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Space-Times and Holonomy Groups

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Abstract. This paper discusses the full holonomy group for space-time manifolds in Einstein's general relativity. Some applications of holonomy theory to Weyl manifolds are presented.

1 Introduction

General Relativity Theory is based on a (4-dimensional) manifold admitting a global Lorentz metric. The corresponding Levi-Civita connection has an associated holonomy group. The object of the first part of this paper is to describe how space-times may be classified according to their holonomy group. The second part of the paper describes Weyl's generalisation of Einstein's concept of space-time and makes some remarks on the holonomy group in Weyl's geometry.

For the first four sections of the paper M will denote a space-time, that is, a 4-dimensional connected hausdorff manifold possessing a global Lorentz metric g with signature $(-+++)$ and corresponding (symmetric) Levi-Civita connection Γ. The manifold structure on M and the metric g will be assumed smooth and M is then necessarily paracompact. The corresponding Riemann and Ricci tensors will be denoted in component form by $R^a{}_{bcd}$ and R_{ab} $(\equiv R^c{}_{acb})$ and the Ricci scalar by $R \equiv R_{ab}g^{ab}$. The energy-momentum tensor is denoted by T_{ab} and Einstein's field equations are

$$G_{ab} \equiv R_{ab} - \frac{1}{2}Rg_{ab} = \kappa T_{ab} \tag{1}$$

for some suitable constant κ.

2 Holonomy Groups

Let $p \in M$ and for a fixed k, $1 \le k \le \infty$, let $C_k(p)$ denote the set of all piecewise C^k closed curves starting and ending at p (although mainly directed at space-times, the remarks in this section apply quite generally to any manifold M with a linear connection Γ). Each $c \in C_k(p)$ gives rise, by means of parallel transport along c by means of Γ, to an isomorphism of the tangent space T_pM to M at p onto itself. If, for $c \in C_k(p)$, $f(c)$ denotes the associated isomorphism of T_pM then, using standard notation for curves, $f(c^{-1}) = f(c)^{-1}$ and $f(c_1 \circ c_2) = f(c_1) \circ f(c_2)$ $(c_1, c_2 \in C_k(p))$ and it follows that the set

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of isomorphisms of T_pM arising from all members of $C_k(p)$ is a subgroup of the group $GL(T_pM)$ of all isomorphisms of T_pM. This subgroup is called the *k-holonomy group of M at p*. Since M is connected, M is necessarily path connected and it then follows (see, e.g. [1]) that the k-holonomy groups at distinct points of M are isomorphic (in fact, conjugate) to each other. Thus it makes sense to speak of the *k-holonomy group of M* and the latter is denoted by Φ_k. If the above operation is repeated using only those members of $C_p(k)$ which are homotopic to zero one easily sees how one arrives firstly at the *restricted k-holonomy group to M at p* and ultimately to the *restricted k-holonomy group of M*. The latter is denoted by Φ_k^0. Finally, it can be shown [1] that the groups Φ_k and Φ_k^0 are independent of k and so one speaks of the *holonomy group Φ of M* and the *restricted holonomy group Φ^0 of M*.

Since Γ is a metric connection on M compatible with the Lorentz metric g it follows that Φ and Φ^0 are each subgroups of the orthogonal group associated with g i.e. the Lorentz group $\mathcal{L}$. Now $\mathcal{L}$ is a 6-dimensional Lie group and, in fact, Φ and Φ^0 can each be given the structure of a Lie subgroup of $\mathcal{L}$ with Φ^0 the identity component of Φ (and so Φ^0 is connected and Φ and Φ^0 have isomorphic Lie algebras). If M is simply connected, $\Phi = \Phi^0$ and so, in this case, the holonomy group is a connected Lie subgroup of $\mathcal{L}$. It should perhaps be remarked here that perhaps the most natural definition of the holonomy group of M is in terms of the frame bundle $L(M)$ of M which is a principal fibre bundle with structure group isomorphic to $GL(4, \mathrm{R})$. Since Γ is a metric connection the structure group can be reduced to the Lorentz group and it is essentially the corresponding reduced subbundle that is being considered here (and the holonomy subbundles that it contains).

Again let $p \in M$ and in some coordinate neighbourhood of p consider the set of matrices of the form

$$R^a{}_{bcd}X^cY^d,\ R^a{}_{bcd;e}X^cY^dZ^e, \ldots \tag{2}$$

where a semi-colon denotes a Γ covariant derivative and where each of the above matrices is evaluated at p with X, Y, Z being arbitrary members of T_pM. Clearly each matrix in (2) is skew-self adjoint with respect to the metric g_{ab} at p and it turns out (see e.g. [1]) that the set in (2) constitutes a Lie algebra under matrix commutation. Thus the set of matrices (2) is a subalgebra of the Lie algebra of $\mathcal{L}$ (the *Lorentz algebra $\mathcal{A}$*). This subalgebra uniquely determines a connected Lie subgroup of $\mathcal{L}$ called the *infinitesimal holonomy group of M at p* and is denoted by $\Phi'(p)$. Each member of $\Phi'(p)$ is, in an obvious sense, a member of the restricted holonomy group of M at p. The link between *infinitesimal* holonomy groups and "true holonomy" is not immediately obvious and is provided by the Ambrose-Singer theorem (see e.g. [1]). However, it should be noted that the infinitesimal holonomy groups at distinct points of M may not be isomorphic. If the *dimension* of the Lie group $\Phi'(p)$ is independent of p then each $\Phi'(p)$ is isomorphic to Φ^0 and so in this case the infinitesimal holonomy groups at distinct points of M *are* isomorphic. Thus if M is simply connected and the dimension of $\Phi'(p)$ is independent of p, Φ, Φ^0 and all the $\Phi'(p)$s are isomorphic.

Now let $p \in M$ and suppose that a non-trivial proper subspace V of T_pM is *holonomy invariant*, that is, V is mapped to itself under parallel transport around any differentiable closed curve starting and ending at p. Then it is clear that parallel transport of V from p to any other point $q \in M$ gives rise to a subspace of T_qM of the same dimension as V which is independent of the differentiable path chosen from p to q and which is itself holonomy invariant. The distribution thus obtained on M can then be shown to be smooth (since Γ is) and integrable and its maximal integral submanifolds are totally geodesic and (since Γ is symmetric) auto-parallel. If such a subspace V exists the holonomy group Γ is called *reducible*. Under these circumstances it is clear that the subspace $V^\perp$ of members of T_pM which are orthogonal to each member of V is also holonomy invariant. The subspace $V^\perp$ is called the *orthogonal complement* of V in spite of the fact that for Lorentz (or any indefinite) metrics V and $V^\perp$ need not be complementary.

Before introducing holonomy into Lorentz manifolds consider first the positive definite case and the well known de Rham decomposition theorem.

Theorem 1 (de Rham) *Let M' be a connected manifold admitting a positive-definite metric and corresponding Levi-Civita connection. Let $p \in M'$ and let V_0 be the maximal subspace of T_pM' on which the associated holonomy group acts trivially. Then the orthogonal complement of V_0 is holonomy invariant and may be decomposed into a direct sum $V_1 \oplus \cdots \oplus V_n$ of subspaces of T_pM' which are mutually orthogonal, holonomy invariant and irreducible under the holonomy (i.e. contain no non-trivial proper holonomy invariant subspaces). Further*

(i) If M_i $(0 \leq i \leq n)$ are the maximal integral submanifolds corresponding to the V_i, through p, there exists an open neighbourhood U of p which, as an open submanifold of M, is the manifold product $U = U_0 \times \cdots \times U_n$ where U_i is an open submanifold of M_i and where the metric on U (restricted from M') is the product of the induced metrics on the U_i. The decomposition of T_pM' given above is unique (up to order) if M' is simply connected.

(ii) If, further, M' is simply connected and complete then each M_i is, with its induced metric, connected, simply connected, totally geodesic and complete and M' is isometric to $M_0 \times \cdots \times M_n$. This decomposition of M is unique up to order.

The manifold M' is said to be locally decomposable if it satisfies the conclusions of the first sentence of (i) above. If further, M' is simply connected, the holonomy group at p can be written as a direct product $\Phi\ (= \Phi^0) = \Phi_0 \times \cdots \times \Phi_n$ of normal subgroups where Φ_i acts trivially on V_j $(i \neq j)$ and is irreducible on V_i, for each i (and so Φ_0 consists only of the identity). In (ii) M'_0 is a Euclidean space and the holonomy group arising in an obvious way from the induced metric in each of the M'_i $(1 \leq i \leq n)$ is Φ_i and is irreducible. If M' satisfies the conclusions of (ii) it is called *globally decomposable*.

3 Space-Time Holonomy

When M is a space-time with Lorentz metric g and if $p \in M$, the non-trivial proper subspaces of T_pM can be divided into three types the members of which are called spacelike, timelike and null, respectively. Such a subspace is called *spacelike* if all its non-zero members are spacelike, *null* if it contains exactly one null vector (up to multiples) and *timelike* otherwise. It follows that all non-zero members of a null subspace are either multiples of the above mentioned special null vector or spacelike and orthogonal to it. It also follows that timelike 2-spaces contain exactly two distinct null vectors and that any timelike subspace contains timelike, spacelike and null members. It is not hard to prove that if V is a non-trivial proper subspace of T_pM then V and $V^\perp$ are complementary if and only if V is non-null (which is equivalent to $V^\perp$ being non-null).

If $p \in M$, a subspace of T_pM is called *weakly irreducible* ([2]— see also [3]) if it is either $\{0\}$, T_pM itself or non-null, and if it contains no non-trivial proper *non-null* holonomy invariant subspaces. It will be seen how the idea of weak ("non-degenerate") reducibility is the required type of holonomy reducibility in order to get a de Rham-type theorem and how difficulties can arise when null holonomy invariant subspaces are admitted. The holonomy group is called *non-degenerately reducible* if some non-trivial, proper *non-null* subspace of T_pM for some (any) $p \in M$ is holonomy invariant. The analogue of theorem 1 when the metric is not positive definite (and, in particular, when it is Lorentz) was given by Wu [2] (and note some corrections this reference makes to earlier articles). To avoid certain problems, especially in the applications to general relativity, attention is focussed on the restricted holonomy group (which, of course, is equal to the holonomy group if M is simply connected).

Theorem 2 (Wu [2]) *Let M be a simply connected space-time with Lorentz metric g and corresponding Levi-Civita connection. Let $p \in M$ and suppose that V_0, the maximal subspace of T_pM on which the holonomy group Φ of M acts trivially, is either $\{0\}$ or proper and non-null (the trivial flat case $V_0 = T_pM$ is excluded). Then the orthogonal complement of V_0 is holonomy invariant and may be decomposed into a direct sum $V_1 \oplus \cdots \oplus V_n$ of subspaces of T_pM which are mutually orthogonal, holonomy invariant and weakly irreducible. Further (and recalling that simply connectedness is assumed here) the statements (i) and (ii) of theorem 1 hold here also provided M' is replaced by M as also do the remarks in the paragraph following theorem 1 if irreducible is replaced by weakly irreducible (in an obvious sense) and if the word Euclidean is replaced by Euclidean or Minkowskian.*

Since M was assumed a simply connected space-time it follows that Φ $(= \Phi^0)$ is a connected Lie subgroup of $\mathcal{L}$ and hence of the identity component $\mathcal{L}^0$ of $\mathcal{L}$. Since there is a one-to-one correspondence between connected Lie subgroups of $\mathcal{L}$ (equivalently $\mathcal{L}^0$) and the subalgebras of the Lorentz algebra $\mathcal{A}$, one can essentially find all candidates for the group Φ by finding all the subalgebras of $\mathcal{A}$. The subalgebras can be conveniently grouped into 15 types labelled R_1— R_{15}, where R_1 is the trivial subalgebra and $R_{15} = \mathcal{A}$ [4,5] (see table 1). Recalling that if one represents $\mathcal{L}$ by $\mathcal{L} = \{A \in \mathrm{GL}(4, \mathrm{R}) : A^T \eta A = \eta\}$ where $\eta = \mathrm{diag}(-1, 1, 1, 1)$ then $\mathcal{A}$ can be represented by the vector space of matrices which are skew-self adjoint with respect to η together with the multiplication given by matrix commutation. Let (R^4, η) denote Minkowski space and let $\{l, n, x, y\}$ be a *null tetrad* of members of R^4, i.e. they satisfy $\eta(l, n) = \eta(x, x) = \eta(y, y) = 1$ with all other η-inner products between them equal to zero. Then using the isomorphisms between R^4 and its dual space (and an obvious abuse of notation) one may regard $l \wedge n$, $l \wedge x$, $l \wedge y$, $n \wedge x$, $n \wedge y$ and $x \wedge y$ as a basis of (decomposable) 2-forms for $\mathcal{A}$. An obvious similar basis may be constructed from a (pseudo-) orthonormal basis $\{u, x, y, z\}$ of members of R^4 satisfying $\eta(x, x) = \eta(y, y) = \eta(z, z) = -\eta(u, u) = 1$. The potential holonomy algebras R_1—R_{15} can then be conveniently described in terms of the members of these bases. One can now collect together the following results [6,7].

Theorem 3 *Let M be a simply connected space-time. Then*

(i) *the group corresponding to the algebra R_5 cannot be the holonomy group of any space-time*

(ii) *the members of the holonomy algebra admit a common eigenvector if and only if the holonomy group of M is reducible (which, in turn, is equivalent to M admitting a global nowhere zero recurrent vector field)*

(iii) *the members of the holonomy algebra admit a common eigenvector with zero eigenvalue if and only if M admits a global nowhere zero covariantly constant vector field.*

The proofs are based on exponentiation of the Lie algebra which links the common eigenvector and a holonomy-invariant direction (i.e. a recurrent or covariantly constant vector field). It is merely remarked that whilst $\mathcal{L}^0$ is an exponential Lie group (i.e. every member of $\mathcal{L}^0$ is the exponential of some member of $\mathcal{A}$) its subgroups are not necessarily exponential. However, every member of a connected Lie group G is a product of finitely many members of G each of which is the exponential of some member of the Lie algebra of G. It is also noted that (ii) relies on both the dimension of M and the signature of g. For a quick proof of (i) note that the earlier remarks about infinitesimal holonomy groups show that, if the holonomy of M corresponds to the algebra R_5 and since the algebra R_5 contains a single non-decomposable 2-form, the Riemann tensor takes the local form $R_{abcd} = \alpha F_{ab}F_{cd}$ where $\alpha \in \mathrm{R}$ and F is a non-decomposable skew symmetric tensor. But then $R_{a[bcd]} = 0 \Rightarrow F_{a[b}F_{cd]} = 0$ which is equivalent to F being decomposable and a contradiction follows.

A geometrical interpretation of the holonomy transformations of T_pM for $p \in M$ can be obtained in terms of a null tetrad l, n, x, y at p by noting that (i) a member $x \wedge y$ in the holonomy algebra leads to a spacelike rotation of T_pM in the xy plane, (ii) a member $l \wedge n$ leads to a boost in the ln plane, (iii) a member $l \wedge x$ leads to a null rotation in the lx plane and (iv) a member $(l \wedge n) + \rho(x \wedge y)$ with $\rho \in \mathrm{R}$ $(\rho \neq 0)$ leads to a screw motion in the orthogonal ℓn and xy planes. These ransformations are, respectively

$$(i)\quad \begin{array}{rcl} l' &=& l \\ n' &=& n \\ x' &=& \cos\theta x - \sin\theta y \\ y' &=& \sin\theta x + \cos\theta y \end{array} \qquad (ii)\quad \begin{array}{rcl} l' &=& Al \\ n' &=& A^{-1}n \\ x' &=& x \\ y' &=& y \end{array}$$

$$(iii)\quad \begin{array}{rcl} l' &=& l \\ n' &=& n - \alpha x + \alpha^2 l/2 \\ x' &=& x + \alpha l \\ y' &=& y \end{array} \qquad (iv)\quad \begin{array}{rcl} l' &=& Al \\ n' &=& A^{-1}n \\ x' &=& \cos\theta x - \sin\theta y \\ y' &=& \sin\theta x + \cos\theta y \end{array}$$

where $\theta, A, \alpha \in \mathrm{R}$, $A > 0$.

¿From the table it can be seen that all possible holonomy groups are reducible except that one associated with type R_{15}. The non-degenerately reducible cases are those corresponding to the algebra types $R_1, R_2, R_3, R_4, R_6, R_7, R_{10}$ and R_{13}. If the holonomy corresponds to an algebra type R_8 or $R11$ a global smooth covariantly constant null vector field is admitted whilst for types R_9, R_{12} or R_{14} a global smooth properly recurrent null vector field l is admitted (i.e. locally l satisfies $l^a{}_{;b} = l^a\omega_b$ for a smooth 1-form

ω but with l not scalable so as to be covariantly constant). This information is collected together in the table.

The holonomy groups themselves can be represented in terms of simpler groups in the following way: $R_1 \to$ the trivial group, $R_2 \to (\mathrm{R}^*, \times)$ or $SO(1,1)$, $R_3 \to (\mathrm{R}, +)$, $R_4 \to SO(2)$, $R_7 \to (\mathrm{R}^*, \times) \times SO(2)$ or $SO(1,1) \times SO(2)$, $R_8 \to (\mathrm{R}^2, +)$, $R_{10} \to SO(1,2)$, $R_{13} \to SO(3)$, $R_{15} \to \mathcal{L}^0$ or $SO(1,3)$ where e.g. $SO(1,1)$ means the component of the identity of $SO(1,1)$. For the remainder one notes that R_{14} (the "null rotation" subgroup of $\mathcal{L}^0$) is just the group of similarities of the Euclidean plane (dilatations, translations and rotations) and R_6, R_9, R_{11} and R_{12} are conveniently described as subgroups of this similarity group. Thus $R_6 \to$ dilatations and translations, $R_9 \to$ dilatations and (two dimensional) translations, $R_{11} \to$ dilatations and (two dimensional) translations and $R_{12} \to$ screw motion and (two dimensional) translations.

Local forms for the metric g in "locally decomposed" coordinate systems can easily be given.

$$(a) \quad ds^2 = -dt^2 + g_{\alpha\beta} dx^\alpha dx^\beta \qquad (b) \quad ds^2 = g_{AB} dx^A dx^B + g_{A'B'} dx^{A'} dx^{B'}$$

$$(c) \quad ds^2 = -dt^2 + dx'^2 + g_{A'B'} dx^{A'} dx^{B'}.$$

The metric (a) arises locally in the R_{13} case and $\partial_a t$ is a covariantly constant unit timelike vector field on this chart domain. In (a) the Greek indices take the values 1,2 and 3 and the $g_{\alpha\beta}$ are simultaneously components of the metric g on M and the components of the positive definite metric induced in the hypersurfaces of constant t and depend only on the x^α coordinates. The local metric in the cases R_3, R_6 and R_{10} can be obtained from (a) by interchanging the coordinates t and x' and making an appropriate change of sign and signature. The local metric (b) occurs in the R_7 case where the indices A, B take the values 0,1 and $A'B'$ the values 2,3 and where g_{AB} (respectively $g_{A'B'}$) depends only on the coordinates x^0, x^1 (respectively x^2, x^3) and are the induced metrics (respectively of Lorentz and positive definite signatures) in the holonomy invariant 2-dimensional distributions obtained from the pairs l, n and xy (see table 1). The local metric (c) occurs in type R_4 and $g_{A'B'}$ depends only on the coordinates $x^2 x^3$ being the positive definite metric induced in the holonomy invariant 2-dimensional distribution arising from the pair xy (see table 1). The local metric for R_2 can be obtained by interchanging t and x^2 in (c) and making the obvious sign and signature changes. This covers all the non-degenerately reducible cases which are not flat. The types R_8 and R_{11} admit a covariantly constant global null vector field and for these types and also for the properly recurrent cases R_9, R_{12} and R_{14} a (slightly less convenient) local form for the metric exists (see e.g. [8]).

For vacuum space-times (i.e. $T_{ab} \equiv 0$ in (1)) only certain holonomy groups are possible. Some time ago it was realised that if a vacuum space-time M had the same infinitesimal holonomy type and the same Petrov type at every $p \in M$ then the Lie algebra of the infinitesimal holonomy group was either of type R_8, R_{14} or R_{15} [4,9,10]. Those of type R_8 were the (Petrov type N) pp-waves [11], those of type R_{14} were the (Petrov type III) metrics of Goldberg and Kerr [9,10] and all other vacuum metrics (if not flat) were of type R_{15} (with any Petrov type being possible). That the dimension of the infinitesimal holonomy group had to be even at any point of any vacuum space-time (and, in fact, of type R_8, R_{14} or R_{15}) is a direct consequence of the self-duality of a vacuum Riemann tensor [12]. The following less restrictive theorem is available [13].

Theorem 4 *Let M be a vacuum space-time. The infinitesimal holonomy group at a point of M is isomorphic to a connected Lie subgroup of $\mathcal{L}^0$ corresponding to a subalgebra of $\mathcal{A}$ of type R_1, R_8, R_{14} or R_{15}. If, further, M is simply connected the holonomy group of M is isomorphic to a connected Lie subgroup of $\mathcal{L}^0$ corresponding to a subalgebra of A of type R_1, R_8, R_{11}, R_{14} or R_{15} with the R_{11} case being impossible if the infinitesimal holonomy group is not of type R_1 at any point of M. The R_{11} case is also impossible if M and g are analytic.*

The author knows of no examples of vacuum space-times for which the holonomy group is of type R_{11} but cannot rule it out in a vacuum space-time containing a point at which the Riemann tensor and each of its covariant derivatives vanish.

To complete this section it is remarked that for a (not necessarily simply connected) space-time M it is generic for the restricted holonomy group to correspond to type R_{15} in the sense that if one puts the Whitney C^∞ topology on the set of all (global smooth) Lorentz metrics on M then this set contains an open dense subset for each member of which the restricted holonomy group is of type R_{15}. This follows from a result in [14]. Examples of space-times with each of the holonomy groups corresponding to $R_1, \ldots, R_{15}$ (except R_5) can be constructed. In fact space-times with holonomy groups corresponding to the algebra types R_2, R_4, R_7, R_{10} and R_{13} can easily be given by constructing globally decomposable space-times along the lines of the locally decomposable ones given earlier. Examples of the other types can be constructed by using the result that if the infinitesimal holonomy groups at the individual points of M have the same dimension, they are isomorphic to each other and to the restricted holonomy group of M. Thus by restricting to an open simply connected chart domain of a space-time with known (fixed) infinitesimal holonomy group, further examples of each of the holonomy group types can be constructed. Thus a simply connected vacuum pp-wave space-time whose Riemann tensor never vanishes is an example of type R_8 (see e.g. [11]) whilst the vacuum metrics in [9,10] give examples of type R_{14}. Examples of type R_3, R_6, R_9, R_{11} and R_{12} can be constructed from the metrics in [15] (see also [16] and, for further general discussion of holonomy in space-times, [17]). The examples are completed by remarking that Minkowski space is of type R_1 and the Schwarzschild metric of type R_{15}.

4 Some Applications

For each of the holonomy types for a simply connected space-time one can find all possible global solutions for a second order symmetric covariant tensor h of the equation $h_{ab;c} = 0$ (see [6] and table 1). If such an h exists which is not a (constant) multiple of g then by adding appropriate constant multiples of g to it, it may, without loss of generality, be assumed non-degenerate and, hence, gives rise to an alternative metric for M compatible with the given connection Γ. Thus the problem of the uniqueness of the metric compatible with Γ is solved [6,7,18]. The alternative metric can (but need not necessarily) be chosen of Lorentz signature.

The work reviewed in this paper on holonomy groups can also be used to study affine symmetries (collineations) in space-times [6,19]. The remark in the previous paragraph is particularly important in this respect. In particular, one has an injective linear map from the (finite-dimensional) vector space of all smooth global affine vector fields on M to the vector space of smooth global second order symmetric covariantly constant tensors on M given by $X \to h_{ab} \equiv X_{(a;b)}$ (X affine $\Rightarrow h_{ab;c} = 0$). The latter vector space has dimension d satisfying $1 \leq d \leq 4$ (table 1) and X is proper affine (i.e. not Killing or homothetic) if and only if h is not a constant multiple of g. (If X is homothetic, $h = cg$, $c \in$ R and if X is Killing, $h \equiv 0$.) This remark enables proper global affine vector fields on M to be "counted" up to additive homothetic or Killing vector fields.

5 Weyl Connections

Let M be a smooth connected hausdorff paracompact manifold of dimension $n > 2$ and let Γ be a smooth symmetric liner connection on M. Then Γ is called a Weyl connection if there exists a smooth metric g (of any signature) on M such that $g_{ab;c} = g_{ab}\phi_c$ for some smooth global 1-form ϕ on M. If M admits a Weyl connection it is called a Weyl manifold and, when the above holds, the pair (g, ϕ) will be said to fit Γ [20].

These definitions are, of course, motivated by Weyl's space-time theory (which is no longer taken seriously by physicists in its original form). Weyl's original creation [21] had M as a space-time on which there existed a Lorentz metric g and a connection Γ such that parallel transport of a non-null vector $k \in T_pM$ along a curve c from p to q in M may result in a change of length of k (with respect to g). However, the ratio of the lengths of k at p and q was assumed to depend only on p, q and c and not on k. This leads to the relation between Γ and g (and introduces ϕ) and so g is recurrent with respect to Γ. The extra freedom built into Γ by not insisting that it is a metric connection compatible with g was used by Weyl to incorporate the electromagnetic field and the Maxwell tensor representing this field was taken to be $F \equiv d\phi$. Thus the absence of electromagnetic fields, $F \equiv 0$, was then equivalent to ϕ being locally a gradient. Weyl also noticed that if (g, ϕ) fits Γ and σ is a global smooth real valued function on M then (g', ϕ') also fits Γ where $g' = e^{\sigma}g$ and $\phi' = \phi + d\sigma$ (and so the corresponding Maxwell tensors $d\phi$ and $d\phi'$ are equal). The changes $(g, \phi) \to (g', \phi')$ are the gauge transformations of Weyl. Thus for the case $F \equiv 0$ (the so called "integrable" case) if U is a starlike coordinate domain U of M one has the existence of a smooth real valued function ψ on U such that, on U, $\phi = d\psi$ and $g' = e^{-\psi}g$ is a metric on U compatible with Γ (i.e. $g'_{\ ab;c} = 0$). In other words, $(g', 0)$ fits Γ on U.

Two alternative approaches to Weyl manifolds were described in [22]. Let M be as above with g a metric (of any signature) on M and let $C(g)$ be the set of all global metrics on M conformal to g (i.e. of the form $e^{\sigma}g$ for some smooth $\sigma : M \to$ R). Denote by S the set of all (generalised) inner products arising from members of $C(g)$ at all points of M. Then S can be given the structure of a principal fibre bundle over M with structure group the positive real numbers under multiplication. A Weyl structure on M is then essentially a connection on S and the corresponding quantities ϕ and $d\phi$ in Weyl's theory are just the pulldown of this connection and curvature to M using a section of S. Next, with M and g as above, define a conformal frame at $p \in M$ to be a frame consisting of n non-null members of T_pM which are mutually orthogonal and (apart from signs) of equal size with respect to any member of $C(g)$ at p. The set of all conformal frames at all points of M can then be given the structure of a principal fibre bundle over M with structure group the conformal group $C(n)$ in n-dimensions appropriate to the signature of g (i.e. the group of all real $n \times n$ matrices A such that $AhA^T = kh$ where h is the Sylvester matrix corresponding to the signature of g, A^T is the transpose of A and $k \in \mathrm{R}$). A symmetric linear connection on M is then a Weyl connection which fits (g, ϕ) for some global 1-form ϕ on M if and only if it is reducible to the bundle of conformal frames on M.

Suppose that a symmetric linear connection Γ is given on the manifold M above. Schmidt [23] has given a criterion, in terms of the holonomy group of Γ, to decide whether Γ is a metric connection. In fact Γ is metric if and only if the holonomy group of Γ is isomorphic to a Lie subgroup of some (pseudo-) orthonormal group in n-dimensions. The possibilities for the associated metric can then be described by techniques similar to that mentioned above for space-times (and a more general approach has been given in [24]). Similar questions can be raised regarding Weyl connections and the following theorem summarises some of the answers [20].

Theorem 5 *Let M be an n-dimensional smooth connected hausdorff paracompact manifold ($n > 2$) admitting a smooth linear symmetric connection Γ. Then*

(i) Γ is a Weyl connection if and only if the holonomy group of Γ is isomorphic to a Lie subgroup of some conformal group in n-dimensions.

(ii) If Γ is a Weyl connection on M and (g_1, ϕ_1) and (g_2, ϕ_2) fit Γ then $d\phi_1 = d\phi_2$.

(iii) If Γ is a Weyl connection on M, such that (g, ϕ) fits Γ, and Γ is compatible with some metric gamma (of any signature) on M then $d\phi = 0$ and Γ is "integrable".

(iv) Let Γ be a Weyl connection on M such that (g_1, ϕ_1) and (g_2, ϕ_2) fit Γ. If g_1 and g_2 are not conformally related at some point of M they are not conformally related at any point of M and the holonomy group of M is reducible.

(v) Let Γ be a Weyl connection on M such that (g, ϕ) fits Γ. If the holonomy group of Γ is non-degenerately (with respect to g) reducible there exists a pair (g', ϕ') which fits Γ such that g' is not conformally rleated to g at any point of M.

(vi) The result in (v) can fail if non-degenerately reducible is replaced by reducible.

In particular, it is pointed out that if Γ is given and if (g, ϕ) and (g', ϕ') fit Γ then one has the possibility of "generalised" gauge transformations $(g, \phi) \to (g, \phi')$' where g' is not necessarily conformally related to g. However, the "electromagnetic field" $d\phi$ ($= d\phi'$) is always invariant. Further, one should note that pairs (g, ϕ) and (g', ϕ) may fit Γ with g and g' of different signatures. Finally it is remarked that the above discussion suggests the obvious definition of a connection being locally metric [20]. It then follows that a Weyl connection on M is locally metric if and only if it is "integrable".

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