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CALCULUS OF FORMS ALONG A MAP ADAPTED TO THE STUDY OF SECOND-ORDER DIFFERENTIAL EQUATIONS

W. Sarlet, E. Martínez and A. Vandecasteele

Abstract. Earlier work, in which we discussed a special class of differential forms associated to a given second-order vector field Γ, has led the roots for a much more extensive and general study of scalar and vector-valued forms along the tangent bundle projection $\tau : TM \to M$ which we briefly review first. The classification of derivations of such forms requires an additional ingredient, namely a non-linear connection. Particularly important in the case of the connection coming from a second-order vector field Γ are the appearance of a degree zero derivation, called the dynamical covariant derivative, and of a type (1,1) tensor field along τ, called the Jacobi endomorphism. We further discuss the extension of this theory to time-dependent equations, for which the general set-up is the calculus of forms along the map $\pi : \mathbb{R} \times TM \to \mathbb{R} \times M$. All essential results of the autonomous case have an analogue in the time-dependent situation, but there are a number of technical complications which make it sometimes hard to decide about the best possible approach. A number of applications are presented. The most successful of these so far concerns a complete characterization of separable equations with the aid of algebraic conditions which are directly verifiable in practice.

Keywords. Forms along a map, derivations, second-order equations, separability.

MS classification. 58A10, 53C05, 70D05.

1. Introduction

In a contribution to the third conference in this series [?], we discussed the rudiments of a calculus of a special type of differential forms on the tangent bundle TM, associated to the vector field Γ which governs a given system of second-order differential equations (Sode). The kind of objects on TM, involved in this theory, for example have the following appearance in coordinates:

$$\begin{aligned}
X &= X^i \frac{\partial}{\partial q^i} + \Gamma(X^i)\frac{\partial}{\partial v^i} \quad \in \mathcal{X}_\Gamma \\
\alpha &= \alpha_i \, dv^i + \Gamma(\alpha_i)\, dq^i \quad \in \mathcal{X}^*_\Gamma \\
U &= u^i_j \left(dq^j \otimes \frac{\partial}{\partial q^i} + dv^j \otimes \frac{\partial}{\partial v^i} \right) + \Gamma(u^i_j)\, dq^j \otimes \frac{\partial}{\partial v^i} \quad \in V^1_\Gamma .
\end{aligned}$$

With respect to the potential relevance of these sets (see also [?]), note that $\mathcal{X}_\Gamma$ contains the symmetries of the system, $\mathcal{X}^*_\Gamma$ contains its adjoint symmetries (see [?] and [?] for

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time-dependent systems) and V^1_Γ contains recursion operators (if they exists) or tensors characterizing complete integrability (see *e.g.* [?]).

What we discovered in 1986 is that: one can give the above mentioned sets the structure of a module over $C^\infty(TM)$, provided one defines a suitable $*$-multiplication; one can wedge forms to get an algebra structure via an adapted $\mathring{\wedge}$ product; basic operations on this algebra, such as a kind of exterior derivative, a Lie derivative, etcetera can be defined, all in such a way that the special association to Γ is preserved. All these constructions, however, were rather ad hoc and thus called for a better explanation, possibly in terms of a more fundamental mechanism.

Observe now that only half of the coefficients of the objects under consideration are relevant and that the main parts can be understood as tensor fields along the map $\tau : TM \to M$. For example, if $S = dq^i \otimes (\partial/\partial v^i)$ denotes the vertical endomorphism on TM, we have that $S(\alpha)$ is a semi-basic form and as such can be identified with a 1-form along τ ($\in \bigwedge^1(\tau)$). Likewise, $\tau_* \circ X$ defines a vector field along τ ($\in \mathcal{X}(\tau)$). The basic mechanism which then emerges is the theory of derivations of forms along τ.

The first few sections of this paper contain a sketch of the theory of derivations of scalar and vector-valued forms along τ and discuss various possibilities for an interplay between objects along τ and objects on TM, with particular emphasis on the role which can be played by a SODE on TM. A full account of this theory can be found in [?] and [?]. We will further indicate how the theory can be generalized for the study of time-dependent dynamical systems and discuss some interesting applications.

2. Calculus of forms along $\tau : TM \to M$

A vector field X along τ (respectively, a 1-form α along τ) is a map defined by the following commutative scheme

$$
\begin{tikzcd}
 & TM \arrow[d, "\tau"] \\
TM \arrow[ur, "X"] \arrow[r, "\tau"'] & M,
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & T^*M \arrow[d, "\pi"] \\
TM \arrow[ur, "\alpha"] \arrow[r, "\tau"'] & M.
\end{tikzcd}
$$

In coordinates, X and α have the following representation:

$$X = X^i(q,v)\frac{\partial}{\partial q^i}, \qquad \alpha = \alpha_i(q,v)\, dq^i.$$

The definition of scalar or vector-valued forms of higher degree (along τ) is similar. We write $\bigwedge(\tau)$ for the set of scalar forms along τ (a module over $C^\infty(TM) \equiv \bigwedge^0(\tau)$) and $V(\tau)$ (with $V^0(\tau) \equiv \mathcal{X}(\tau)$) for the set of vector-valued forms along τ (a graded module over $\bigwedge(\tau)$). Typically, an element $L \in V^\ell(\tau)$ is of the form

$$L = \lambda^i \otimes (\partial/\partial q^i), \qquad \text{with} \quad \lambda^i = \lambda^i_{j_1\cdots j_\ell}\, dq^{j_1} \wedge \cdots \wedge dq^{j_\ell} \in \textstyle\bigwedge^\ell(\tau).$$

The definition of a derivation of $\bigwedge(\tau)$ follows the standard pattern.

Definition. *$D : \bigwedge(\tau) \to \bigwedge(\tau)$ is a derivation of degree r if*

1. $D(\bigwedge^p(\tau)) \subset \bigwedge^{p+r}(\tau)$
2. $D(\alpha + \lambda\,\beta) = D\alpha + \lambda\, D\beta, \qquad \lambda \in \mathbb{R}$
3. $D(\alpha \wedge \gamma) = D\alpha \wedge \gamma + (-1)^{pr}\alpha \wedge D\gamma, \qquad \alpha \in \bigwedge^p(\tau).$

A derivation D (of degree r) of $V(\tau)$ has an associated derivation of $\bigwedge(\tau)$, also denoted by D, and essentially must satisfy an additional Leibnitz-type rule (cf. [?]), namely: for $L \in V^\ell(\tau)$, $\omega \in \bigwedge^p(\tau)$,

$$D(\omega \wedge L) = D\omega \wedge L + (-1)^{pr}\omega \wedge DL.$$

As in the standard theory of Frölicher and Nijenhuis [?], one easily proves that every D of $\bigwedge(\tau)$ is completely determined by its action on *functions* and on *basic 1-forms* ($\in \bigwedge^1(M)$). Moreover, if such a D has been obtained and one wants to extend its action to $V(\tau)$, one merely has to specify some action on *basic vector fields* ($\in \mathcal{X}(M)$), in a way which fulfills the requirements for a derivation on the module of vector fields on M.

There is a natural degree 1 derivation on $\bigwedge(\tau)$, called the "vertical exterior derivative" d^V, which is in fact just the translation of the derivation d_S on $\bigwedge(TM)$, for what its restriction to semi-basic forms is concerned. Explicitly, d^V is determined by the following rules:

$$\begin{aligned} &d^V F = V_i(F)\, dq^i, \quad V_i = \frac{\partial}{\partial v^i}, \qquad \forall F \in C^\infty(TM) \\ &d^V\alpha = 0 \qquad \text{for} \quad \alpha \in \textstyle\bigwedge^1(M). \end{aligned}$$

The notion of an $i_\star$-derivation (vanishing on functions) is defined in the usual way and referring again to [?], one will probably expect that the classification of all derivations could be done in terms of $i_\star$-derivations and derivations commuting with d^V. This, however, only works for D's vanishing on $C^\infty(M)$. For a full classification, one further needs a kind of "horizontal exterior derivative", which in turn requires a connection on $\tau : TM \to M$. Assuming such a connection is given, with coefficients Γ^i_j and putting $H_i = (\partial/\partial q^i)^H \; = \; \partial/\partial q^i - \Gamma^j_i(\partial/\partial v^j)$, d^H on $\bigwedge(\tau)$ is defined by

$$\begin{aligned} &d^H F = H_i(F)\, dq^i, \qquad \forall F \in C^\infty(TM) \\ &d^H\alpha = d\alpha \qquad \text{for} \quad \alpha \in \textstyle\bigwedge^1(M). \end{aligned}$$

A few comments are in order about the extension of these primary derivations to vector-valued forms. The action of $i_\star$-derivations and of d^V is most easily extended to $V(\tau)$ by requiring that they vanish on $\partial/\partial q^i$. For the extension of d^H, one can think of different possibilities, but what seems for us to be the most natural procedure (cf. [?]) is obtained as follows. Observe first that the familiar notion of horizontal and vertical lifts of vector fields on M has an immediate extension to vector fields along τ. Then, for $X \in \mathcal{X}(\tau)$, we define the type (1,1) tensor field $d^H X$ by the rule: $\forall Z \in \mathcal{X}(\tau)$ (and with P_V the vertical projector),

$$(d^H X(Z))^V = P_V([Z^H, X^V]).$$

Eventually, we are then led to consider 4 types of derivations:

- type $i_\star$ derivations: vanish on functions, are determined by some $L \in V(\tau)$ and written as i_L
- type $d^V_\star$ derivations: are of the form $d^V_L = [i_L, d^V]$
- type $d^H_\star$ derivations: are of the form $d^H_L = [i_L, d^H]$
- type $a_\star$ derivations: vanish on the whole of $\bigwedge(\tau)$ (hence only relevant for the extension to $V(\tau)$), are determined by some $Q \in \bigwedge(\tau) \otimes V^1(\tau)$ and written as a_Q.

The classification and decomposition theorem which emerges reads as follows.

Theorem. *Every derivation D of $V(\tau)$, of degree r, has a unique decomposition in the form*

$$D = i_{L_1} + d^V_{L_2} + d^H_{L_3} + a_Q,$$

for some $L_1 \in V^{r+1}(\tau)$, $L_2, L_3 \in V^r(\tau)$, $Q \in \bigwedge^r(\tau) \otimes V^1(\tau)$.

In the work of many authors (see *e.g.* [**?**, **?**, **?**, **?**, **?**, **?**, **?**]), important tensor fields associated to a connection, such as *torsion* and *curvature*, appear as vertical-vector-valued forms on TM. In our approach, they are just elements of $V(\tau)$. The torsion and curvature tensors in coordinates are of the form

$$\begin{aligned} T &= \frac{1}{2}(V_j(\Gamma^k_i) - V_i(\Gamma^k_j))\, dq^i \wedge dq^j \otimes \frac{\partial}{\partial q^k} \\ R &= \frac{1}{2}(H_j(\Gamma^k_i) - H_i(\Gamma^k_j))\, dq^i \wedge dq^j \otimes \frac{\partial}{\partial q^k}, \end{aligned}$$

whereas intrinsically, they are defined by the commutators $[d^H, d^V]$ and $[d^H, d^H]$ (for the action on scalar forms) as follows:

$$[d^H, d^V] = d^V_T, \qquad \frac{1}{2}[d^H, d^H] = -i_{d^V R} + d^V_R.$$

Needless to say, we have extensively studied the decomposition of many other commutators [**?**]. For example, the fact that $[d^V_\star, d^V_\star]$ is again of type $d^V_\star$ leads to a bracket operation, which *e.g.* for vector fields reads: $[d^V_X, d^V_Y] = d^V_{[X,Y]_V}$ with

$$[X, Y]_V = (X^k V_k(Y^i) - Y^k V_k(X^i))\frac{\partial}{\partial q^i}.$$

Since $d^V \circ d^V = 0$, such a bracket satisfies the Jacobi identity. There are equally interesting "horizontal brackets", *e.g.*

$$[X, Y]_H = (X^k H_k(Y^i) - Y^k H_k(X^i))\frac{\partial}{\partial q^i}.$$

In general, however, unless $R = 0$, we have $d^H \circ d^H \neq 0$ and therefore no Jacobi identity.

To conclude the general discussion on derivations, we wish to put some emphasis on an important subclass of derivations D of degree 0, namely those which are *self-dual*, by which we refer to those D which have a Leibnitz-type property with respect to the pairing of 1-forms and vector fields:

$$D\langle X, \alpha\rangle = \langle DX, \alpha\rangle + \langle X, D\alpha\rangle.$$

Their relevance in part lies in the fact that they can be given an extended action on tensor fields of arbitrary type. Therefore, when a derivation D of degree 0 is encountered not having this property, it may be of interest to define two new derivations by the process of self-dualization. On the one hand, one can start from the restriction $D|_{\bigwedge(\tau)}$ and define a new action on $\mathcal{X}(\tau)$ by imposing the above duality rule. Similarly, one can take $D|_{\mathcal{X}(\tau)}$ as the starting point and extend this to $\bigwedge^1(\tau)$ by the same rule. As an example, consider the basic degree 0 derivations emerging from the classification theory, namely d^V_X and d^H_X. The first procedure then creates Lie-derivative type derivations, while the second one defines derivations D^V_X and D^H_X which are linear in X (over $C^\infty(TM)$) and therefore are called *vertical and horizontal covariant derivatives*.

3. Lifts to TM and the Sode connection

While doing calculations involving objects along τ, there is some benefit to be gained from keeping an eye on corresponding things happening on TM all the time, in a way we will briefly sketch now.

As said before, for $X = X^i(\partial/\partial q^i) \in \mathcal{X}(\tau)$, we have corresponding lifted vector fields $X^V = X^i V_i$, $X^H = X^i H_i$ on TM. Every $Z \in \mathcal{X}(TM)$ has a unique decomposition of the form $Z = X^V + Y^H$, for some $X, Y \in \mathcal{X}(\tau)$. Lifts of other objects accordingly are easily defined by their action on horizontal and vertical vector fields. A few examples will suffice to illustrate this idea:

- for $U \in V^1(\tau)$, we define the horizontal lift (cf. [?]) $U^H \in V^1(TM)$ by requiring that $U^H(X^V) = U(X)^V$ and $U^H(X^H) = U(X)^H$.
- for g a symmetric (0,2) tensor field along τ, the *Kähler lift* $g^K \in \bigwedge^2(TM)$ is defined by

$$\begin{aligned} g^K(X^H, Y^H) &= g^K(X^V, Y^V) = 0 \\ g^K(X^V, Y^H) &= g(X,Y) = -g^K(X^H, Y^V). \end{aligned}$$

Going the other way around, a decomposition of known tensor fields on TM may lead to the discovery of interesting tensor fields along τ. Take for example the Nijenhuis tensor of U^H. Acting with N_{U^H} on say X^H and Y^V, the resulting vector field on TM has a unique decomposition into a horizontal and a vertical part and reveals this way two tensor fields N^H_U and N^V_U along τ. To be specific, we have

$$N_{U^H}(X^H, Y^V) = (N^H_U(X,Y))^V - (N^V_U(X,Y))^H.$$

So far, none of the above considerations has anything to do with second-order differential equations. One aspect of the presence of a Sode is that it makes other lifting procedures available and here we will in fact close the circle with the considerations of the introduction. Observe first that elements of $\bigwedge(\tau)$ and $V(\tau)$ can be prolonged to elements of $\bigwedge(\tau_{21})$ and $V(\tau_{21})$, where τ_{21} is the projection of T^2M onto TM. On the other hand, a Sode Γ can be regarded as a section $\gamma : TM \to T^2M$ of this bundle. Composition of these two operations gives rise to maps $I_\Gamma : \bigwedge(\tau) \to \bigwedge(TM)$ and $J_\Gamma : V(\tau) \to V(TM)$. The diagram below illustrates the construction for vector fields

$$\begin{tikzcd} & T(TM) \arrow[d, "\tau_*"] \\ T^2M \arrow[ur, "X^1"] \arrow[r, shift left, "\tau_{21}"] & TM \arrow[l, shift left, "\gamma"] \end{tikzcd} \qquad X^1 \circ \gamma \in \mathcal{X}(TM)$$

and in coordinates, we have $J_\Gamma X = X^1 \circ \gamma = X^i \partial/\partial q^i + \Gamma(X^i)\partial/\partial v^i$. This is essentially the mechanism by which the theory referred to at the beginning [?] is fully recovered in a much more complete way.

The most important aspect of a given SODE is that the availability of a connection is no longer an extra assumption, as Γ induces one, defined by the following horizontal lift construction (see *e.g.* [?]): $X \in \mathcal{X}(M) \mapsto X^H = \frac{1}{2}(X^c + [X^V, \Gamma])$, where X^c stands for the complete lift of X. If the coordinate expression of Γ reads $\Gamma = v^i\,\partial/\partial q^i + f^i(q,v)\,\partial/\partial v^i$, the coefficients of the associated connection are given by: $\Gamma^i_j = -\frac{1}{2}\partial f^i/\partial v^j$. In virtually every construction one can think of now, two operations will keep showing up, which therefore must reflect important concepts, carrying so to speak most of the information of the dynamics under consideration. For example, for $X \in \mathcal{X}(\tau)$, $\alpha \in \bigwedge^1(\tau)$, $J_\Gamma X$ and $I_\Gamma \alpha$ turn out to have the following decomposition: $J_\Gamma X = X^H + (\nabla X)^V$, $I_\Gamma \alpha = (\nabla \alpha)^H + \alpha^V$. ∇ is a self-dual derivation of degree 0, which we call the *dynamical covariant derivative*. In coordinates: $\nabla F = \Gamma(F)$, for $F \in C^\infty(TM)$ and

$$\nabla\left(\frac{\partial}{\partial q^i}\right) = \Gamma^k_i \,\frac{\partial}{\partial q^k}\,, \qquad \nabla(dq^i) = -\Gamma^i_k\, dq^k.$$

Likewise, we have $\mathcal{L}_\Gamma X^H = (\nabla X)^H + \Phi(X)^V$, where this time an element $\Phi \in V^1(\tau)$ presents itself, called the *Jacobi endomorphism* and having the following matrix coefficients,

$$\Phi^i_j = -\frac{\partial f^i}{\partial q^j} - \Gamma^k_j \Gamma^i_k - \Gamma(\Gamma^i_j).$$

We further limit ourselves here to listing just a few interesting properties of these operations. We have $d^V\Phi = 3R$, $d^H\Phi = \nabla R$, and the decomposition of ∇ reads:

$$\nabla = -i_{d^H\mathbf{T}} + d^V_{\nabla\mathbf{T}} + d^H_{\mathbf{T}} - a_{d^H\mathbf{T}}.$$

Here, $\mathbf{T} = v^i\,\partial/\partial q^i$ is the canonical vector field along τ and $\mathbf{t} = -d^H\mathbf{T}$ is the *tension* of the connection (cf. [?], or [?]), with the following coordinate expression:

$$\mathbf{t} = \left(\Gamma^j_i - v^k \frac{\partial \Gamma^j_i}{\partial v^k}\right) dq^i \otimes \frac{\partial}{\partial q^j}.$$

4. Time-dependent systems

A time-dependent second-order equation can be regarded as a vector field living on the manifold $\mathbb{R} \times TM$, *i.e.* the first jet space of the bundle $\mathbb{R} \times M \to \mathbb{R}$. Again, there is a natural way of associating with such a SODE a connection on π, implicitly defined by the following horizontal lift constuction: $X \in \mathcal{X}(\mathbb{R} \times M) \mapsto X^H = \frac{1}{2}(X^{(1)} + [X^V, \Gamma] + \langle X, dt\rangle\Gamma)$, where $X^{(1)}$ stands for the prolongation of X. Clearly then, the general calculus to be developed first is the theory of derivations of forms along π. It is no surprise that such an extension of the preceding theory poses a number of technical complications. More important, however, is the fact that there are problems of conceptual nature, in the sense that there does not seem to be a unique "natural" way of constructing this generalization: certain selections have to be made and these, inevitably, are sometimes a matter of personal taste. We refer to forthcoming publications for a full account of the theory. For the present survey, rather than repeating the story for this new situation, we will spend some time commenting on just a few reasons why the extension is not an entirely straightforward matter.

To begin with, recall that our d^V for the autonomous theory was modelled on d_S and as such vanishes on the subring $C^\infty(M)$ of $C^\infty(TM)$. The first potential branching point in the non-autonomous case comes from the fact that different (1,1) tensors of "vertical endomorphism"-type are available on $\mathbb{R} \times TM$. Each of the following tensor fields could be used to model d^V on forms along π:

$$S = \theta^i \otimes \frac{\partial}{\partial v^i}, \qquad \text{with} \quad \theta^i = dq^i - v^i dt,$$

$$S_a = dq^i \otimes \frac{\partial}{\partial v^i}, \qquad \tilde{S} = S_a + dt \otimes \frac{\partial}{\partial t}.$$

The latter one, for example, characterizes a so-called almost stable tangent structure (see [**?**, **?**]). To complicate matters, there are other ways of reasoning, in which d^V need not directly correspond to some tensor field on $\mathbb{R} \times TM$, but merely has to be a derivation relative to the subring $C^\infty(\mathbb{R} \times M)$ (and d^H then likewise could be selected in any suitable way to be a derivation relative to the subring $C^\infty(\mathbb{R})$). Our choice has been to preserve the link with some (1,1) tensor and to give preference to the tensor field S for that purpose. As a result, d^V is fully determined by the following properties:

$$\begin{aligned} d^V F &= \frac{\partial F}{\partial v^i}\theta^i, \qquad \forall F \in C^\infty(\mathbb{R} \times TM) \\ d^V(dt) &= 0, \qquad d^V\theta^i = dt \wedge \theta^i. \end{aligned}$$

Whereas the preference for S is of course justified by the jet bundle structure of $\mathbb{R}\times TM$, one subsequently has to live with a few perhaps rather unpleasant features of this choice, such as the fact that $d^V \circ d^V \neq 0$ and that the commutator of two $d^V_\star$-derivations is not of type $d^V_\star$.

Concerning the selection problem for d^H, assume we have a connection on $\pi : \mathbb{R} \times TM \to \mathbb{R} \times M$ at our disposal, *i.e.* a splitting of the sequence $0 \to V(\mathbb{R} \times TM) \to T(\mathbb{R} \times TM) \to \pi^*\big(T(\mathbb{R} \times M)\big) \to 0$. Such a connection provides us with two lots of connection coefficients Γ^j_0 and Γ^j_i, whereby

$$\begin{aligned} \left(\frac{\partial}{\partial q^i}\right)^H &= \frac{\partial}{\partial q^i} - \Gamma^j_i \frac{\partial}{\partial v^j} = H_i \\ \left(\frac{\partial}{\partial t}\right)^H &= \frac{\partial}{\partial t} - \Gamma^j_0 \frac{\partial}{\partial v^j} = H_0. \end{aligned}$$

One can now use the full projector $P_H = dq^i \otimes H_i + dt \otimes H_0$ for inducing a d^H-derivation on $\bigwedge(\pi)$, or use instead the two parts of its natural decomposition

$$P_H = \theta^i \otimes H_i + dt \otimes \mathbf{T}^H, \quad \mathbf{T} = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial q^i} \in \mathcal{X}(\pi),$$

to define two separate derivations (cf. the discussion about 'horizontal' versus 'strong horizontal' lifts in [**?**]). We choose for the first option and accordingly define d^H by

$$d^H F(X) = X^H(F), \;\; F \in C^\infty(\mathbb{R} \times TM), \; X \in \mathcal{X}(\pi)$$

$$d^H(dt) = 0, \qquad d^H \theta^i = \Gamma^i_k \, \theta^k \wedge dt.$$

As a remark, observe that every vector-valued form L along π has a decomposition in the form

$$L = \overline{L} + L^0 \otimes \mathbf{T}, \quad \text{with} \quad L^0 = i_L \, dt.$$

In particular, we have for the identity tensor

$$I = \overline{I} + dt \otimes \mathbf{T}, \qquad \overline{I} = \theta^i \otimes \frac{\partial}{\partial q^i},$$

from which it follows that

$$d^V = d^V_{dt \otimes \mathbf{T}} + d^V_{\overline{I}}, \qquad d^H = d^H_{dt \otimes \mathbf{T}} + d^H_{\overline{I}}.$$

It is to be expected that $d^V_{\overline{I}}$ and $d^H_{\overline{I}}$ may well be more important than the full d^V and d^H in applications. As a matter of fact, $d^V_{\overline{I}}$ for example, is to a large extent the nice vertical derivative, analogous to the one in the autonomous theory and enjoying the property $d^V_{\overline{I}} \circ d^V_{\overline{I}} = 0$.

Needless to say, there is again some freedom when it comes to extending the basic derivations from scalar to vector-valued forms, although this freedom is merely a question of 'fixing the time-component' and, generally speaking, a most natural approach here readily presents itself. For example: for $X \in \mathcal{X}(\pi)$, we define $d^H X$ as before by

$$(d^H X(Z))^V = P_V([Z^H, X^V]),$$

requiring in addition that

$$i_{d^H X} dt = d^H(i_X dt).$$

Note that this construction implies that $d^H \mathbf{T} = 0$ here.

In the end, whatever choices have been made underway, *all results* of the autonomous theory will have some kind of analogue in the time-dependent case. For example, the classification theorem in our approach reads as follows.

Theorem. *Every derivation D of $V(\pi)$ has a decomposition in the form*

$$D = i_{L_1} + d^V_{L_2} + d^H_{L_3} + a_Q, \qquad \text{with } \; i_{L_2} dt = 0.$$

Let us finally list just a few items related to the case of interest, where the connection is the one coming from a given Sode Γ. With Γ this time of the form

$$\Gamma = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial q^i} + f^i(t, q, v) \frac{\partial}{\partial v^i},$$

we have

$$\Gamma^j_i = -\frac{1}{2}\frac{\partial f^j}{\partial v^i}, \qquad \Gamma^j_0 = -f^j - v^i \Gamma^j_i.$$

It is worth observing that Γ is horizontal, in fact we have $\Gamma = \mathbf{T}^H = J_\Gamma \mathbf{T}$. The *dynamical covariant derivative* ∇ is defined as before, with the additional specification that $i_{\nabla X} dt = \Gamma(i_X dt)$, from which it actually follows that $\nabla = d^H_{\mathbf{T}}$ and $\nabla \mathbf{T} = 0$. Similarly, the *Jacobi endomorphism* Φ is again determined by $\mathcal{L}_\Gamma X^H = (\nabla X)^H + \Phi(X)^V$, together with the supplementary requirement $i_\Phi dt = 0$. In coordinates, Φ thus becomes $\Phi = \Phi^i_j\, \theta^j \otimes (\partial/\partial q^i)$, with Φ^i_j as before.

5. Applications

For simplicity, we will go back for most of this section to the autonomous case, but all results have corresponding versions in the non-autonomous case.

Consider first the so-called *inverse problem of Lagrangian mechanics*. The conditions for a multiplier matrix to turn the given second-order equations into the form of Euler-Lagrange equations are known as the *Helmholtz conditions*. We take this problem just to illustrate how using the tools which have been described above, one is formulating questions in intrinsic geometrical terms, while the resulting conditions at the same time are as close as possible to the analytical equations which must be tackled for local solutions. A most economical formulation of the Helmholtz conditions thus reads: find a (non-degenerate) symmetric (0,2) tensor field g along τ, such that $\nabla g = 0$, $\Phi \lrcorner g$ is symmetric and $\mathrm{D}^V_X g$ is symmetric for all X (see [**?**] and [**?**]). Under these conditions, the g_{ij} will locally be of the form $g_{ij} = \partial^2 L/\partial v^i \partial v^j$ for some L and the Kähler lift g^K becomes the Cartan 2-form.

In a forthcoming publication, Martínez will present a new geometrical characterization of *linearizable second-order equations*. Briefly, his results can be summarized as follows. It can be shown that the horizontal and vertical covariant derivatives induce a linear connection on the pull back bundle $\tau^*\tau : \tau^*(TM) \to TM$ in the following way. For $X, Y \in \mathcal{X}(\tau)$, putting $Z = X^H + Y^V \in \mathcal{X}(TM)$, $\tilde{D}_Z = \mathrm{D}^H_X + \mathrm{D}^V_Y$ is a linear connection in the sense of a derivation of the set of sections of the bundle $\tau^*\tau$. In the case of a Sode connection, it turns out that linearizability of Γ is equivalent to this induced linear connection being flat, together with the condition that the Jacobi endomorphism Φ is parallel.

The most comprehensive application so far concerns the characterization of *complete separability* of a Sode Γ. A full account of this theory can be found in [**?**]. To be able to state the main theorem of that paper, we need one extra concept: for a general $U \in V^1(\tau)$, define a type (1,2) tensor field C^V_U (not skew-symmetric) by

$$C^V_U(X,Y) = [\mathrm{D}^V_X U, U](Y), \qquad \forall X, Y \in \mathcal{X}(\tau).$$

Theorem. *There exist coordinates in which a given* Sode Γ *separates into* n *decoupled single second-order equations if and only if the following conditions hold:*

Φ *is diagonalizable*, $R = 0$
$[\nabla\Phi, \Phi] = 0$, $C^V_\Phi = 0$
$\mathbf{t}$ *is diagonalizable*, $C^V_{\mathbf{t}} = 0$.

Most importantly, this certainly non-trivial result is also effectively applicable in practice. In the first place, all conditions of the theorem are algebraic and can, for given Γ, in principle be verified. Doing all this algebra of course can be a horrendous task, but computer algebra packages offer a great relief in that respect. Secondly, the proof of the theorem is constructive. Hence, once the requirements are satisfied, the proof will actually indicate how coordinates in which the system decouples must be obtained.

For an illustration of the applicability of this theory, let's take a fairly simple example. Consider the following system of differential equations:

$$\begin{aligned} \ddot{q}_1 &= -c(t)q_1 + b(t){q_1}^2 - a(t){q_2}^2 + k(t)\dot{q}_1 + \ell_1(t)\dot{q}_2 \\ \ddot{q}_2 &= -c(t)q_2 + 2b(t)q_1q_2 + k(t)\dot{q}_2 + \ell_2(t)\dot{q}_1. \end{aligned}$$

This is of course a time-dependent system so that the theorem to be applied is slightly different from the one stated above. The explicit functions of time are as yet arbitrary and we insist that the non-linear terms should not vanish, *i.e.* a and b are not zero. The separability analysis then tells us that decoupling is possible if and only if we have $a = -\omega^2 b$ and $\ell_1 = \omega^2\ell_2$ for some constant ω. The coordinate transformation which will do the job reads: $Q_1 = q_1 - \omega\, q_2$, $Q_2 = q_1 + \omega\, q_2$.

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Willy Sarlet, Ann Vandecasteele
Instituut voor Theoretische Mechanica
Universiteit Gent
Krijgslaan 281 S9
B-9000 Gent, Belgium

Eduardo Martínez
Departamento de Matemática Aplicada
Universidad de Zaragoza
Avda. María de Luna 3
E-50015 Zaragoza, Spain

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