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# COMPLEX POISSON MANIFOLDS

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Abstract. We discuss some elementary properties of complex Poisson manifolds which are introduced by analogy to (real) Poisson ones. The properties are formulated in the language of sheaf theory and concern holomorphic geometrical objects. In particular, we speak of Hamiltonian morphisms which should be characteristic of a future formulation of complex Hamiltonian mechanics. Moreover, we consider the canonical complex symplectic structure on the holomorphic cotangent manifold of a complex manifold which can be regarded as a complex analogue of the real symplectic structure of a phase space.
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## Introduction

Traditionally, Hamiltonian mechanics has a modern formulation relative to symplectic manifolds (see [1]) and, recently, relative to more general objects such as Poisson manifolds (see [11]) which are here referred to as real Poisson ones. Both these kinds of manifolds are of real differential geometry nature. For example, a symplectic structure on a smooth manifold is a real closed differential 2-form. However, as one knows Kähler manifolds afford basic examples of symplectic ones and, in this sense, we can speak of the complex origin of real symplectic structures. On the other hand, all realizations of the Penrose programme show that we can succeed in applying complex analytic methods to many problems first formulated in real variable systems (see [2]). This gives us a base to expect that also in Hamiltonian mechanics the passing from real variable data to complex ones can be useful. More generally, our programme consists in applying complex analysis methods to the study of Hamiltonian mechanics in the traditional real formulation as well as in a new complex formulation. To be precise, one can find papers in this direction (see [4], [5] and [6]).

In this paper we introduce the notion of a complex Poisson manifold and then discuss some elementary properties of such manifolds. Complex Poisson manifolds should be regarded as direct complex analogues of real Poisson manifolds. Roughly speaking, they arise by replacing the corresponding smooth geometrical objects with holomorphic ones. This involves the necessity of using the language of sheaf theory, so we should speak of sheaves of germs of holomorphic functions, holomorphic forms, holomorphic tensor fields, etc. Clearly, the full passing from real to complex Poisson manifolds is more complicated than such a replacing procedure. We must be careful whether

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new analogical complex formulations can hold for holomorphic geometrical objects. An appropriate way to present properties of complex Poisson manifolds is a systematic development of the corresponding theory.

There is another approach to complex Poisson manifolds which is characteristic of complex differential geometry. Namely, we can regard such manifolds as real smooth ones, which means that holomorphic objects are controlled by the corresponding standard differential equations. But here we can lose a clear analogy between properties of real and complex Poisson manifolds. On the other hand, such an approach is important because it allows us to describe relations between both the kinds of Poisson manifolds. In particular, in this way we can give new interesting examples of real Poisson manifolds (see [8]).

The paper obviously bears an introductory character and it can be considered as to some extent parallel to paper [10]. One can observe that a weak point of this paper are few examples of complex symplectic manifolds. The reason of such a situation comes from the fact that many examples of real symplectic manifolds are those of Kähler manifolds which cannot determine complex symplectic ones. However, we have $\mathbb{C}^{2n}$ and, more generally, the holomorphic cotangent manifold of a complex one to be a very interesting example of a complex symplectic manifold. Other complex symplectic manifolds can be obtained from the given ones by taking the operations of Poisson products and open submanifolds (Proposition 6.1). Finally, let us pay attention to the fact that besides complex Poisson manifolds which are considered to be direct complex analogues of real Poisson manifolds, one can define other complex manifolds which can be regarded as Kähler analogues of real Poisson ones (see [9]).

**0. Preliminaries.** Let $\mathbb{K}$ be a fixed field which can be the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex ones. Following [3], by a *Poisson* $\mathbb{K}$-*algebra* we shall mean a pair $(\mathcal{A}, \{\cdot,\cdot\})$ where $\mathcal{A}$ is an associative and commutative $\mathbb{K}$-algebra with identity and $\{\cdot,\cdot\}$ is a *Poisson bracket* on $\mathcal{A}$, that is, a $\mathbb{K}$-bilinear map from $\mathcal{A}\times\mathcal{A}$ to $\mathcal{A}$ satisfying the following axioms $(\alpha,\beta,\gamma\in\mathcal{A})$:

$(P1)$ $\{\alpha,\beta\} = -\{\beta,\alpha\}$ (antisymmetry);

$(P2)$ $\{\alpha,\beta\gamma\} = \{\alpha,\beta\}\gamma + \beta\{\alpha,\gamma\}$ (insertion differentiability);

$(P3)$ $\{\alpha,\{\beta,\gamma\}\} + \{\beta,\{\gamma,\alpha\}\} + \{\gamma,\{\alpha,\beta\}\} = 0$ (The Jacobi identity).

We can regard every Poisson $\mathbb{K}$-algebra as a $\mathbb{K}$-linear space equipped with a $\mathbb{K}$-algebra structure defined by an associative multiplication as well as a Lie one defined by a Poisson bracket. A *Poisson homomorphism* from a Poisson $\mathbb{K}$-algebra to another one is a homomorphism of both the algebra structures. We shall say a *real* (*complex*) Poisson algebra instead of a Poisson $\mathbb{R}$-algebra ($\mathbb{C}$-algebra).

By a complex manifold we shall always mean a finite dimensional one with a countable topological base. If $M$ is such a manifold, we denote by $\mathcal{O}_M$ the sheaf of germs of holomorphic functions on $M$. This means that for any nonempty open subset $U$ of $M$, $\mathcal{O}_M(U)$ is the complex algebra of all holomorphic functions on $U$, under the pointwise operations. By a (*holomorphic*) *Poisson structure* on $M$ we shall mean an assignment $\{\cdot,\cdot\}$ which associates to any open subset $U$ of $M$ a Poisson bracket $\{\cdot,\cdot\}_U$ on $\mathcal{O}_M(U)$ such that this assignment makes $\mathcal{O}_M$ be a sheaf of complex Poisson algebras on $M$. A *complex Poisson manifold* is a complex manifold $M$ equipped with a Poisson structure, denoted by $\{\cdot,\cdot\}_M$ and called the *Poisson* one of $M$, which means that $\mathcal{O}_M$ is a sheaf of complex Poisson algebras such that for any open subset $U$ of $M$ it is given the Poisson bracket $\{\cdot,\cdot\}_{M,U}$ on $\mathcal{O}_M(U)$. Clearly, there is a Poisson structure on every complex manifold, namely the zero one, denoted by $\{\cdot,\cdot\}_\theta$, which defines the zero Poisson bracket on $\mathcal{O}_M(U)$ for each open subset $U$ of $M$. Note that if $\dim M \leq 1$, then $\{\cdot,\cdot\}_\theta$ is a unique Poisson structure on $M$.

If $M$ and $N$ are complex Poisson manifolds, then a holomorphic map $f\colon M \to N$ is said to be *Poisson* in the case when for any open subsets $U$ of $M$ and $V$ of $N$, respectively, such that $f(U) \subseteq V$, the pullback map $f^*_{U,V}\colon \mathcal{O}_N(V) \to \mathcal{O}_M(U)$, defined by $\alpha \mapsto (\alpha \circ f)|U$, is a homomorphism of complex Poisson algebras. Complex Poisson manifolds with morphisms given by holomorphic Poisson maps form a category, called the category of *complex Poisson manifolds*. If $M$ is such a manifold, we can obviously speak of complex Poisson submanifolds of $M$. It is seen that every open submanifold $U$ of $M$ is again a complex Poisson manifold under the Poisson structure induced from $M$, which makes $U$ be a complex Poisson submanifold of $M$.

If $M$ is a complex manifold, $U$ an open subset of $M$ and $p \in M$, then we adopt the following notations:

$TM$: the holomorphic tangent manifold (bundle) of $M$;

$T_pM$: the complex tangent vector space of $M$ at $p$;

$T^*M$: the holomorphic cotangent manifold (bundle) of $M$;

$T^*_pM$: the complex cotangent vector space of $M$ at $p$;

$\mathcal{X}_M$: the sheaf of germs of holomorphic vector fields on $M$;

$\mathcal{X}_M(U)$: the $\mathcal{O}_M(U)$-module of all holomorphic vector fields on $U$;

$\mathcal{D}_M$: the sheaf of germs of holomorphic differential forms on $M$;

$\mathcal{D}_M(U)$: the $\mathcal{O}_M(U)$-module of all holomorphic differential forms on $U$;

$\mathcal{O}_n = \mathcal{O}_{\mathbb{C}^n}$ : the sheaf of germs of holomorphic functions on $\mathbb{C}^n$.

If $M$ is a complex manifold, we denote by $\otimes^2 TM$ ($\otimes^2 T^*M$) the tensor product $TM \otimes TM$ ($T^*M \otimes T^*M$) of the holomorphic tangent (cotangent) bundle of $M$ by itself as well as the total complex manifold of this bundle. The bundle will be called the *holomorphic bitangent* (*bicotangent*) *tensor bundle* of $M$. By a *bicovariant* (*bicontravariant*) *tensor field* on $M$ we shall mean a global holomorphic section of this bundle. We can consider the following list of tensor products of holomorphic vector bundles

$$\otimes^2 T^*M = T^*M \otimes T^*M,\ T^*M \otimes TM,\ TM \otimes T^*M,\ \otimes^2 TM = TM \otimes TM$$

which are called the *canonical holomorphic* 2*-tensor bundles* of $M$. Any global holomorphic section of such a bundle is called a 2*-tensor field* on $M$. In particular, so is any *bicovariant* (*bicontravariant*) tensor field on $M$.

Throughout this paper, all tensor fields are assumed to be *holomorphic*, and so, this concerns all vector fields and differential forms, too. For any tensor field $X$ defined on an open subset $U$ of $M$ and any point $p$ of $U$, we denote by $X_p$ the value of $X$ at $p$. If in addition $X$ is a vector field and $\alpha \in \mathcal{O}_M(U)$, then by $X(\alpha)$ we denote the action of $X$ on $\alpha$. Moreover, by $d\alpha$ we denote the *holomorphic differential* of $\alpha$.

Let $\Phi$ ($F$) be a bicovariant (bicontravariant) tensor field on $M$. For each $p \in M$ we denote by $\Phi_p$ ($F_p$) the value of $\Phi$ ($F$) at $p$ as well as the *associated* $\mathbb{C}$-bilinear map

$$\Phi_p\colon T_pM \times T_pM \to \mathbb{C} \quad (F_p\colon T^*_pM \times T^*_pM \to \mathbb{C}).$$

In turn, for any open subset $U$ of $M$ we have the *associated* $\mathcal{O}_M(U)$-bilinear map

$$\Phi_U\colon \mathcal{X}_M(U)\times\mathcal{X}_M(U)\to\mathcal{O}_M(U)\quad (F_U\colon \mathcal{D}_M(U)\times\mathcal{D}_M(U)\to\mathcal{O}_M(U))$$

defined by

$$\Phi_U(X,Y)(p)=\Phi_p(X_p,Y_p)\quad (F_U(\xi,\eta)(p)=F_p(\xi_p,\eta_p))\quad \text{for } p\in U.$$

For any $p\in M$ we have defined the *holomorphic pairing* of $M$ *at* $p$ which is the canonical $\mathbb{C}$-bilinear map $\langle\cdot,\cdot\rangle_{M,p}\colon T_p^*M\times T_pM\to\mathbb{C}$ satisfying the condition:

*if $U$ is an open subset of $M$ containing $p$, then*

$$\langle (d\alpha)_p, X_p\rangle_{M,p}=X(\alpha)(p) \textit{ for any } \alpha\in\mathcal{O}_M(U) \textit{ and } X\in\mathcal{X}_M(U).$$

In the pointwise manner, for any open subset $U$ of $M$ this pairing defines another one, called the *holomorphic pairing* of $M$ *over* $U$ which is the canonical $\mathcal{O}_M(U)$-bilinear map $\langle\cdot,\cdot\rangle_{M,U}\colon \mathcal{D}_M(U)\times\mathcal{X}_M(U)\to\mathcal{O}_M(U)$ satisfying the condition

$$\langle d\alpha, X\rangle_{M,U}=X(\alpha) \textit{ for any } \alpha\in\mathcal{O}_M(U) \textit{ and } X\in\mathcal{X}_M(U).$$

If $U$ is an open subset of $M$, then for any $X,Y\in\mathcal{X}_M(U)$ we denote by $[X,Y]_{M,U}$ the *Lie bracket* of vector fields on $U$ which makes $\mathcal{X}_M(U)$ be a Lie algebra. More generally, in this way we can regard $\mathcal{X}_M$ as a sheaf of Lie algebras on $M$.

Let $f\colon N\to M$ be a holomorphic map of complex manifolds. The (*holomorphic*) *differential* of $f$ is a morphism $f_*\colon TN\to TM$ of holomorphic vector bundles defined in the usual manner. In turn, this differential induces the corresponding one of degree 2 for holomorphic bitangent vector bundles of $N$ and $M$, respectively, which is denoted by $f_*\colon \otimes^2 TN\to\otimes^2 TM$ too. On the other hand, for any $q\in N$ the *pullback map* $f_q^*\colon T_p^*M\to T_q^*N$ ($p=f(q)$) is defined by

$$\langle f_q^*\xi, X\rangle_{N,q}=\langle \xi, f_*X\rangle_{M,p} \text{ for each } \xi\in T_p^*M\ (X\in T_qN).$$

Next, in the pointwise manner, for any open subset $U$ of $M$ we have defined the *pullback map* $f_U^*\colon \mathcal{D}_M(U)\to\mathcal{D}_N(V)$ ($V=f^{-1}(U)$) which can be regarded as $\mathcal{O}_M(U)_f$*-linear*, i.e.

$$f_U^*(\alpha\xi)=(f_U^*\alpha)f_U^*\xi \text{ for } \alpha\in\mathcal{O}_M(U) \text{ and } \xi\in\mathcal{D}_M(U)$$

where $f_U^*\alpha=\alpha\circ f\in\mathcal{O}_M(V)$ and $(f_U^*\xi)(q)=f_q^*\xi_p$ for $q\in V$, $p=f(q)$.

If $K$ is an arbitrary holomorphic tensor field on $M$, then by $f^\# K=K\circ f$ we denote the *pullback tensor field* of $K$ via $f$. In particular, if $K$ is a vector field, then for any open subset $U$ of $M$ we can associate with $f^\# K$ the linear map $(f^\# K)_U\colon \mathcal{D}_M(U)\to\mathcal{O}_N(V)$ defined by

$$(f^\# K)_U(\xi)=f_U^*K_U(\xi)=f_U^*\langle\xi, K|U\rangle_{M,U}.$$

Moreover, $(f^\# K)_U$ is $\mathcal{O}_M(U)_f$*-linear*, which means that

$$(f^\# K)_U(\alpha\xi)=(f_U^*\alpha)(f^\# K)_U(\xi) \text{ for } \alpha\in\mathcal{O}_M(U) \text{ and } \xi\in\mathcal{D}_M(U).$$

Let us take a bicontravariant tensor field $F$ on $M$ and consider the pullback tensor field $f^\# F=F\circ f$. Similarly, for any open subset $U$ of $M$ we can associate with $f^\# F$ the bilinear map $(f^\# F)_U\colon \mathcal{D}_M(U)\times\mathcal{D}_M(U)\to\mathcal{O}_N(V)$ defined by

$$(f^\# F)_U(\xi,\eta)=f_U^*F_U(\xi,\eta).$$

Moreover, $(f^\# F)_U$ is $\mathcal{O}_M(U)_f$*-bilinear*, which means that

$$(f^\# F)_U(\alpha\xi,\beta\eta)=(f_U^*\alpha)(f_U^*\beta)(f^\# F)_U(\xi,\eta) \text{ for } \alpha,\beta\in\mathcal{O}_M(U) \text{ and } \xi,\eta\in\mathcal{D}_M(U).$$

**1. Local structure.** We begin our considerations with the following examples of holomorphic Poisson structures.

**1.1. Example.** Consider the complex manifold $\mathbb{C}^{2n}$ with the global coordinate system $q_1, \dots, q_n, p_1, \dots, p_n$. If $U$ is an open subset of $\mathbb{C}^{2n}$ and if $\alpha, \beta \in \mathcal{O}_{2n}(U)$, we set

$$\{\alpha, \beta\}_U = \sum_{i=1}^{n} \left( \frac{\partial \alpha}{\partial p_i} \frac{\partial \beta}{\partial q_i} - \frac{\partial \alpha}{\partial q_i} \frac{\partial \beta}{\partial p_i} \right).$$

It is easy to verify that this formula defines a Poisson bracket on $\mathcal{O}_{2n}(U)$ in such a manner that $\mathcal{O}_{2n}$ becomes a sheaf of complex Poisson algebras on $\mathbb{C}^{2n}$, which makes $\mathbb{C}^{2n}$ be a complex Poisson manifold. The corresponding Poisson structure is called the *standard* one on $\mathbb{C}^{2n}$.

**1.2. Example.** Consider now the complex manifold $\mathbb{C}^n$ with the global coordinate system $z_1, \dots, z_n$. Let $c_{ij}$ $(i, j = 1, \dots, n)$ be an antisymmetric complex matrix. For any open subset $U$ of $\mathbb{C}^n$ and functions $\alpha, \beta \in \mathcal{O}_n(U)$ we set

$$\{\alpha, \beta\}_U = \sum_{i,j=1}^{n} c_{ij} \cdot \frac{\partial \alpha}{\partial z_i} \frac{\partial \beta}{\partial z_j}.$$

Similarly as in the previous example, this formula defines a Poisson structure on $\mathbb{C}^n$, called a *constant* one.

**1.3. Example.** Consider the complex manifold $\mathbb{C}^n$ as in above. Let $\{c_{ijk}\}$ $(i, j, k = 1, \dots, n)$ be a system of complex numbers satisfying the identities

$$c_{ijk} + c_{jik} = 0, \; c_{ijk} + c_{jki} + c_{kij} = 0.$$

Consider the matrix $\{F_{ij}\}$ of holomorphic linear functionals on $\mathbb{C}^n$ where

$$F_{ij}(z) = \sum_{k=1}^{n} c_{ijk} z_k \text{ for } z = (z_1, \dots, z_n).$$

For any open subset $U$ of $\mathbb{C}^n$ and functions $\alpha, \beta \in \mathcal{O}_n(U)$ we set

$$\{\alpha, \beta\}_U = \sum_{i,j=1}^{n} F_{ij} \cdot \frac{\partial \alpha}{\partial z_i} \frac{\partial \beta}{\partial z_j}.$$

¿From the above identities for $\{c_{ijk}\}$ it follows that this formula defines a Poisson structure on $\mathbb{C}^n$, called a *Lie-Poisson* one.

Let $M$ be a complex manifold. By a *Poisson system* in a local chart $(U; z_1, \dots, z_n)$ on $M$ we shall mean a matrix $\{F_{ij}\}$ $(i, j = 1, \dots, n)$ of holomorphic functions on $U$ satisfying the following identities $(i, j, k = 1, \dots, n)$:

$$F_{ij} = -F_{ji} \text{ and } \sum_{s=1}^{n} \left( F_{si} \cdot \frac{\partial F_{kj}}{\partial z_s} + F_{sj} \cdot \frac{\partial F_{ik}}{\partial z_s} + F_{sk} \cdot \frac{\partial F_{ji}}{\partial z_s} \right) = 0.$$

If $F = \{F_{ij}\}$ is such a Poisson system, then for any open subset $V$ of $U$ and functions $\alpha, \beta \in \mathcal{O}_M(V)$ we set

$$\{\alpha, \beta\}_{F;V} = \sum_{i,j=1}^{n} F_{ij} \cdot \frac{\partial \alpha}{\partial z_i} \frac{\partial \beta}{\partial z_j}.$$

Clearly, we have $F_{ij} = \{z_i, z_j\}_{F;U}$. Moreover, one can prove that this formula defines a Poisson structure on $U$, denoted by $\{\cdot, \cdot\}_F$. More precisely, we have

**1.4. Proposition.** *Let $(U; z_1, \dots, z_n)$ be a local chart on a complex manifold $M$. Then the assignment $F \mapsto \{\cdot,\cdot\}_F$ defines a one-to-one correspondence between Poisson systems in $(U; z_1, \dots, z_n)$ and Poisson structures on $U$.*

Let $(U; z_1, \dots, z_n)$ be a local chart on a complex Poisson manifold $M$. By Proposition 1.4 there is a unique Poisson system $F = \{F_{ij}\}$ in $(U; z_1, \dots, z_n)$ such that the Poisson structure $\{\cdot,\cdot\}_F$ coincides with that induced from $M$ on $U$. We call $F$ the *representing* (*Poisson*) *system* of $\{\cdot,\cdot\}_M$ in $(U; z_1, \dots, z_n)$. ¿From Proposition 1.4 we obviously get (compare [11])

**1.5. Corollary.** *Every Poisson structure on the complex manifold $\mathbb{C}^n$ is given by a Poisson system of global holomorphic functions on $\mathbb{C}^n$, and conversely.*

**2. Poisson products.** Let $M$ and $N$ be complex Poisson manifolds. Denote by $M \times N$ their Cartesian complex manifold product. From the well-known Osgood lemma it follows that if $U$ and $V$ are open subsets of $M$ and $N$ respectively, then $\mathcal{O}_{M\times N}(U \times V)$ consists of all continuous functions $\alpha\colon U \times V \to \mathbb{C}$ such that for any $u \in U$ and $v \in V$ we have $\alpha(u, \cdot) \in \mathcal{O}_N(V)$ and $\alpha(\cdot, v) \in \mathcal{O}_M(U)$. It is easy to verify

**2.1. Proposition.** *Let $M$ and $N$ be complex Poisson manifolds. Then there is a unique Poisson structure $\{\cdot,\cdot\}$ on $M \times N$ satisfying the following condition:*

*If $U$ and $V$ are open subsets of $M$ and $N$ respectively and if $\alpha, \beta \in \mathcal{O}_{M\times N}(U \times V)$, then*

$$\{\alpha, \beta\}_{U\times V}(u, v) = \{\alpha(\cdot, v), \beta(\cdot, v)\}_{M,U}(u) + \{\alpha(u, \cdot), \beta(u, \cdot)\}_{N,V}(v).$$

A unique Poisson structure $\{\cdot,\cdot\}$ on $M \times N$ satisfying this proposition is called the *product Poisson* one of $\{\cdot,\cdot\}_M$ and $\{\cdot,\cdot\}_N$, denoted by $\{\cdot,\cdot\}_{M\Box N}$. Similarly, the complex Poisson manifold $(M \times N, \{\cdot,\cdot\}_{M\Box N})$ will be called the *Poisson product* of $M$ and $N$, denoted shortly by $M\Box N$. Let $p_M$ and $p_N$ be the canonical projections from $M \times N$ onto $M$ and $N$, respectively. It is seen that both these projections are Poisson holomorphic maps from $M\Box N$ to $M$ and $N$, respectively. By Proposition 2.1 we obviously get

**2.2. Corollary.** *Let $M$ and $N$ be complex Poisson manifolds and let $(U; u_1, \dots, u_n)$ and $(V; v_1, \dots, v_m)$ be local charts on $M$ and $N$, respectively. Suppose that $\{G_{ij}\}$ $(i, j = 1, \dots, n)$ $(\{H_{kl}\}$ $(k, l = 1, \dots, m))$ is the representing Poisson system of $\{\cdot,\cdot\}_M$ $(\{\cdot,\cdot\}_N)$ in $(U; u_1, \dots, u_n)$ $((V; v_1, \dots, v_m))$. Then the representing one $\{F_{rs}\}$ $(r, s = 1, \dots, n + m)$ of $\{\cdot,\cdot\}_{M\Box N}$ in $(U \times V; u_1, \dots, u_n, v_1, \dots, v_m)$ is given by*

$$F_{rs} = \begin{cases} p_M^* G_{rs} & \text{if } r, s \le m; \\ 0 & \text{if } r \le m \text{ and } s > m \text{ or if } r > m \text{ and } s \le m; \\ p_N^* H_{r-m,s-m} & \text{if } r, s > m, \end{cases}$$

*where $p_M^*$ and $p_N^*$ are the corresponding pullback maps for the canonical projections $p_M$ and $p_N$ on $M\Box N$.*

The following example shows that in general the product $M\Box N$ of complex Poisson manifolds need not be a product in the category of such manifolds.

**2.3. Example.** Let $M = N = \mathbb{C}$ be the standard one-dimensional complex manifold regarded as a Poisson one under the zero Poisson structure on $\mathbb{C}$. Then the Poisson product $M\Box N$ is the complex Poisson manifold $(\mathbb{C}^2, \{\cdot,\cdot\}_\theta)$. On the other hand, we can choose a Poisson structure $\{\cdot,\cdot\}$ in $\mathbb{C}^2$ defined in Example 1.1 for $n = 1$. Let $p_1$ and $p_2$ denote the canonical projections from $\mathbb{C}^2$ onto the first and second factor respectively. Clearly, these projectors are morphisms of complex Poisson manifolds from $(\mathbb{C}^2, \{\cdot,\cdot\})$ onto $M$ and $N$ respectively. If $M\Box N$ were a product in the category of complex Poisson manifolds, we would conclude from the universal property of such a product that the identity map of $\mathbb{C}^2$ is a Poisson holomorphic map from $(\mathbb{C}^2, \{\cdot,\cdot\})$ to $(\mathbb{C}^2, \{\cdot,\cdot\}_\theta)$, which is impossible.

We can regard the Poisson product $\Box$ of complex Poisson manifolds as a bifunctor from the category of pairs of complex Poisson manifolds to the category of such manifolds (see [7]). More precisely, if $f\colon M \to M'$ and $g\colon N \to N'$ are morphisms of complex Poisson manifolds, then so is the map $f\Box g = f \times g\colon M\Box N \to M'\Box N'$. Moreover, if in addition $f'\colon M' \to M''$ and $g'\colon N' \to N''$ are such morphisms too, we have

$$(f'\Box g') \circ (f\Box g) = (f \circ f')\Box(g \circ g').$$

It turns out that the category of complex Poisson manifolds is relaxed commutatively monoidal with respect to the identity object $E$ defined to be a fixed one-point complex Poisson manifold. This means that we have the following list of canonical isomorphisms of complex Poisson manifolds:

$$(M\Box N)\Box P \cong M\Box(N\Box P),\ E\Box M \cong M \cong M\Box E,\ M\Box N \cong N\Box M$$

where $M$, $N$ and $P$ are arbitrary such manifolds (compare [10], Section 4).

**3. Regularity.** Let $(U; u_1, \dots, u_n)$ be a local chart on a complex Poisson manifold $M$ and let $\{F_{ij}^U\}$ be the representing Poisson system of $\{\cdot,\cdot\}_M$ in this chart. This implies that the bicontravariant tensor field

$$F_U = \sum_{i,j=1}^{n} F_{ij}^U (\partial/\partial u_i) \otimes (\partial/\partial u_j)$$

is antisymmetric, and so, it is a bivector field on $U$, called the *representing bivector field* of $\{\cdot,\cdot\}_M$ in $(U; u_1, \dots, u_n)$. It is seen that for any $\alpha, \beta \in \mathcal{O}_M(U)$ we have

$$\{\alpha, \beta\}_{M,U} = F_U(d\alpha, d\beta).$$

Similarly, if $(V; v_1, \dots, v_n)$ is another local chart on $M$, we have the representing bivector field $F_V$ of $\{\cdot,\cdot\}_M$ in this chart. For $U \cap V \neq \emptyset$ it follows that $F_U|U \cap V = F_V U \cap V$ because $\{\cdot,\cdot\}_{M,U}|U \cap V = \{\cdot,\cdot\}_{M,U}|U \cap V = \{\cdot,\cdot\}_{M,U}|U \cap V$. Consequently, there is a bivector field $F$ defined on $M$ such that $F|U = F_U$ for any local chart $(U; u_1, \dots, u_n)$ on $M$, called the (*global*) *representing bivector field* of $\{\cdot,\cdot\}_M$.

In the sequel, we shall frequently write $(M, F)$ to denote a complex Poisson manifold. This means that $M$ is a complex Poisson manifold and $F$ is a *Poisson* bivector field on $M$, that is, $F$ satisfies the following condition:

*If $U$ is an open subset of $M$ and if $\alpha, \beta \in \mathcal{O}_M(U)$, then the expression*

$$\{\alpha, \beta\}_{F;U} = F_U(d\alpha, d\beta)$$

*defines a Poisson bracket on $\mathcal{O}_M(U)$.*

In this case we have defined the *Poisson structure* $\{\cdot,\cdot\}_F$ on $M$ induced by $F$ such that $F$ is its global representing bivector field.

Let $(M,F)$ be a complex Poisson manifold. It is clear that $F$ as a global holomorphic section of the bundle $\otimes^2 TM$ defines a holomorphic function $\alpha_F$ on the complex bicotangent manifold $\otimes^2 T^*M$ as follows

$$\alpha_F(\omega) = \langle \omega, F_p\rangle^2_{M,p} \text{ for } \omega \in \otimes^2 T^*_p M$$

where $\langle\cdot,\cdot\rangle^2_{M,p}\colon \otimes^2 T^*_pM \times \otimes^2 T_pM \to \mathbb{C}$ is the canonical holomorphic 2-pairing of $M$ at $p$. Clearly, $\alpha_F$ is $\mathbb{C}$-linear on each fibre of $\otimes^2 T^*M$. Note that the assignment $F \mapsto \alpha_F$ is a one-to-one correspondence between global holomorphic sections of $\otimes^2 TM$ and holomorphic functions on $\otimes^2 T^*M$ which are $\mathbb{C}$-linear on fibres. Thus, from the corresponding property of holomorphic functions we get

**3.1. Proposition.** *Let $(M,F)$ and $(M,G)$ be connected complex Poisson manifolds. If there is an open subset $U$ of $M$ such that $F|U = G|U$, then $F = G$.*

Let now $F$ be a 2-tensor field on a complex manifold $M$. If $(U; z_1,\dots,z_n)$ is a local chart on $M$, then $F|U$ can be represented by a matrix $\{F^U_{ij}\}$ $(i,j = 1,\dots,n)$ of holomorphic functions on $U$ which are components of $F$ relative to this chart. For example, if $F$ is a bicovariant (bicontravariant) tensor field on $M$, then

$$F|U = \sum_{ij} F^U_{ij}(dz_i)\otimes(dz_j) \quad \Big(F|U = \sum_{ij} F^U_{ij}(\partial/\partial z_i)\otimes(\partial/\partial z_j)\Big).$$

The matrix $\{F^U_{ij}\}$ is called the *representing* one of $F$ in a local chart $(U; z_1,\dots,z_n)$ on $M$. For any point $p$ of $M$ we define the rank of $F$ at $p$ to be $\rho_F(p) = \operatorname{rank}\{F^U_{ij}(p)\}$ where $\{F^U_{ij}\}$ is the representing matrix of $F$ in some local chart $(U; z_1,\dots,z_n)$ on $M$ containing $p$. Clearly, the definition of $\rho_F(p)$ does not depend from a chosen local chart containing $p$. Thus we can regard that the *rank-dimension* function $\rho_F\colon x \mapsto \rho_F(x)$ is well-defined on $M$. We say that $p$ is a *regular* point of $F$ in $M$ in case $\rho_F$ is locally constant at $p$ and we denote by reg $F$ the set of all such points. It is easy to verify

**3.2. Proposition.** *Let $F$ be a 2-tensor field defined on a complex manifold $M$. Then the rank-dimension function $\rho_F$ is lower semicontinuous on $M$, which implies that the set reg $F$ is open and dense in $M$.*

Let $U$ be an open subset of $M$. We say that $F$ is of *constant rank $k$* (on $U$) in case $\rho_F = k$ $(\rho_F|U = k)$. In turn, $F$ is called *regular* (on $U$) provided that it is locally of constant rank at each point of $M$ $(U)$, i.e. reg $F = M$ $(U \subseteq \text{reg } F)$, otherwise $F$ is called *irregular* (on $U$). Clearly, Proposition 3.2 implies

**3.3. Corollary.** *Every regular 2-tensor field defined on a connected complex manifold is of constant rank.*

Let $n = \dim M$. We say that $F$ is *nondegenerate* at a point $p$ of $M$ in case $\rho_F(p) = n$. Let us set $\text{reg}^\dagger F = \{x \in M\colon \rho_F(x) = n\}$ and note that it is an open subset of $M$. If $U$ is an open subset of $M$, we call $F$ *nondegenerate* (on $U$) in case $F$ is nondegenerate at each point of $M$ $(U)$, i.e. $\text{reg}^\dagger F = M$ $(U \subseteq \text{reg}^\dagger F)$.

Let now $\{\cdot,\cdot\}_F$ be a Poisson structure defined on a complex manifold $M$ and let $U$ be an open subset of $M$. We call $\{\cdot,\cdot\}_F$ *regular*, *irregular*, *nondegenerate* and of *constant* *rank* (on $U$) provided that so is the representing bivector field $F$, respectively. Clearly, every nondegenerate Poisson structure on $M$ is regular but not conversely in general. It is seen that the standard Poisson structure on $\mathbb{C}^{2n}$ (Example 1.1) is nondegenerate. Next, note that a constant Poisson structure on $\mathbb{C}^n$ (Example 1.2) is of constant rank equal to rank $\{c_{ij}\}$ , which means that it is nondegenerate provided that rank $\{c_{ij}\} = n$. In turn, a Lie-Poisson structure on $\mathbb{C}^n$ (Example 1.3) is irregular unless it is the zero Poisson one.

**4. Hamiltonian morphisms.** Let $\Phi$ $(F)$ be a bicovariant (bicontravariant) tensor field defined on a complex manifold $M$. For any $x = a, b$ we define the morphisms $\Phi^x \colon TM \to T^*M$ $(F^x \colon T^*M \to TM)$ of holomorphic vector bundles as follows:

$$\langle \Phi^a X, Y \rangle_{M,p} = \langle \Phi^b Y, X \rangle_{M,p} = \Phi_p(X, Y)$$
$$(\langle \eta, F^a \xi \rangle_{M,p} = \langle \xi, F^b \eta \rangle_{M,p} = F_p(\xi, \eta))$$

where $p$ ranges over $M$ and $X, Y \in T_pM$ $(\xi, \eta \in T_p^*M)$. We call $\Phi^a$ $(F^a)$ and $\Phi^b$ $(F^b)$ the *canonical left* and *right morphisms* of $\Phi$ $(F)$, respectively. By definition they are mutually adjoint with respect to the holomorphic pairing of $M$ and note that each of them determines $\Phi$ $(F)$. In particular, $\Phi^a$ $(F^a)$ is an isomorphism if and only if so is $\Phi^b$ $(F^b)$. Clearly, for each $p \in M$ we have defined the $\mathbb{C}$-linear map

$$\Phi_p^x \colon T_pM \to T_p^*M \ (F_p^x \colon T_p^*M \to T_pM) \text{ where } \Phi_p^x = \Phi^x | T_pM \ (F_p^x = F | T_p^*M).$$

By an easy verification we get

**4.1. Proposition.** *Let $\Phi$ $(F)$ be a bicovariant (bicontravariant) tensor field defined on a complex manifold $M$.*

*(1) $\Phi$ $(F)$ is symmetric if and only if $\Phi^a = \Phi^b$ $(F^a = F^b)$;*
*(2) $\Phi$ $(F)$ is antisymmetric if and only if $\Phi^a = -\Phi^b$ $(F^a = -F^b)$;*
*(3) $\Phi$ $(F)$ is nondegenerate if and only if $\Phi^a$ and $\Phi^b$ $(F^a$ and $F^b)$ are isomorphisms.*

Recall that if $\Phi$ $(F)$ is a bicovariant (bicontravariant) tensor field on $M$, then for any open subset $U$ of $M$ we have defined the associated $\mathcal{O}_M(U)$-bilinear map

$$\Phi_U \colon \mathcal{X}_M(U) \times \mathcal{X}_M(U) \to \mathcal{O}_M(U) \quad (F_U \colon \mathcal{D}_M(U) \times \mathcal{D}_M(U) \to \mathcal{O}_M(U)).$$

Next, for $x = a, b$ we can define the $\mathcal{O}_M(U)$-module homomorphism

$$\Phi_M^x(U) \colon \mathcal{X}_M(U) \to \mathcal{D}_M(U) \quad (F_M^x(U) \colon \mathcal{D}_M(U) \to \mathcal{X}_M(U))$$

as follows:

$$\langle \Phi_M^a(U)(X), Y \rangle_{M,U} = \langle \Phi_M^b(U)(Y), X \rangle_{M,U} = \Phi_U(X, Y)$$
$$(\langle \eta, F_M^a(U)(\xi) \rangle_{M,U} = \langle \xi, F_M^b(U)(\eta) \rangle_{M,U} = F_U(\xi, \eta)).$$

It can be seen that we get morphisms $\Phi_M^x \colon \mathcal{X}_M \to \mathcal{D}_M$ and $F_M^x \colon \mathcal{D}_M \to \mathcal{X}_M$ of sheaves of $\mathcal{O}_M$-modules, which means that the systems $\Phi_M^x = \{\Phi_M^x(U)\}$ and $F_M^x = \{F_M^x(U)\}$ are subject to the following condition:

*for any open subsets $U$ and $V$ of $M$ such that $V \subseteq U$, the following diagrams commute*:

$$\begin{array}{ccc} \mathcal{X}_M(U) & \xrightarrow{\Phi^x_M(U)} & \mathcal{D}_M(U) \\ \rho^U_V \downarrow & & \downarrow \rho^U_V \\ \mathcal{X}_M(V) & \xrightarrow{\Phi^x_M(V)} & \mathcal{D}_M(V) \end{array} \qquad \begin{array}{ccc} \mathcal{D}_M(U) & \xrightarrow{F^x_M(U)} & \mathcal{X}_M(U) \\ \rho^U_V \downarrow & & \downarrow \rho^U_V \\ \mathcal{D}_M(V) & \xrightarrow{F^x_M(V)} & \mathcal{X}_M(V) \end{array}$$

*where $\rho^U_V$ denotes the corresponding restriction morphisms.*

Moreover, note that if $\Phi$ and $F$ are nondegenerate, then $\Phi^x_M$ and $F^x_M$ become isomorphisms of sheaves of $\mathcal{O}_M$-modules. By an easy computation we get

**4.2. Proposition.** *Let $\Phi$ ($F$) be a bicovariant (bicontravariant) tensor field on a complex manifold $M$. Let $\Phi|U = \sum_{ij} \phi_{ij}(dz_i)\otimes(dz_j)$ ($F|U = \sum_{ij} f_{ij}(\partial/\partial z_i)\otimes(\partial/\partial z_j)$) be the expression of $\Phi$ ($F$) in a local chart $(U; z_1, \dots, z_n)$ on $M$. Then the following conditions hold:*

*(1)* $\Phi^a_M(U)(X) = \sum_{ij} \lambda_i \phi_{ij} dz_j$, $\Phi^b_M(U)(X) = \sum_{ij} \lambda_j \phi_{ij} dz_i$;

*(2)* $F^a_M(U)(\xi) = \sum_{ij} \alpha_i f_{ij}(\partial/\partial z_j)$, $F^b_M(U)(\xi) = \sum_{ij} \alpha_j f_{ij}(\partial/\partial z_i)$;

*where* $X = \sum_k \lambda_k(\partial/\partial z_k) \in \mathcal{X}_M(U)$ *and* $\xi = \sum_k \alpha_k dz_k \in \mathcal{D}_M(U)$.

Let $F$ be a bicontravariant tensor field on $M$. If $U$ is an open subset of $M$ and if $\alpha \in \mathcal{O}_M(U)$, we define the *canonical left* (*right*) *$F$-vector field* of $\alpha$ on $U$ to be $F^{U,a}_\alpha = F^a_M(U)(d\alpha)$ ($F^{U,b}_\alpha = F^b_M(U)(d\alpha)$). By the *canonical left* (*right*) *vector valued $F$-derivation* of $\mathcal{O}_M(U)$ we shall mean the assignment $F^{U,a}\colon \alpha \mapsto F^a_M(U)(d\alpha)$ ($F^{U,b}\colon \alpha \mapsto F^b_M(U)(d\alpha)$) which is an $\mathcal{X}_M(U)$-valued derivation, i.e.

$$F^{U,x}(\alpha\beta) = \alpha F^{U,x}_\beta + \beta F^{U,x}_\alpha \text{ for } x = a, b \text{ and any } \alpha, \beta \in \mathcal{O}_M(U).$$

Moreover, it is seen that

$$F^{U,a}_\alpha(\beta) = \langle d\beta, F^a_M(U)(d\alpha)\rangle_{M,U} = F_U(d\alpha, d\beta) = \langle d\alpha, F^b_M(U)(d\beta)\rangle_{M,U} = F^{U,b}_\beta(\alpha).$$

In turn, Proposition 4.1 implies that if $F$ is symmetric (antisymmetric), then $F^{U,a}_\alpha = F^{U,b}_\alpha$ ($F^{U,a}_\alpha = -F^{U,b}_\alpha$) for each $\alpha \in \mathcal{O}_M(U)$.

Let $M$ be a complex Poisson manifold and let $F$ be the global representing bivector field of $\{\cdot,\cdot\}_M$. If $U$ is an open subset of $M$ and if $\alpha \in \mathcal{O}_M(U)$, then by the *Hamiltonian vector field* of $\alpha$ on $U$ we shall mean the vector field $H^U_\alpha = F^{U,a}_\alpha$. In turn, we define the *Hamiltonian morphism* $H^U$ of $M$ over $U$ to be the canonical left vector valued $F$-derivation of $\mathcal{O}_M(U)$, i.e. $H^U = F^{U,a}$. Clearly, $H^U_\alpha(\beta) = \{\alpha,\beta\}_{M,U} = -\{\beta,\alpha\}_{M,U} = -H^U_\beta(\alpha)$ for $\alpha, \beta \in \mathcal{O}_M(U)$. Recall that the Poisson bracket $\{\cdot,\cdot\} = \{\cdot,\cdot\}_{M,U}$ makes $\mathcal{O}_M(U)$ be a Lie algebra. Similarly, $\mathcal{X}_M(U)$ is a Lie algebra under the usual bracket $[\cdot,\cdot]$ of vector fields. It is easy to verify

**4.3. Proposition.** *Let $M$ be a complex Poisson manifold and let $U$ be an open subset of $M$. The Hamiltonian morphism $H^U$ of $M$ over $U$ is a linear map from $\mathcal{O}_M(U)$ to $\mathcal{X}_M(U)$ satisfying the following conditions ($\alpha, \beta \in \mathcal{O}_M(U)$):*

*(1)* $H^U$ *is an* $\mathcal{X}_M(U)$*-valued derivation, i.e.* $H^U_{\alpha\beta} = \alpha H^U_\beta + \beta H^U_\alpha$;

*(2) $H^U$ is a Lie algebra homomorphism, i.e.* $H^U_{\{\alpha,\beta\}} = [H^U_\alpha, H^U_\beta]$.

Remark that one could define the Hamiltonian morphism of $M$ over $U$ to be the canonical right vector valued $F$-derivation of $\mathcal{O}_M(U)$, that is, to be $-H^U$. But such a definition causes that the corresponding Hamiltonian morphism of $M$ over $U$ would be a Lie algebra antihomomorphism. However, note that in the case of real symplectic (Poisson) manifolds many authors prefer such a definition (see [1] and [3]).

It is seen that the system $H = H^U$ of Hamiltonian morphisms of $M$ over open subsets of $M$ defines a morphism $H\colon \mathcal{O}_M \to \mathcal{X}_M$ of sheaves of complex Lie algebras on $M$, called the *Hamiltonian morphism* of $M$. This means that $H$ is subject to the condition:

*for any open subsets $U$ and $V$ of $M$ such that $V \subseteq U$, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{O}_M(U) & \xrightarrow{H^U} & \mathcal{X}_M(U) \\ {\scriptstyle \rho^U_V}\downarrow & & \downarrow{\scriptstyle \rho^U_V} \\ \mathcal{O}_M(V) & \xrightarrow{H^V} & \mathcal{X}_M(V) \end{array}$$

*where $\rho^U_V$ denotes the corresponding restriction morphisms.*

By an easy computation we get

**4.4. Proposition.** *Let $H$ be the Hamiltonian morphism of a complex Poisson manifold $(M, F)$. If $(U; z_1, \dots, z_n)$ is a local chart on $M$ and $F|U = \sum f_{ij}(\partial/\partial z_i) \otimes (\partial/\partial z_j)$, then for any $\alpha \in \mathcal{O}_M(U)$ we have*

$$H^U_\alpha = \sum f_{ij}(\partial\alpha/\partial z_i)(\partial/\partial z_j).$$

For convenience, if $\Phi$ $(F)$ is a bicovariant (bicontravariant) tensor field on a complex manifold $M$, we accept the notations $(x = a, b)$:

$$F^x = F^x_M(M)\colon \mathcal{D}_M(M) \to \mathcal{X}_M(M) \text{ and } \Phi^x = \Phi^x_M(M)\colon \mathcal{X}_M(M) \to \mathcal{D}_M(M).$$

**4.5. Example.** Consider the standard Poisson structure on the complex manifold $\mathbb{C}^{2n}$ with the global coordinate system $q_1, \dots, q_n, p_1, \dots, p_n$ (Example 1.1). It is seen that its representing bivector field is of the form

$$F = \sum_{i=1}^{n}((\partial/\partial p_i) \otimes (\partial/\partial q_i) - (\partial/\partial q_i) \otimes (\partial/\partial p_i)) = 2\sum_{i=1}^{n}(\partial/\partial p_i) \wedge (\partial/\partial q_i).$$

Moreover, note that $F^a(dp_i) = -F^b(dp_i) = \partial/\partial q_i$ and $F^a(dq_i) = -F^b(dq_i) = -\partial/\partial p_i$ for $i = 1, \dots, n$. By Proposition 4.4, if $U$ is an open subset of $M$ and $\alpha \in \mathcal{O}_{2n}(U)$, we get

$$H^U_\alpha = \sum_{i=1}^{n}((\partial\alpha/\partial p_i)(\partial/\partial q_j) - (\partial\alpha/\partial q_i)(\partial/\partial p_i)).$$

One can prove (compare [10], Theorem 4.1)

**4.6. Proposition.** *Let $(M,F)$ and $(N,G)$ be complex Poisson manifolds. If $f\colon N\to M$ is a holomorphic map, then the following conditions are equivalent:*

*(a) $f$ is a Poisson map;*

*(b) $G_V(f_U^*\xi, f_U^*\eta) = (f^\# F)_U(\xi,\eta)$ for any open subset $U$ of $M$ and $\xi,\eta\in\mathcal{D}_M(U)$, where $V=f^{-1}(U)$;*

*(c) $f_*\circ H^V_{f^*\alpha} = f^\# H^U_\alpha$ for any open subset $U$ of $M$, $V=f^{-1}(U)$ and $\alpha\in\mathcal{O}_M(U)$;*

*(d) $f_*\circ G = f^\# F$, where $H$ denotes the Hamiltonian morphisms of $(M,F)$ and $(N,G)$ simultaneously.*

Clearly, conditions (c) and (d) of this proposition mean that the following diagrams are respectively commutative:

$$\begin{tikzcd} TN \arrow[r, "f_*"] & TM \\ V \arrow[u, "H^V_{f^*\alpha}"] \arrow[r, "f|V"] & U \arrow[u, "H^U_\alpha"'] \end{tikzcd} \qquad \begin{tikzcd} \otimes^2 TN \arrow[r, "f_*"] & \otimes^2 TM \\ N \arrow[u, "G"] \arrow[r, "f"] & M \arrow[u, "F"'] \end{tikzcd}$$

**5. Conjugate pairs.** Let $\Phi$ $(F)$ be a nondegenerate bicovariant (bicontravariant) tensor field on a complex manifold $M$. From Proposition 4.1 it follows that $\Phi^x\colon TM\to T^*M$ and $F^x\colon T^*M\to TM$ $(x=a,b)$ are isomorphisms of holomorphic vector bundles. We can regard the induced isomorphisms

$$\Phi^x\otimes\Phi^x\colon\ \otimes^2 TM\to\otimes^2 T^*M \text{ and } F^x\otimes F^x\colon\ \otimes^2 T^*M\to\otimes^2 TM$$

of such bundles too. Clearly, these isomorphisms define other ones between the $\mathcal{O}_M(M)$-modules of holomorphic global sections of $\otimes^2 TM$ and $\otimes^2 T^*M$ which are also denoted by $\Phi^x\otimes\Phi^x$ and $F^x\otimes F^x$, respectively. One can prove (compare [10], Proposition 2.5)

**5.1. Proposition.** *Let $M$ be a complex manifold.*

*(1) For any nondegenerate bicovariant (bicontravariant) tensor field $\Phi$ $(F)$ on $M$ there is a unique bicontravariant (bicovariant) one $\Phi^\sim$ $({}^\sim F)$ such that*

$$(\Phi^a\otimes\Phi^a)\Phi^\sim = (\Phi^b\otimes\Phi^b)\Phi^\sim = \Phi \quad ((F^a\otimes F^a)^\sim F = (F^b\otimes F^b)^\sim F = F).$$

*(2) The assignments $\Phi\mapsto\Phi^\sim$ and $F\mapsto{}^\sim F$ are mutually inverse bijections between nondegenerate bicovariant and bicontravariant tensor fields on $M$, that is, ${}^\sim(\Phi^\sim)=\Phi$ and $({}^\sim F)^\sim = F$.*

Under assumption of this proposition, we call $\Phi^\sim$ $({}^\sim F)$ the *conjugate bicontravariant (bicovariant) tensor field* of $\Phi$ $(F)$. By a *conjugate pair* of 2-tensor fields on $M$ we shall mean a pair $(\Phi,F)$ such that $\Phi$ $(F)$ is a bicovariant (bicontravariant) tensor field on $M$ and $\Phi^\sim = F$ $({}^\sim F=\Phi)$. This means that if $\Phi$ $(F)$ is a bicovariant (bicontravariant) tensor field on $M$, then $(\Phi,\Phi^\sim)$ $(({}^\sim F, F))$ is such a conjugate pair. From Proposition 5.1 we obviously get

**5.2. Corollary.** *Let $(\Phi,F)$ be a conjugate pair of 2-tensor fields on $M$.*

*(1) $F_p(\Phi^a X,\Phi^a Y) = F_p(\Phi^b X,\Phi^b Y) = \Phi_p(X,Y)$ for each $p\in M$ and any $X,Y\in T_pM$;*

*(2)* $\Phi_p(F^a\xi, F^a\eta) = \Phi_p(F^b\xi, F^b\eta) = F_p(\xi, \eta)$ *for each* $p \in M$ *and any* $\xi, \eta \in T_p^*M$*;*

*(3)* $\Phi_U(F_\alpha^{U,a}, F_\beta^{U,a}) = \Phi_U(F_\alpha^{U,b}, F_\beta^{U,b}) = F_U(d\alpha, d\beta)$ *for each open subset* $U$ *of* $M$ *and any functions* $\alpha, \beta \in \mathcal{O}_M(U)$.

Denote by $\mathbf{1}$ and $\mathbf{1}^*$ the identity map of $TM$ and $T^*M$, respectively. By an easy verification we get

**5.3. Proposition.** *Let* $\Phi$ $(F)$ *be a bicovariant (bicontravariant) tensor field defined on a complex manifold* $M$*. Then the following conditions are equivalent:*

*(a)* $(\Phi, F)$ *is a conjugate pair of 2-tensor fields on* $M$*;*
*(b)* $F^b \circ \Phi^a = \mathbf{1}$*;*
*(c)* $F^a \circ \Phi^b = \mathbf{1}$*;*
*(d)* $\Phi^b \circ F^a = \mathbf{1}^*$*;*
*(e)* $\Phi^a \circ F^b = \mathbf{1}^*$*;*
*(f)* $\langle \Phi^a X, F^a\xi\rangle_{M,p} = \langle \xi, X\rangle_{M,p}$ *for each* $p \in M$ *and any* $X \in T_pM$*,* $\xi \in T_p^*M$*;*
*(g)* $\langle \Phi^b X, F^b\xi\rangle_{M,p} = \langle \xi, X\rangle_{M,p}$ *for each* $p \in M$ *and any* $X \in T_pM$*,* $\xi \in T_p^*M$*.*

Remark that this proposition could be the base for the definition of a conjugate pair of 2-tensor fields on $M$ (compare [10], Proposition 2.1). Clearly, Proposition 5.3 implies the following corollaries.

**5.4. Corollary.** *Let* $(\Phi, F)$ *be a conjugate pair of 2-tensor fields on* $M$*. Then* $\Phi$ *is symmetric (antisymmetric) if and only if so is* $F$*.*

**5.5. Corollary.** *If* $(\Phi, F)$ *is a conjugate pair of 2-tensor fields on* $M$*, then*

$$\Phi^x \otimes \Phi^x \colon \otimes^2 TM \to \otimes^2 T^*M \quad (F^x \otimes F^x \colon \otimes^2 T^*M \to \otimes^2 TM)$$

*is an isomorphism of holomorphic vector bundles for* $x = a, b$*. Moreover, the isomorphisms* $\Phi^a \otimes \Phi^a$ $(\Phi^b \otimes \Phi^b)$ *and* $F^b \otimes F^b$ $(F^a \otimes F^a)$ *are mutually inverse.*

**5.6. Example.** Let $F$ be the representing bivector field of the standard Poisson structure on $\mathbb{C}^{2n}$ with the global coordinate system $q_1, \ldots, q_n, p_1, \ldots, p_n$ (Example 4.5). Consider the conjugate bicovariant tensor field $\Phi$ of $F$, which means that $(\Phi, F)$ is a conjugate pair of 2-tensor field on $M$. Since $F^a(dp_i) = \partial/\partial q_i$ and $F^a(dq_i) = -\partial/\partial p_i$ $(i = 1, \ldots, n)$, it follows from Proposition 5.3 that $\Phi^b(\partial/\partial p_i) = -dq_i$ and $\Phi^b(\partial/\partial q_i) = -dp_i$. Hence and from Proposition 5.1 we get

$$\begin{aligned}\Phi = (\Phi^b \otimes \Phi^b)F &= (\Phi^b \otimes \Phi^b)\Big(\sum_{i=1}^n ((\partial/\partial p_i) \otimes (\partial/\partial q_i) - (\partial/\partial q_i) \otimes (\partial/\partial p_i)\Big) \\ &= \sum_{i=1}^n ((dp_i) \otimes (dq_i) - (dq_i) \otimes (dp_i)) = 2\sum_{i=1}^n (dp_i) \wedge (dq_i).\end{aligned}$$

**6. Symplectic structures.** By a complex *symplectic* manifold we shall mean a complex Poisson manifold such that its global representing bivector field is symplectic, i.e. nondegenerate. A *symplectic structure* on a complex manifold $M$ is a nondegenerate Poisson one which makes $M$ be a complex symplectic manifold. It is seen that the complex manifold $\mathbb{C}^{2n}$ with the standard Poisson structure (Example 1.1) is a complex symplectic one, called the *standard complex symplectic manifold* $\mathbb{C}^{2n}$. Clearly, every complex symplectic manifold is of even dimension. By an easy verification we get

**6.1. Proposition.** *Every open Poisson submanifold of a complex symplectic manifold is again such a manifold. The Poisson product of complex symplectic manifolds is a complex symplectic one.*

Let $M$ be a complex manifold. A bicovariant tensor field $\Phi$ on $M$ is said to be *symplectic* provided that it is nondegenerate and the conjugate bicontravariant tensor field $\Phi^{\sim}$ is *symplectic*, i.e. $(M, \Phi^{\sim})$ is a complex symplectic manifold. In this case $\Phi^{\sim}$ is also called the *conjugate symplectic bivector field* of $\Phi$. Clearly, by Corollary 5.4 every symplectic bicovariant tensor field on $M$ is antisymmetric, and so, it is a differential 2-form. Analogously as for real symplectic manifolds, one can prove

**6.2. Proposition.** *A nondegenerate differential 2-form on a complex manifold is symplectic if and only if it is closed.*

A *symplectic pair* on a complex manifold $M$ is a conjugate pair $(\Phi, F)$ of 2-tensor fields on $M$ where both $\Phi$ and $F$ are symplectic. If $(\Phi, F)$ is such a pair, then $(M, F)$ is clearly a complex symplectic manifold. From Corollary 5.2 we obviously get

**6.3. Corollary.** *Let $(\Phi, F)$ be a symplectic pair on M. If $H$ is the Hamiltonian morphism of the complex symplectic manifold $(M, F)$, then for any open subset $U$ of $M$ and any functions $\alpha, \beta \in \mathcal{O}_M(U)$ we have*

$$\Phi_U(H_\alpha^U, H_\beta^U) = \{\alpha, \beta\}_{F;U}.$$

Let $U$ be an open subset of $M$ and let $p \in U$. We say that functions $\alpha^1, \dots, \alpha^n \in \mathcal{O}_M(U)$ are *holomorphically independent* at $p$ in case the holomorphic differentials $d\alpha^1, \dots, d\alpha^n$ are linearly independent at $p$. If these functions are holomorphically independent at each point of $U$, we call they *holomorphically independent* on $U$. Similarly, we can say that vector fields $X^1, \dots, X^n \in \mathcal{X}_M(U)$ are *linearly independent* at $p$ (on $U$). Applying Proposition 4.4 and the theorem on biholomorphism one can prove

**6.4. Proposition.** *Let $M$ be a complex symplectic manifold. Suppose that $U$ is an open subset of $M$ and $p \in U$. Let us take functions $\alpha^1, \dots, \alpha^n \in \mathcal{O}_M(U)$. Then these functions are holomorphically independent at $p$ (on $U$) if and only if the Hamiltonian vector fields $H_{\alpha^1}, \dots, H_{\alpha^n}$ are linearly independent at $p$ (on $U$). Moreover, if $\alpha^1, \dots, \alpha^n$ are holomorphically independent on $U$, then the corresponding Hamiltonian vector fields form a base of the $\mathcal{O}_M(U)$-module $\mathcal{X}_M(U)$.*

Let now $M$ be an arbitrary complex manifold. Consider the holomorphic cotangent manifold $T^*M$ together with the canonical projection $\pi \colon T^*M \to M$. Let $(U; q_1, \dots, q_n)$ be a local chart on $M$. It is known that this chart induces the canonical another one on $T^*M$ of the form $(\pi^{-1}(U), q_1, \dots, q_n, p_1, \dots, p_n)$ where $p_1, \dots, p_n$ should be regarded as the components of a point $p \in T_q^*M$ $(q = (q_1, \dots, q_n))$ with respect to the canonical base $(dq_1)_q, \dots, (dq_n)_q$ where $q_1, \dots, q_n$ are here regarded as the corresponding holomorphic coordinate functions on $U$. We call $(\pi^{-1}(U), q_1, \dots, q_n, p_1, \dots, p_n)$ a *local canonical coordinate system* on $T^*M$. By using such systems we can first define locally the *canonical differential 1-form* $\vartheta^U$ on $T^*M$ by

$$\vartheta^U = 2 \sum_i p_i dq_i.$$

The transformation rules for local canonical coordinate systems on $T^*M$ guarantee that we have defined globally the canonical differential 1-form $\vartheta$ on $T^*M$. Next, we define the *canonical differential 2-form* $\Omega = d\vartheta$ on $T^*M$ which is clearly closed and nondegenerate. It is easy to verify

**6.5. Proposition.** *The canonical differential 2-form $\Omega$ on $T^*M$ is symplectic. If $\Gamma = \Omega^\sim$ is the conjugate symplectic bivector field, then for any local canonical coordinate system $(\pi^{-1}(U), q_1, \ldots, q_n, p_1, \ldots, p_n)$ on $T^*M$ we have*

$$\Omega|\pi^{-1}(U) = 2\sum_i (dp_i) \wedge (dq_i) \text{ and } \Gamma|\pi^{-1}(U) = 2\sum_i (\partial/\partial p_i) \wedge (\partial/\partial q_i).$$

*Moreover, the Poisson structure $\{\cdot,\cdot\}_\Gamma$ on $T^*M$ satisfies the condition:*

$$\{\alpha, \beta\}_{\Gamma;\pi^{-1}(U)} = \sum_{i=1}^{n} \left( \frac{\partial\alpha}{\partial p_i}\frac{\partial\beta}{\partial q_i} - \frac{\partial\alpha}{\partial q_i}\frac{\partial\beta}{\partial p_i} \right)$$

*for any $\alpha, \beta \in \mathcal{O}_{T^*M}(\pi^{-1}(U))$.*

Under the assumptions of this proposition, we call $\Omega$ $(\Gamma)$ the *canonical symplectic* differential 2-form (bivector field) on $T^*M$. In turn, we say that $(T^*M, \Gamma)$ is the *complex symplectic cotangent manifold* of $M$, denoted shortly by $T^*M$. It is obviously a complex analogue of the real symplectic cotangent manifold of a smooth one. Since the latter manifold is regarded as a phase space in Hamiltonian mechanics, we can regard its complex analogue as a complex phase space (compare [4] and [6]).

One can see that Proposition 6.5 means that the complex symplectic cotangent manifold $T^*M$ is locally isomorphic to an open Poisson submanifold of the standard complex symplectic manifold $\mathbb{C}^{2n}$ ($n = \dim M$). Observe that our definitions of the canonical differential 1-form and 2-form on $T^*M$ differ from the standard ones by the coefficient 2. This coefficient is accepted by reason of symmetry between the canonical differential 2-form and the canonical bivector field on $T^*M$ which are assumed to form a symplectic pair of 2-tensor fields on $T^*M$.

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