{EH} = -\frac{R\sqrt{-g}}{16\pi G} \tag{2.4}$$

where $g = \det g_{ij}$ is the determinant of the metrical tensor. For equation (2.2) we have $\sqrt{-g} = N\exp(3\alpha)$, and (2.4) reads (in units where $8\pi G = 1$)

$$\mathbf{L}_{EH} = (3\ddot{\alpha}/N - 3\dot{\alpha}\dot{N}/N^2 + 6\dot{\alpha}^2/N)e^{3\alpha} \tag{2.5}$$

By a coordinate transformation it is always possible to reach $N = 1$. But in connection with variational derivatives it is not arbitrary whether we do this before or after the variation. To show the difference we first set $N = 1$. Then equation (2.5) becomes

$$\mathbf{L}_{EH} = 3(\ddot{\alpha} + 2\dot{\alpha}^2)e^{3\alpha} \tag{2.6}$$

The variational derivative of (2.6) with respect to α is

$$\begin{aligned} 0 \;\; = \;\; & \delta\mathbf{L}_{EH}/\delta\alpha = (\partial\mathbf{L}_{EH}/\partial\alpha) - \frac{d}{dt}(\partial\mathbf{L}_{EH}/\partial\dot{\alpha}) + \frac{d^2}{dt^2}(\partial\mathbf{L}_{EH}/\partial\ddot{\alpha}) \\ & = 9(\ddot{\alpha} + 2\dot{\alpha}^2)e^{3\alpha} - \frac{d}{dt}(12\dot{\alpha}e^{3\alpha}) + \frac{d^2}{dt^2}(3e^{3\alpha}) \end{aligned}$$

The equation $0 = \delta\mathbf{L}_{EH}/\delta\alpha$ is a necessary but not a sufficient condition for the stationarity of the Einstein–Hilbert action. The reason is that we fixed one coordinate before performing the variation. The better way to proceed is the following: We perform the variational derivative of equation (2.3) with respect to N:

$$0 = \delta\mathbf{L}_{EH}/\delta N = \partial\mathbf{L}_{EH}/\partial N - \frac{d}{dt}(\partial\mathbf{L}_{EH}/\partial\dot{N})$$

Afterwards we put $N = 1$ and get

$$0 = (-3\ddot{\alpha} - 6\dot{\alpha}^2)e^{3\alpha} + (3\ddot{\alpha} + 9\dot{\alpha}^2)e^{3\alpha}, \quad \text{i.e.} \quad \dot{\alpha}^2 = 0$$

$\alpha(t)$ is a solution if and only if $\alpha = const.$ This corresponds to the fact that (2.2) is the flat Minkowski space-time if and only if $\alpha = const.$ and this is the only solution to the Einstein vacuum equation and a Friedmann model. To cancel the second derivative in (2.5) we add a divergence

$$\mathbf{L}_{div} = -\frac{d}{dt}(3\dot{\alpha}e^{3\alpha}/N) \tag{2.7}$$

$$\begin{aligned} \mathbf{L} \;\; = \;\; & \mathbf{L}_{div} + \mathbf{L}_{EH} \\ & = (3\ddot{\alpha}/N - 3\dot{\alpha}\dot{N}/N^2 + 6\dot{\alpha}^2/N - 3\ddot{\alpha}/N + 3\dot{\alpha}\dot{N}/N^2 - 9\dot{\alpha}^2/N)e^{3\alpha} \\ & = (-3\dot{\alpha}^2/N)e^{3\alpha} \end{aligned} \tag{2.8}$$

The canonical coordinate in superspace is α, the corresponding velocity is $\dot{\alpha}$, the momentum π_α corresponding to α is defined by

$$\pi_\alpha = \frac{\partial \mathbf{L}}{\partial \dot{\alpha}}, \quad \text{one gets}$$

$$\pi_\alpha = -6\dot{\alpha}e^{3\alpha}/N.$$

The Hamiltonian density is $\mathbf{H} = \dot{\alpha}\pi_\alpha - \mathbf{L}$, one gets

$$\mathbf{H} = -3\dot{\alpha}^2 e^{3\alpha}/N = -N\pi_\alpha^2 e^{-3\alpha}/12 \tag{2.9}$$

The Hamiltonian constraint is $0 = \frac{\partial \mathbf{H}}{\partial N}$, i.e.: $H = 0$, which is equivalent to $\alpha = const.$

Equation (2.2) is thought to represent a cosmological model with spatial torus-topology $(S^1)^3$, i.e.: x, y, z are periodic coordinates $0 \le x, y, z \le 1$ with identified endpoints. Then the proper Hamiltonian $\bar{\mathbf{H}}$ is the invariant integral of the Hamiltonian density $\mathbf{H}$ over the space-like hypersurface $t = const.$:

$$\bar{\mathbf{H}} = \int_0^1\int_0^1\int_0^1 \mathbf{H}\sqrt{h}\ dxdydz \tag{2.10}$$

$h = \det h_{\gamma\delta}$ is the determinant of the spatial metric. Here

$$h_{\gamma\delta} = \text{diag}\ (e^{2\alpha}, e^{2\alpha}, e^{2\alpha}), \quad \text{i.e.:} \quad \sqrt{h} = e^{3\alpha}$$

From (2.9) and (2.10) we get

$$\bar{\mathbf{H}} = -\frac{N}{12}\int_0^1\int_0^1\int_0^1 \pi_\alpha^2\ dxdydz = -\frac{N}{12}\pi_\alpha^2 \tag{2.11}$$

In this manner, we often cancel the operator $\int_0^1\int_0^1\int_0^1\ dxdydz$

2.2 The Quantization

The quantization rule is that all physical quantities are replaced by operators denoted by a hat. Only the coordinates (here: N and α) go over to identical C-numbers and no hat is necessary. The usual way to proceed is to replace π_α by $-i\hbar\nabla_\alpha$, where ∇_α is the derivative with respect to α. Therefore equation (2.11) goes over to

$$\hat{\mathbf{H}} = -\frac{N}{12}\hat{\pi}_\alpha^2 \tag{2.12}$$

It is connected with the uncertainty relation that α and $\hat{\pi}_\alpha$ do not comute. So we have the following situation: Let $f(\alpha)$ and $g(\alpha)$ be arbitrary functions, then for example let:

$$\mathbf{H}_{\mathbf{f},\mathbf{g}} = -\frac{N}{12}e^f\pi_\alpha e^{g-f}\pi_\alpha e^{-g} \tag{2.13}$$

it holds: $\mathbf{H_{f,g}} = \mathbf{H}$. But the quantized versions differ: this is the factor-ordering problem. The Wheeler-DeWitt equation is then

$$\hat{\mathbf{H}}_{\mathbf{f},\mathbf{g}}\psi(\alpha) = 0 \tag{2.14}$$

It can be integrated once to:

$$e^{g-f}\nabla_\alpha(e^{-g}\psi(\alpha)) = C = const. \tag{2.15}$$

The complete solution reads, the second integration constant being D:

$$\psi(\alpha) = [D + C\int_0^\alpha e^{f(\beta)-g(\beta)}]e^{g(\alpha)} \tag{2.16}$$

This is not a satisfactory solution because we have two free functions in the whole system. This situation can be improved by requiring some coordinate invariances of the Wheeler-DeWitt equation. We perform a coordinate transformation: $x \to xe^\tau$, $y \to ye^\tau$, $z \to ze^\tau$ working at equation (2.2) one can see that this coordinate transformation leads to $\alpha \to \alpha - \tau$. The requirement of this coordinate invariance leads to the condition that for each real τ and each solution $\psi(\alpha)$ of the Wheeler-DeWitt equation also $\psi(\alpha - \tau)$ is a solution of it. One can show that this condition is fulfilled for solution (2.16) if and only if f and g are linear functions. For linear functions this statement is trivial, but the other direction needs a not fully trivial proof. The absolute part of f and g can be absorbed by a redefinition of C and D resp. So we may insert $f(\alpha) = f\alpha$, $g(\alpha) = g\alpha$ with constants f and g. With this we find from equation (2.16)

$$\psi(\alpha) = [D + \frac{C}{f-g}(e^{(f-g)\alpha} - 1)]e^{g\alpha} \quad \text{for } f \neq g \tag{2.17}$$

and

$$\psi(\alpha) = [D + C\alpha]e^{g\alpha} \quad \text{for } f = g \tag{2.18}$$

By a redefinition of the constants we can write (2.17) as

$$\psi(\alpha) = De^{g\alpha} + Ce^{f\alpha} \quad \text{with } f \neq g \tag{2.19}$$

If we interpret $|\psi|^2$ as probability density then the normalization condition

$$\int_{-\infty}^{+\infty} |\psi|^2 \, d\alpha = 1$$

should be fulfilled, but for (2.19) this is not possible for any values of the constants. This fact sometimes occur in quantum mechanics. Nevertheless a probability interpretation is available for some cases. Our situation corresponds to a free particle solution, which also is not normalizable.

From the above chosen linear functions f and g we find the Wheeler-DeWitt equation as

$$\begin{aligned} 0 \quad = \quad & \hat{\mathbf{H}}_{\mathbf{f},\mathbf{g}}\psi(\alpha) = -\frac{N}{12}e^{f\alpha}\hat{\pi}_\alpha e^{(g-f)\alpha}\hat{\pi}_\alpha e^{-g\alpha} \ \psi(\alpha) \\ = & +\frac{N}{12}\hbar^2(e^{f\alpha}\nabla_\alpha e^{(g-f)\alpha}\nabla_\alpha e^{(-g)\alpha} \ \psi(\alpha) \end{aligned} \tag{2.20}$$

The most natural values for f and g seem to be $f = g = 0$, because this makes equation (2.20) most simple. But this is a dangerous argument as can be seen by the following.

If we set $f = g = 0$ then we get:

$$0 = \hat{\mathbf{H}}_{\mathbf{f},\mathbf{g}}\psi(\alpha) = +\frac{N}{12}\hbar^2\nabla_\alpha^2\psi(\alpha) \tag{2.21}$$

Here all linear functions $\psi(\alpha)$ are solutions. To see the problem, we make the same procedure once more: a will be taken as coordinate, but now with the constraint $a > 0$. So we insert $\alpha = \ln a$ and $\dot{\alpha} = \dot{a}/a$ into equation (2.8) :

$$\mathbf{L} = -\frac{3}{N}a\dot{a}^2, \qquad \pi_\alpha = \frac{\partial\mathbf{L}}{\partial\dot{a}} = -\frac{6}{N}a\dot{a}$$

$$\mathbf{H} = \dot{a}\pi_a - \mathbf{L} = -\frac{3}{N}a\dot{a}^2 = -\frac{N}{12}\pi_a^2 a^{-1}$$

$$\bar{\mathbf{H}} = -\frac{N}{12}\pi_a^2 a^3, \qquad \text{because now} \qquad \sqrt{h} = a^3.$$

$$\text{Because of} \quad \frac{d\alpha}{da} = \frac{1}{a} \qquad \text{we have} \quad \nabla_\alpha = a\nabla_a.$$

The "most natural factor ordering" in (2.21) leads now to

$$0 = \hat{\mathbf{H}}_{\mathbf{f},\mathbf{g}}\psi(\alpha) = \frac{N}{12}\hbar^2 a\nabla_a a\nabla_a\psi(\alpha), \qquad \text{which looks less natural.}$$

2.3 A scalar field

The main application for quantum cosmology with a scalar field Φ is the inflationary phase where Φ is approximately constant and the potential $V(\Phi)$ is approximately a cosmological term. Then the Lagrangian for the cosmological term is simply

$$\mathbf{L}_\Lambda = -\Lambda\sqrt{-g} \tag{2.22}$$

We add this $\mathbf{L}_\Lambda$ to equation (2.8) and get

$$\tilde{\mathbf{L}} = \mathbf{L}_{EH} + \mathbf{L}_\Lambda = -e^{3\alpha}\,(\frac{3\dot{\alpha}^2}{N} + \Lambda N) \tag{2.23}$$

The classical solution follows from:

$$\frac{\partial\tilde{\mathbf{L}}}{\partial N} = 0, \quad \text{i.e.:} \quad 3\dot{\alpha}^2 = \Lambda, \quad \alpha = Ht, \quad 3H^2 = \Lambda,$$

where H is the Hubble parameter. This is the deSitter space-time. Instead of (2.9) we get for the Hamiltonian density:

$$\mathbf{H} = (\Lambda e^{6\alpha} - \frac{\pi_\alpha^2}{12})Ne^{-3\alpha} \tag{2.24}$$

and from (2.11)

$$\bar{\mathbf{H}} = (\Lambda e^{6\alpha} - \frac{\pi_\alpha^2}{12})N \tag{2.25}$$

The analogous equation to (2.21) then reads

$$0 = \hat{\mathbf{H}}\psi(\alpha) = N(\frac{\hbar^2}{12}\nabla_\alpha^2 + \Lambda e^{6\alpha})\psi(\alpha) \tag{2.26}$$

the solution is much more complicated. By a suitable translation of α we get the equation

$$(\nabla_\alpha^2 + e^{6\alpha})\psi(\alpha) = 0 \tag{2.27}$$

With $\psi = e^{\int \nu d\alpha}$ this goes over to

$$\frac{d\nu}{d\alpha} + \nu^2 + e^{6\alpha} = 0 \qquad \text{hence follows} \quad \frac{d\nu}{d\alpha} < 0 \quad \text{always.} \tag{2.28}$$

We use some approximations : let at $\alpha \geq \alpha_0$: $\nu(\alpha) \to +\infty$ and at $\alpha \leq \alpha_1 > \alpha_0$: $\nu(\alpha) \to -\infty$. At $\alpha \geq \alpha_0$

we have $\psi(\alpha) \approx \sin[e^{3\alpha_0}(\alpha - \alpha_0)]$

$$\nu(\alpha) = \frac{d}{d\alpha} \ln \sin[e^{3\alpha_0}(\alpha - \alpha_0)] = e^{3\alpha_0} \cot[e^{3\alpha_0}(\alpha - \alpha_0)]$$

For $\alpha \to \infty$ the oscillations are quite good approximated by

$$\psi(\alpha) = C \sin[e^{3\alpha}(\alpha - \alpha_0)] \tag{2.29}$$

The space of solutions has topology $S^1 \times \mathbf{R}$, characterized e. g. by phase α_0 and amplitude C. For $\alpha \to -\infty$ we make an ansatz ensuring a compensation of $\frac{d\nu}{d\alpha}$ with $e^{6\alpha}$ so that $\nu \to 0$ as $\alpha \to -\infty$:

$$\nu = \sum_{j=1}^{+\infty} c_j e^{6j\alpha} \tag{2.30}$$

Inserting (2.30) into (2.28) we get step by step $c_1 = \frac{1}{6}$, $c_2 = \frac{1}{432}$ and each c_j is uniquely determined.

The leading term is $\nu = \frac{1}{6}e^{6\alpha}, \qquad \psi(\alpha) = C \exp(-\frac{1}{36}e^{6\alpha})$

which becomes approximately constant for $\alpha \to -\infty$. All other solutions have the behaviour $\nu = \frac{1}{\alpha - \alpha_1}$ for $\alpha \to -\infty$, which leads to asymptotically linear $\psi(\alpha)$. Here also for this situation the solution ψ can not be normalized.

Next, let us introduce a minimally coupled scalar field $\mathbf{\Phi}$. This means for the Lagrangian:

$$\mathbf{L}_{\mathbf{\Phi}} = \sqrt{-g}\,[\frac{1}{2}g^{ij}\Phi_{,i}\Phi_{,j} - V(\Phi)] \tag{2.31}$$

A comparison with (2.22) shows that for a constant Φ, $\Lambda = V(\Phi)$ is the effective cosmological term. The total Lagrangian is now

$$\begin{aligned} \mathbf{L} &= \mathbf{L}_{\mathbf{\Phi}} + \mathbf{L}_{\mathbf{EH}} \\ &= \sqrt{-g}\,[-\frac{R}{2} + \frac{1}{2} g^{ij}\Phi_{,i}\Phi_{,j} - V(\Phi)] \end{aligned} \tag{2.32}$$

We restrict again to a spatially flat Friedmann model (2.2) and restrict Φ to depend on t only. We add again the divergence (2.7), use (2.8) and the relation

$$g^{ij}\Phi_{,i}\Phi_{,j} = \frac{1}{N^2}\dot{\Phi}^2 \tag{2.33}$$

to get

$$\mathbf{L} = e^{3\alpha}\,(-\frac{3\dot{\alpha}^2}{N} + \frac{1}{2N}\dot{\Phi}^2 - NV(\Phi)\,) \tag{2.34}$$

Now the superspace is two-dimensional, its coordinates can be chosen as $q^1 = \alpha$, $q^2 = \Phi$, the corresponding velocities are $\dot{q}^A$ $(A = 1, 2)$. The momentum π_A corresponding to q^A is defined by:

$$\pi_A = \frac{\partial \mathbf{L}}{\partial \dot{q}^A}, \qquad \text{i.e. :} \qquad \pi_1 = -\frac{6\dot{\alpha}e^{3\alpha}}{N}, \quad \pi_2 = \frac{\dot{\Phi}e^{3\alpha}}{N}.$$

At this moment it would be more appropriate to change the units by a factor 6: from $8\pi G = 1$ to $4\pi G = 3$. So we write instead of equation (2.34):

$$\mathbf{L} = e^{3\alpha}\,(-\frac{\dot{\alpha}^2}{2N} + \frac{1}{2N}\dot{\Phi}^2 - NV(\Phi)\,) \tag{2.35}$$

$$\pi_1 = -\frac{\dot{\alpha}e^{3\alpha}}{N}, \quad \pi_2 = \frac{\dot{\Phi}e^{3\alpha}}{N}.$$

The Hamiltonian density is:

$$\mathbf{H} = \dot{q}^A\pi_A - \mathbf{L} = Ne^{-3\alpha}\,(-\frac{\pi_1^2}{2} + \frac{\pi_2^2}{2} + V(\Phi)e^{6\alpha}) \tag{2.36}$$

and the Hamiltonian is

$$\bar{\mathbf{H}} = -\frac{N}{2}\pi_1^2 + \frac{N}{2}\pi_2^2 + NV(\Phi)e^{6\alpha}$$

again $\frac{\partial \bar{\mathbf{H}}}{\partial N} = 0$ is the $00-$component of the Einstein equation. So the Wheeler-DeWitt equation becomes:

$$[\frac{\hbar^2}{2}\,(\nabla_\alpha^2 - \nabla_\Phi^2) + V(\Phi)e^{6\alpha}]\;\psi(\alpha, \Phi) = 0 \tag{2.37}$$

Now we introduce the superspace metric g_{AB} by

$$g_{AB} = \frac{\partial^2 \mathbf{L}}{\partial \dot{q}^A \partial \dot{q}^B} = \frac{\partial \pi_A}{\partial \dot{q}^B} \tag{2.38}$$

We get $g_{11} = -\frac{e^{3\alpha}}{N}$, $g_{22} = \frac{e^{3\alpha}}{N}$, $g_{12} = 0$ and can write

$$\mathbf{L} = \frac{1}{2} g_{AB} \dot{q}^A \dot{q}^B - NV(\Phi) e^{3\alpha} \tag{2.39}$$

and

$$\mathbf{H} = \frac{1}{2} g^{AB} \pi_A \pi_B + NV(\Phi) e^{3\alpha} \tag{2.40}$$

That g_{AB} is not positive definite expresses the fact that Einstein gravity is not stable [1] [10]: g_{11}, the coefficient describing the volume-expansion, is negative. In the present case the superspace is a space of non-vanishing constant curvature. The analogous proper superspace metric f_{AB} for the proper Hamiltonian is conformally related to g_{AB} with the conformal factor $e^{3\alpha}$:

$f_{11} = -\frac{1}{N}$, $f_{22} = \frac{1}{N}$, $f_{12} = 0$ and then we get

$$\bar{\mathbf{H}} = \frac{1}{2} f^{AB} \pi_A \pi_B + NV(\Phi) e^{3\alpha} \tag{2.41}$$

At this point it is indeed appropriate to introduce a proper time coordinate i.e.: $N = 1$. Then the proper superspace metric is: $f_{AB} = diag(-1, 1)$, i.e. simply the metric the flat two-dimensional Minkowski space-time. The Wheeler-deWitt equation can be expressed as:

$$(\frac{\hbar^2}{2} \Box - V(\Phi) e^{6\alpha})\ \psi(A) = 0, \tag{2.42}$$

where $\Box$ is the flat space d'Alembertian of the metric f_{AB}. f_{AB} is a tensor in superspace and so the usual tensor transformation rules can be applied. For example: a usual coordinate transformation in the superspace is to replace e^α by a. Then we know without any calculation that also here the proper superspace is flat. A "Super-Lorentz-transformation" this is a Lorentz-transformation in superspace is possible in the $\alpha - \Phi-$plane. It mixes geometrical and matter objects. Sometimes this transformation may be helpful in solving the Wheeler-deWitt equation. Now we look for solutions of equation (2.37) for a constant potential $V(\Phi) = \Lambda \geq 0$.

First case: $\Lambda = 0$

The general solution is: $\psi(\alpha, \Phi) = \psi_1(\alpha - \Phi) + \psi_2(\alpha + \Phi)$ with arbitrary functions ψ_1 and ψ_2. The corresponding classical equations are $\ddot{\Phi} + 3\dot{\alpha}\dot{\Phi} = 0$ and $\dot{\alpha}^2 = \dot{\Phi}^2$. They can be solved in closed form by $\alpha = \pm\Phi + C$, $\Phi = \frac{1}{3}\ln(t - t_0)$. The solution with upper sign corresponds to $\psi_1(z) = \delta(z - C)$, $\psi_2 = 0$, i.e.: $\psi(\alpha, \Phi) = \delta(\alpha - \Phi - C)$, where δ is the delta distribution. Analogously one can describe the solution with lower sign. But this simple analogy to the classical solutions is not to be expected for more complicated situations: already for the second case this becomes apparent.

Second case: $\Lambda > 0$: We have to solve

$$[\frac{\hbar^2}{2}\ (\nabla_\alpha^2 - \nabla_\Phi^2) + 2\Lambda e^{6\alpha}]\ \psi(\alpha, \Phi) = 0 \tag{2.43}$$

(In comparison to equation (2.26) one has to observe that meanwhile we had changed the units by a factor 6.) The classical equations are:

$$\ddot{\Phi} + 3\dot{\alpha}\dot{\Phi} = 0 \qquad \text{and} \qquad \dot{\alpha}^2 = \dot{\Phi}^2 + 2\Lambda \tag{2.44}$$

The solution with $\Phi = 0$ is the deSitter space time:

$$\alpha = \pm\sqrt{2\Lambda}(t - t_0) \tag{2.45}$$

It corresponds to the solution of (2.43): $\psi(\alpha, \Phi) = \psi(\alpha)\delta(\Phi)$, where $\psi(\alpha)$ is approximately given by (2.29) and (2.30). Classically α increases linearly with time and therefore the probabilities $|\psi|^2$ to find a special value are approximately constant. For the other solutions of (2.34) we make the ansatz:

$$\psi(\alpha, \Phi) = \psi(\alpha)e^{\lambda\Phi}, \qquad \lambda = const.$$

This leads to the equation:

$$(\nabla_\alpha^2 - \lambda^2 + \frac{2\Lambda}{\hbar^2}e^{6\alpha})\ \psi(\alpha) = 0$$

The corresponding classical solutions are calculated as follows: let $y = \dot{\Phi}$, then $\dot{\alpha} = \pm\sqrt{y^2 + 2\Lambda}$. We choose the time direction so that $\dot{\alpha} > 0$ and get:

$$\dot{y} = -3y^2\sqrt{y^2 + 2\Lambda}$$

This can be integrated to:

$$3t\sqrt{2\Lambda} = \ln y - \ln[\sqrt{2\Lambda} + \sqrt{y^2 + 2\Lambda}] \tag{2.46}$$

(If $\dot{\Phi} < 0$ we take $-\Phi$ instead of Φ.) Equation (2.46) can be inverted to:

$$y = \frac{-\sqrt{2\Lambda}}{\sinh(3t\sqrt{2\Lambda})}, \quad \Phi = -\frac{1}{3}\ln\tanh(\frac{3}{2}t\sqrt{2\Lambda}) \tag{2.47}$$

and then we get from (2.44)

$$\dot{\alpha} = \sqrt{2}\coth(3t\sqrt{2\Lambda}), \quad \alpha = \frac{1}{3}\ln\sinh(3t\sqrt{2\Lambda}) + C \tag{2.48}$$

For $t \to \infty$ (2.48) is asymptotically given by (2.45), this is the attractor property of the deSitter space-time.

3 Fourth Order Gravity

Now we start from an other theory and deduce the Wheeler-deWitt equation for the same geometry. The problem with deducing the Wheeler-DeWitt equation from higher order field equations is: the transformation to Hamiltonian formalism is not a priori possible. One has to introduce auxiliary fields which bring the field equation to second order. There exist different possibilities to define such auxiliary fields.

The simplest Lagrangian of fourth order gravity is:

$$\mathbf{L_4} = R^2\sqrt{-g} \tag{3.1}$$

To maintain the advantages of the Hamiltonian formalism we introduce an additional auxiliary field:

$$\beta = \ddot{\alpha} \tag{3.2}$$

Then the 00−component of the field equation following from (3.1) reads [8] [11] for the metric (2.2) with N = 1:

$$0 = 2\dot{\alpha}\dot{\beta} - \beta^2 + 6\dot{\alpha}^2\beta \tag{3.3}$$

The classical solution $\dot{\alpha} = const.$ corresponds to the deSitter space-time. The classical solution $\beta = -2\dot{\alpha}^2$, i.e.: $\alpha = \frac{1}{2}\ln t$ corresponds to Friedmann's radiation model $R = 0$. All other classical solutions have the following behaviour: $\alpha \approx \frac{1}{2}\ln t$ for $t \to 0$ and $\dot{\alpha} = const.$ for $t \to \infty$. Again this represents the attractor property of the deSitter space-time.

Now we proceed like in section two. The metric is:

$$ds^2 = dt^2 - e^{2\alpha(t)}(dx^2 + dy^2 + dz^2) \tag{3.4}$$

The Lagrangian is up to unessential divergences:

$$\mathbf{L} = (2\dot{\alpha}\dot{\beta} + 6\dot{\alpha}^2\beta + \beta^2)e^{3\alpha}, \tag{3.5}$$

where (3.2) has been applied. $q^1 = \alpha$, $q^2 = \beta$ are the coordinates: $q^A \quad (A = 1, 2)$ of superspace. The corresponding momenta are:

$$\pi_A = \frac{\partial \mathbf{L}}{\partial \dot{q}^A}, \quad \text{i.e.:} \quad \pi_\alpha = (2\dot{\beta} + 12\dot{\alpha}\beta)e^{3\alpha}, \quad \pi_\beta = 2\dot{\alpha}e^{3\alpha}$$

The Hamiltonian is:

$$\mathbf{H} = \pi_A\dot{q}^A - \mathbf{L} = (2\dot{\alpha}\dot{\beta} + 6\dot{\alpha}^2\beta - \beta^2)e^{3\alpha}. \tag{3.6}$$

As it should be $\mathbf{H} = 0$ is the 00−component of the field equation and:

$$\mathbf{H} = (\frac{1}{2}\pi_\alpha\pi_\beta - \frac{3}{2}\beta\pi_\beta^2)e^{-3\alpha} - \beta^2 e^{3\alpha} \tag{3.7}$$

The superspace metric is, as in (2.38):

$$g_{AB} = \frac{\partial^2 \mathbf{L}}{\partial \dot{q}^A \dot{q}^B} \tag{3.8}$$

$g_{11} = 12\beta e^{3\alpha}$, $g_{12} = 2e^{3\alpha}$, $g_{22} = 0$ and $g = -4e^{6\alpha}$. This can be inverted to: $g^{11} = 0$, $g^{12} = \frac{1}{2}e^{-3\alpha}$ and $g^{22} = -3\beta e^{-3\alpha}$.
With this notation we get:

$$\bar{\mathbf{H}} = \frac{1}{2}g^{AB}\pi_A\pi_B - \beta^2 e^{3\alpha} \tag{3.9}$$

Accordingly we get the Wheeler-deWitt equation:

$$0 = (-\frac{1}{2}\hbar^2\nabla_\alpha\nabla_\beta + \frac{3}{2}\hbar^2\beta\nabla_\beta^2 - \beta^2 e^{6\alpha})\ \psi(\alpha, \beta) \tag{3.10}$$

As in the former section, the superspace metric has Lorentzian signature, i.e.: The Wheeler-deWitt equation is a normal hyperbolic wave equation. One sees that the superspace is flat by calculating the curvature scalar. But we want to get an explicit transformation leading to the usual flat space-time.

$$\text{Let} \quad \sigma = e^{-3\alpha} \qquad \text{and} \qquad \tau = \frac{2\beta}{3}e^{3\alpha} \tag{3.11}$$

Then $d\sigma = -3e^{-3\alpha}d\alpha$, $d\tau = \frac{2}{3}e^{3\alpha}d\beta + 2\beta e^{3\alpha}d\alpha$ and

$$-2d\sigma d\tau = 4d\alpha d\beta + 12\beta d\alpha^2, \tag{3.12}$$

which represents just the above metric up to factor $e^{3\alpha}$.

4 Discussion

From (3.11) one can see that only the range $\sigma > 0$ is covered, i.e.: one half of the flat space-time only. Equation (3.11) can be inverted to:

$$\alpha = -\frac{1}{3}\ln(\sigma), \qquad \beta = \frac{3}{2}\sigma\tau \tag{4.1}$$

With superspace coordinates σ and τ the superspace metric g is simply $g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. One should expect the Wheeler-deWitt equation to be:

$$0 = (\hbar^2 \nabla_\sigma \nabla_\tau - \frac{9}{4}\tau^2)\ \psi(\sigma, \tau) \tag{4.2}$$

But one must be very careful in dealing in this manner.

First: Inserting (3.11) in (3.12) one expects to get (3.10), but instead of $T_1 = \beta\nabla_\beta^2$ there appears the term $T_2 = \nabla_\beta \beta \nabla_\beta = \nabla_\beta + \beta\nabla_\beta^2$. If one adopts (4.2) to be "the most natural factor ordering", then we have to replace the term T_1 in (3.10) by Term T_2.

Second: The corresponding solutions $\psi(\sigma, \tau)$ and $\psi(\alpha, \beta)$ are not directly comparable: because of the possible interpretation of $|\psi|^2$ as probability density, the solutions must be multiplied by the determinant of the Jacobian. This determinant can be expressed by the quotient of the determinants of the two superspace metrics. For our example we have the lucky circumstance that this quotient becomes constant and so we may identify $\psi(\alpha, \beta)$ with $\psi(\sigma, \tau)$ at respective points.

Third: The transformation $\tau \to -\tau$ changes the sign of the first term in (4.2), so that there is no possibility to denote the second term: $\frac{9}{4}\tau^2$ to be the "positive effective potential", because it is with the same right a "negative" one.

To solve equation (4.2) we first perform a transformation:

$$\begin{aligned} \psi(\sigma, \tau) &= e^{A(\sigma,\tau)}, \qquad \text{which yields} \\ \hbar^2[\nabla_\sigma \nabla_\tau A(\sigma, \tau) + \nabla_\sigma A(\sigma, \tau)\nabla_\tau A(\sigma, \tau)] &= \frac{9}{4}\tau^2. \end{aligned}$$

Then we use following ansatz: $A = \sigma B(\tau) + C(\tau)$ From this we get:

$$\frac{\partial B(\tau)}{\partial \tau} + \sigma B(\tau)\frac{\partial B(\tau)}{\partial \tau} + B(\tau)\frac{\partial C(\tau)}{\partial \tau} = \frac{9}{4}\tau^2. \tag{4.3}$$

If we set $B(\tau) = \rho_1 = const.$ we get $A = \rho_1\sigma + C(\tau)$ and $\hbar^2 \rho_1 \frac{\partial C(\tau)}{\partial \tau} = \frac{9}{4}\tau^2$. Therefore the result for $C(\tau)$ is:

$$C = \frac{3}{4\rho_1 \hbar^2}\tau^3 + \rho_2 \qquad \text{and}$$

ρ_2 is an integration constant, which can be determined by the normalization condition. Finally we get for ψ:

$$\psi(\sigma, \tau) = Me^{C\sigma + D\tau^3} \qquad \text{with} \quad M = e^{\rho_2} \quad \text{and} \quad CD = \frac{3}{4\hbar^2}. \tag{4.4}$$

A further discussion and a more general calculation, together with a large reference list can be found in reference [7].

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