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HOLOMORPHICALLY–PROJECTIVE MAPPINGS OF T–QUASISEMISYMMETRIC AND GENERALLY SYMMETRIC KÄHLERIAN SPACES

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Abstract. The conditions on T-quasisemisymmetric Kählerian spaces to admit holomorphically-projective mappings are found. It is proved that m-symmetric spaces do not admit nontrivial holomorphically-projective mappings.
Key words: holomorphically-projective mapping, Kählerian spaces, quasisemisymmetric spaces.
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1. Introduction

Investigation of holomorphically-projective mappings of symmetric and semisymmetric spaces was initiated by T. Sakaguchi [6] and continued in works of V.V. Domašev and J. Mikeš [1], J.Mikeš [4], [6], [8], I.N. Kurbatova [3].

A Riemannian space K_n is said to be *Kählerian* if there exists, together with the metric tensor $g_{ij}(x)$, a structure $F_i^h(x)$ with the properties:

$$F_\alpha^h F_i^\alpha = e\delta_i^h; \quad F_{(ij)} = 0; \quad F_{i,j}^h = 0, \tag{1}$$

where $e = \pm 1$; $F_{ij} \equiv g_{i\alpha}F_j^\alpha$, δ_i^h is the Kronecker symbol, and comma denotes the covariant derivative with respect to the connection on K_n.

If $e = -1$, then K_n is called an *(elliptical) Kählerian space* and denoted by K_n^-, and if $e = +1$ then K_n is called a *hyperbolic Kählerian space* and denoted by K_n^+.

By an *analytically planar curve* in K_n we shall mean a curve defined by the equations $x^h = x^h(t)$ such that the following conditions hold

$$d\lambda^h/dt + \Gamma_{\alpha\beta}^h \lambda^\alpha \lambda^\beta = \rho_1(t)\lambda^h + \rho_2(t)F_\alpha^h \lambda^\alpha,$$

where $\lambda^h \equiv dx^h/dt$, ρ_1, ρ_2 are functions of the argument t.

A bijective correspondence between points of Kählerian spaces K_n and $\bar{K}_n$ is called a *holomorphically-projective mapping*, if any analytically planar curve in K_n is transformed into an analytically planar curve in $\bar{K}_n$.

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A Kählerian space K_n admits a holomorphically-projective mapping onto $\bar{K}_n$ if and only if one of the following conditions is satisfied (via a diffeomorphism $K_n \to \bar{K}_n$):

$$\bar{g}_{ij,k} = 2\bar{g}_{ij}\psi_k + \psi_{(i}\bar{g}_{j)k} - e\bar{\psi}_{(i}\bar{F}_{j)k}; \tag{2}$$

$$a_{ij,k} = \lambda_{(i}g_{j)k} - e\bar{\lambda}_{(i}F_{j)k}, \tag{3}$$

where $\bar{g}_{ij}$ is a metric tensor of $\bar{K}_n$, ψ_i, λ_i are gradient vectors, a_{ij} is a tensor,

$$a_{ij} = a_{ji} = -ea_{\alpha\beta}F_i^\alpha F_j^\beta;\ det\|a_{ij}\| \neq 0;\ \bar{\psi}_i \equiv \psi_\alpha F_i^\alpha;\ \bar{\lambda}_i \equiv \lambda_\alpha F_i^\alpha;\ \bar{F}_{ij} \equiv \bar{g}_{i\alpha}F_j^\alpha.$$

The vector λ_i has the following properties: $\lambda_{i,j} = \lambda_{j,i} = -e\lambda_{\alpha,\beta}F_i^\alpha F_j^\beta$.

2. Holomorphically-projective mappings of T-quasisemisymmetric Kählerian spaces

In what follows T will be an m-covariant tensor, which cannot be expressed as a tensorial sum of products of invariants, the metric and the structure tensor. Later on let us define the tensor

$$Z^h_{ijk} \equiv R^h_{ijk} - B(\delta^h_k g_{ij} - \delta^h_j g_{ik} + eF^h_k F_{ij} - eF^h_j F_{ik} + 2eF^h_i F_{jk}),$$

where R^h_{ijk} is the Riemann tensor, B is a function.

It takes place the following

Theorem 1. *Let in a Kählerian space K_n the condition*

$$\sum_{r=1}^{m} T_{i_1\ldots i_{r-1}\alpha i_{r+1}\ldots i_m} Z^\alpha_{i_r jk} = 0 \tag{4}$$

be satisfied.

Then if K_n $(n > 4(m-1))$ admits holomorphically-projective mappings, the basic equations of the theory of holomorphically-projective mappings take the form

$$(a)\quad a_{ij,k} = \lambda_{(i}g_{j)k} - e\bar{\lambda}_{(i}F_{j)k};\quad (b)\quad \lambda_{i,j} = \mu g_{ij} + Ba_{ij}, \tag{5}$$

where μ is a function.

Proof. Let T in K_n be as above, satisfying the conditions (4). Further, we shall suppose K_n to admit a holomorphically-projective mapping. According to [1],[5] the equations (3) hold and the corresponding integrability conditions are

$$a_{\alpha(i}R^\alpha_{j)kl} = \lambda_{l,(i}g_{j)k} - \lambda_{k,(i}g_{j)l} - e\bar{\lambda}_{l,(i}F_{j)k} - \bar{\lambda}_{k,(i}F_{j)l}. \tag{6}$$

In [4] the above theorem is proved for the case $e = -1$, $m = 1, 2$. The proof for the case e=+1, m=1,2, is analogous.

By induction we shall prove our theorem for an arbitrary $m > 2$ assuming that it is correct for $1, 2, ..., m-1$. Let us suppose the opposite, i.e. that for a holomorphically-projective mapping the conditions (5) do not hold. Evidently, any convolution of T with g^{ij} ($\|g^{ij}\| \equiv \|g_{ij}\|^{-1}$), or $F^{ij} \equiv F^i_\alpha g^{\alpha j}$ satisfies the relations (4) and by our assumption these convolutions are of the form of a tensorial sum of products of invariants, metric and the structure tensor. For m odd the pointed convolutions are equal to zero. For m even and $n \geq 2m-1$ one easily can construct a tensor T^*, being a sum of the tensor T and the tensorial sums of products of definite invariants, metric and the structure tensor, such that any convolution of T^* with g^{ij} or F^{ij} is zero. The tensor T^* satisfies the conditions (4); hence, without loss of generality, in what follows we shall consider T^* instead of T.

We shall convolve the relation (4) with $a_{k\alpha}g^{\alpha j}$, and then symmetrize respectively to l and k. Taking into account the structure of the tensor Z^h_{ijk}, the conditions (6), and the properties of the Riemann tensor, after a corresponding grouping we get

$$\sum_{r=1}^{m}\{\Lambda_{h_r(i}T_{|h_1\dots h_{r-1}|j)h_{r+1}\dots h_m} - e\Lambda_{h_r\alpha}F^{\alpha}_{(i}F^{\beta}_{j)}T_{h_1\dots h_{r-1}\beta h_{r+1}\dots h_m} -$$

$$- g_{h_r(i}T^r_{|h_1\dots h_{r-1}|j)h_{r+1}\dots h_m} + eF_{h_r(i}F^{\beta}_{j)}T^r_{h_1\dots h_{r-1}\beta h_{r+1}\dots h_m}\} = 0, \tag{7}$$

where

$$\Lambda_{ij} \equiv \lambda_{i,j} - \mu\, g_{ij} - B\, a_{ij}; \tag{8}$$

$$\mu \equiv (\lambda_{\alpha,\beta} - Ba_{\alpha\beta})g^{\alpha\beta}/n;$$

$$T^r_{h_1\dots h_m} \equiv \Lambda_{\alpha h_r}g^{\alpha\beta}T_{h_1\dots h_{r-1}\beta h_{r+1}\dots h_m}.$$

Contracting (7) in turn with $g^{ih_p}\,g^{jh_q}$, $g^{ih_p}\,F^{jh_q}$, $F^{ih_p}\,F^{jh_q}$, where $p,q = 1,2,\dots,m$ $(p \neq q)$, we shall obtain equations which for $n > 4(m-1)$ give rise to

$$\begin{aligned} &g^{\alpha\beta}T^r_{h_1\dots h_{p-1}\alpha h_{p+1}\dots h_{r-1}\beta h_{r+1}\dots h_m} = 0;\\ &F^{\alpha\beta}T^r_{h_1\dots h_{p-1}\alpha h_{p+1}\dots h_{r-1}\beta h_{r+1}\dots h_m} = 0. \end{aligned} \tag{9}$$

Then we contract (7) with g^{ih_p} and F^{ih_p}. From the obtained relations taking into account (9) we shall get, that $T^r_{h_1\dots h_m} = 0$. Then the formulas (7) are simplified. From them one can show that either $\Lambda_{ij} = 0$, or $T_{h_1\dots h_m} = 0$. Taking into account formulas (8), the theorem is completely proved.

It takes place

Theorem 2. *Let T in a Kählerian space K_n $(n \geq 4(m-1))$ be as above, and satisfying the conditions $T_{,lm} - T_{ml} = 0$. If a K_n admits a holomorphically-projective mapping then it holds*

$$\lambda_{i,j} = \mu g_{ij}, \tag{10}$$

where $\mu = const$, i.e. there exists a convergent vector field in K_n.

The proof follows from Theorem 1 for $B = 0$. But it remains to prove that μ is constant. The latter easily follows from the integrability conditions of equations (10) and (6).

Note that Theorems 1 and 2 have been proved earlier by J. Mikeš [4], [6] for m=1,2; $n \geq 4$, and for a special case where m=4, $n \geq 4$ and T is the Riemann tensor. These theorems a priori are incorrect for $n = m$.

In the paper [4],[6] the spaces K_n, admitting holomorphically-projective mappings satisfying the condition (5) are called $K_n[B]$-*spaces.* Spaces where the conditions (4) hold are called T-quasisemisymmetric $\{TPs_n[B]\}$-spaces and the spaces $TPs_n[0]$ are called

T-semisymmetric $\{TPs_n\}$-spaces.The mentioned classes of spaces naturally develop the idea of a semisymmetric space.

Taking into account [7], one can show easily that the compact TPs_n-spaces, the tensor T of which at each point cannot be represented as a sum of products of invariants, metric and the structure tensor for $n \geq 4(m-1)$, do not admit a global holomorphically-projective mapping.

3. Holomorphically-projective mappings of generally symmetric spaces

Let us prove the following theorems

Theorem 3. *Let T in K_n be as above, and suppose that it satisfies the following condition:*

$$T_{h_1 h_2 \ldots h_m, l_1 l_2 \ldots l_r} = 0, \tag{11}$$

for $n \geq 4(m+r-3)$, $r > 1$ or $n \geq 4(m-1)$, $r = 1$.

Then K_n admits a holomorphically-projective mapping satisfying the condition (10).

Proof.

a) First suppose that $m + r$ is odd and $r > 1$. Then alternate (11) with respect to l_{r-1} and l_r. In virtue of Theorem 2 either the conditions (10) hold, or

$$T_{h_1 h_2 \ldots h_m, l_1 l_2 \ldots l_{r-2}} = 0.$$

Applying this procedure consecutively we shall get either

$$T_{h_1 h_2 \ldots h_m} = 0 \quad \text{or} \quad T_{h_1 h_2 \ldots h_m, l} = 0.$$

b) Consider the case $m + r$ is even and $r > 1$. Then by an analogy of the above we can see that $T_{h_1 h_2 \ldots h_m, l_1 l_2 \ldots l_{r-2}}$ is a tensorial sum of products of invariants, metric and the structure tensor. The obtained expression shall be covariantly differentiated with respect to x^{r-1} and x^r. Then we shall convince that the second covariant derivative of the mentioned invariants is zero. Hence, either these invariants are constant, or the conditions (10) hold. But then

$$T_{h_1 h_2 \ldots h_m, l_1 l_2 \ldots l_{r-1}} = 0,$$

i.e. the question under consideration is reduced to a).

c) It remains to investigate the case $T_{h_1 h_2 \ldots h_m, l} = 0$. Let us covariantly differentiate this relation respectively to x^k and then alternate with respect to k and l. For m odd the tensor T is vanishing by Theorem 2, and Theorem 3 is proved. In case of even m even (as in b)) we get that the tensor T is tensorial sum of products of constants, the metric and the structure tensor, what was required to prove.

Let us recall that the space where

$$R^h_{ijk, l_1 l_2 \ldots l_m} = 0.$$

is called *m-symmetric* [2].

Theorem 4. *An $m-$symmetric Kählerian space K_n ($n \geq 4(m+3)$) of nonconstant holomorphically sectional curvature does not admit a nontrivial holomorphically-projective mapping.*

In virtue of Theorem 3 it follows that such spaces could admit a holomorphically-projective mapping only if the condition (6) be satisfied. But this is impossible by results of J. Mikeš [8].

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