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THE INVERSION OF WHITEHEAD'S THEOREM ON CONVEX REGIONS [1]

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Abstract. In 1932 J. H. C. Whitehead [2] proved the existence of convex neighbourhoods with respect to the geodesic arcs corresponding to a given connection on a manifold. The present paper gives necessary and sufficient conditions for a set of paths to be the set of all geodesics of some connection. The main condition is the existence of convex neighbourhoods in a more general sense.
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Fundamental notions: We suppose that all manifolds and mappings used throughout this paper are smooth. We consider a closed one-dimensional interval $\langle 0, 1\rangle \subset R$ and a n-dimensional manifold X. The derivatives with respect to the global coordinate on $\langle 0, 1\rangle$ are denoted by dots. On X we use local coordinate systems

$$\varphi : V \to \varphi(V) \subset R^n, V \subset X, \tag{1}$$

with coordinates Q, q or x, where for example $Q = (Q^1, Q^2, \dots, Q^n) = (Q^k) \in \varphi(V) \subset R^n$, $k = 1, 2, \dots, n$. On the tangent bundle TX with the natural projection $\pi : TX \to X$ we use local coordinate systems

$$T\varphi : TV \to T\varphi(TV) = \varphi(V) \times R^n \tag{2}$$

associated with (1) and denote the corresponding coordinates by $(x, v) \in \varphi(V) \times R^n$. Everywhere we use the standard summation convention with the summation from 1 to n. The most important notions, which are considered in this work, are described in the following two paragraphs.

A path γ on X is a mapping $\gamma : \langle 0, 1\rangle \to X$. Its coordinate expression is denoted by $x(t)$ or $q(t)$. Let P be a set of all polynomial mappings $\langle 0, 1\rangle \to \langle 0, 1\rangle$ whose degree is less than two, that is, the mappings of the form $\theta(\tau) = (1 - \tau)T + \tau t$, where $T, t \in \langle 0, 1\rangle$. We say that a path $\gamma \circ \theta$, where $\theta \in P$, is *a subpath* of the path γ. We understand *a linear connection* Γ on X in a usual way given for example in [1]. Notice that a linear connection Γ on X exists if and only if there is, for each point $c \in X$, a

[1] This paper is in final form and no version of it will be submitted for publication elsewhere.

local coordinate system (1), $c \in V$, and functions $\Gamma^k_{lm}(x)$ on $\varphi(V)$ satisfying, for every coordinate transformation $\bar\varphi \circ \varphi^{-1}\big|_{\varphi(\bar V \cap V)}$, the relations

$$\frac{\partial^2 \bar x^k}{\partial x^l \partial x^m} + \bar\Gamma^k_{rs}(\bar x)\frac{\partial \bar x^r}{\partial x^l}\frac{\partial \bar x^s}{\partial x^m} = \frac{\partial \bar x^k}{\partial x^r}\Gamma^r_{lm}(x). \tag{3}$$

A geodesic of a linear connection Γ on X is a path each coordinate expression of which satisfies the differential equations

$$\ddot x^k(t) + \Gamma^k_{lm}(x(t))\dot x^l(t)\dot x^m(t) = 0. \tag{4}$$

Let $W(X)$ be a set of paths on X. We say that $W(X)$ is *closed on subpaths* if and only if $\gamma \in W(X)$, $\theta \in P$ implies $\gamma \circ \theta \in W(X)$. We say that an open set $U \subset X$ is *convex with respect to* $W(X)$ if and only if

a) for each two points $a, b \in U$ there exists a unique path $\gamma_{ab} \in W(X)$ such that $\gamma_{ab}(0) = a, \gamma_{ab}(1) = b, \gamma_{ab}(\langle 0,1\rangle) \subset U$,

b) the mapping $\rho : U \times U \times \langle 0,1\rangle \ni (a,b,\tau) \to \gamma_{ab}(\tau) = \rho(a,b,\tau) \in U$ is smooth.

We say that X is *locally convex with respect to* $W(X)$ if and only if for each $c \in X$ there exists an open set U such that $c \in U \subset X$, U is convex with respect to $W(X)$. We say that $W(X)$ is *a maximal set of paths on X satisfying the conditions* c_1, c_2 if and only if for every set $\bar W(X) \supset W(X)$ of paths on X satisfying the conditions c_1, c_2 the relation $\bar W(X) = W(X)$ holds.

Theorem: Let $W(X)$ be a set of paths on a manifold X. The following two assertions are equivalent:

1) There is a linear connection Γ on X such that $W(X)$ is a set of all its geodesics,
2) $W(X)$ is a maximal set of paths on X satisfying the conditions:
   a) X is locally convex with respect to $W(X)$,
   b) $W(X)$ is closed on subpaths.

Lemma 1 (Whitehead): Let $W(X)$ be a set of all geodesics of some linear connection Γ on a manifold X. Then X is locally convex with respect to $W(X)$.

Proof[2] of Lemma 1: Considering (3) we can define a vector field η on TX by the chart expression

$$(TT\varphi) \circ \eta \circ (T\varphi)^{-1} = v^k \frac{\partial}{\partial x^k} - \Gamma^k_{lm}(x) v^l v^m \frac{\partial}{\partial v^k} \tag{5}$$

in each of the local coordinate systems (2). Let $\exp(\tau\eta)$ be the local flow of this vector field, $h \in \omega$, where ω is the zero section of π. From (5) we obtain $\exp(\tau\eta)(h) = h$. So the local flow $\exp(\tau\eta)$ is defined on ω for each $\tau \in R$. Then there is an open set $\bar M$, $\omega \subset \bar M \subset TX$, such that the flow $\exp(\tau\eta) : \bar M \to TX$ is defined for each $\tau \in \langle 0,1\rangle$.

Let us consider a point $c \in X$ and a coordinate system (1) such that $c \in V$, $\varphi(c) = 0$. A matrix, whose components are $A_{kl}(x) = g_{kl} - \Gamma^m_{kl}(x) g_{mr} x^r$, where $g_{kl} = 1$ for $k = l$ and $g_{kl} = 0$ for $k \neq l$, is positive definite for $x = 0$. Hence there is an open set $\bar U$, $c \in \bar U \subset V$, such that the matrix, whose components are $A_{kl}(\varphi(a))$, is positive definite for each point $a \in \bar U$.

[2]The proof of Lemma 1 is merely a simplification of the proof of Theorem 8.7 given in [1].

Let us consider a function $\chi : \varphi(V) \ni x \to g_{kl}x^k x^l \in \langle 0, \infty)$. From (5) the mapping $\bar{\sigma} : \bar{M} \ni m \to (\pi(m), \pi \circ \exp(\eta)(m)) \in X \times X$ is regular at a point $h \in \omega$, $\pi(h) = c$, and $\pi \circ \exp(\tau\eta)(h) = c$ for each $\tau \in \langle 0, 1\rangle$. Hence there are a real number $\delta > 0$ and an open set $M \subset X$, $h \in M \subset \bar{M}$, such that

a) $\varphi(U)$, $c \in U = \{a \in \bar{U} | \chi \circ \varphi(a) < \delta\} \subset X$, is an open ball on $\varphi(V)$,

b) $\sigma : M \ni m \to (\pi(m), \pi \circ \exp(\eta)(m)) \in U \times U$ is a diffeomorphism,

c) $\pi \circ \exp(\tau\eta)(M) \subset \bar{U}$ for each $\tau \in \langle 0, 1\rangle$.

From (4) and (5) we have

d) the mapping $\gamma_{ab} : \langle 0, 1\rangle \to X$, defined by

$$\gamma_{ab}(\tau) = \pi \circ \exp(\tau\eta) \circ \sigma^{-1}(a, b), \tag{6}$$

satisfies the conditions $\gamma_{ab} \in W(X)$, $\gamma_{ab}(0) = a$, $\gamma_{ab}(1) = b$,

e) $\ddot{F}(\tau) = 2A_{kl}(x(\tau))\dot{x}^k(\tau)\dot{x}^l(\tau) \geq 0$, where $F = \chi \circ \varphi \circ \gamma_{ab}$, $x(\tau) = \varphi \circ \gamma_{ab}(\tau)$, which implies $F(\tau) \leq (1-\tau)F(0) + \tau F(1) < (1-\tau)\delta + \tau\delta = \delta$ and so $\gamma_{ab}(\langle 0, 1\rangle) \subset U$.

Because σ is a diffeomorphism, for each two points $a, b \in U$ there exists the unique path (6) such that $\gamma_{ab} \in W(X), \gamma_{ab}(0) = a, \gamma_{ab}(1) = b, \gamma_{ab}(\langle 0, 1\rangle) \subset U$, and the mapping $\rho : U \times U \times \langle 0, 1\rangle \ni (a, b, \tau) \to \gamma_{ab}(\tau) = \rho(a, b, \tau) \in U$ is smooth. Consequently the open set $U \subset X$ is convex with respect to $W(X)$. In this way for each $c \in X$ we obtain an open set U such that $c \in U \subset X$ and U is convex with respect to $W(X)$. This completes the proof of Lemma 1.

Lemma 2: Let $W(X)$ be a set of all geodesics of some linear connection Γ on a manifold X. Then $W(X)$ is closed on subpaths.

Proof of Lemma 2: Let $\gamma \in W(X)$. Since the coordinate expression $q(t)$ of γ satisfies the differential equations (4), the coordinate expression $x(t)$ of $\gamma \circ \theta$, $\theta \in P$, satisfies, by $\theta(t) = (1-t)\theta(0) + t\theta(1)$, $x(t) = q(\theta(t))$, $\dot{x}(t) = \dot{q}(\theta(t))(\theta(1) - \theta(0))$, $\ddot{x}(t) = \ddot{q}(\theta(t))(\theta(1) - \theta(0))^2$, the differential equations (4) too, and so $\gamma \circ \theta \in W(X)$. Therefore, $\gamma \in W(X), \theta \in P$ implies $\gamma \circ \theta \in W(X)$ and the proof of Lemma 2 is finished.

Lemma 3: Let $W(X)$ be a set of paths on X satisfying the conditions a), b) of the Theorem. Then there is a linear connection Γ on X such that $W(X)$ is a set of its geodesics.

Proof of Lemma 3: Let an open set $U \subset X$ be convex with respect to $W(X)$, $a, b \in U$. Since $W(X)$ is closed on subpaths, it holds that $\gamma_{ab}(1-\tau) = \gamma_{ba}(\tau)$, $\gamma_{ab}(0) = \gamma_{ab}(0\tau) = \gamma_{aa}(\tau)$. From this and the definition of convexity, the smooth mapping $\rho(a, b, \tau) = \gamma_{ab}(\tau)$ satisfies

$$\rho(a, b, 1-\tau) = \rho(b, a, \tau), \tag{7}$$

$$\rho(a, a, \tau) = a. \tag{8}$$

Let $c \in X$. From the condition a) of the Theorem there exist a coordinate system (1) and an open convex with respect to $W(X)$ set $U \subset X$ so that $c \in V \subset U$. Let D be a diagonal of $V \times V$. From (8) there is an open set N, $D \subset N \subset V \times V$, such that $\rho(N \times \langle 0, 1\rangle) \subset V$. We denote $(\rho^k(Q, q, \tau)) = \varphi \circ \rho \circ \psi^{-1}(Q, q, \tau)$, where

$$\psi : N \times \langle 0, 1\rangle \ni (a, b, \tau) \to (\varphi(a), \varphi(b), \tau) \in \psi(N \times \langle 0, 1\rangle) \subset R^{2n+1}.$$

From (7) and (8) by considering

$$\left(\frac{\partial \rho^k(Q, q, \frac{1}{2})}{\partial Q^l} + \frac{\partial \rho^k(Q, q, \frac{1}{2})}{\partial q^l}\right)\Bigg|_{Q=q=x} = \frac{\partial(\rho^k(Q, q, \frac{1}{2})|_{Q=q=x})}{\partial x^l}$$

it follows that

$$\left.\frac{\partial\rho^k(Q,q,\frac{1}{2})}{\partial Q^l}\right|_{Q=q=x} = \left.\frac{\partial\rho^k(Q,q,\frac{1}{2})}{\partial q^l}\right|_{Q=q=x} = \frac{1}{2}\delta^k_l, \tag{9}$$

where δ^k_l is the Kronecker symbol. So for each point $c \in X$ there is a local coordinate system (1), $c \in V$, and functions

$$\Gamma^k_{lm}(x) = -4\left.\frac{\partial^2\rho^k(Q,q,\frac{1}{2})}{\partial Q^l \partial q^m}\right|_{Q=q=x} \tag{10}$$

on $\varphi(V)$ satisfying, by (9) and (10), the relations (3). Therefore, the linear connection Γ on X, given by the relations (10), exists.

Let $\gamma \in W(X)$, $\tau_0 \in \langle 0,1\rangle$. Then there are a closed interval $I = \langle\tau_1,\tau_2\rangle$, $\tau_1 < \tau_2$, a coordinate system (1) and an open convex with respect to $W(X)$ set $U \subset X$ such that $\tau_0 \in I \subset \langle 0,1\rangle$, $\gamma(I) \subset V \subset U \subset X$. From the condition b) of the Theorem we have $\gamma\circ\theta \in W(X)$, $\gamma\circ\theta(\langle 0,1\rangle) \subset U$, $\gamma\circ\theta(0) = \gamma(T)$, $\gamma\circ\theta(1) = \gamma(t)$, where $\theta(\tau) = (1-\tau)T + \tau t$, $T, t \in I$. From the condition a) of the Theorem we get $\gamma\circ\theta(\tau) = \rho(\gamma(T),\gamma(t),\tau)$. The coordinate expression of this equation is

$$x^k((1-\tau)T + \tau t) = \rho^k(x(T), x(t), \tau). \tag{11}$$

From (10) by differentiating (11) with respect to T, t and putting $T = t$, $\tau = \frac{1}{2}$ we obtain that $x(t)$ satisfies the differential equations (4). Therefore, each path $\gamma \in W(X)$ is a geodesic of the connection Γ on X. This completes the proof of Lemma 3.

Proof of the Theorem: Let 1) hold. Then from Lemma 1 and Lemma 2 $W(X)$ satisfies the conditions a), b). Every set $\bar{W}(X) \supset W(X)$ of paths on X satisfying the conditions a), b) satisfies, by Lemma 3, also the condition $\bar{W}(X) \subset W(X)$, and so $\bar{W}(X) = W(X)$. Therefore, $W(X)$ is a maximal set of paths on X satisfying conditions a), b), and 2) holds.

Conversely, suppose 2). Then from Lemma 3 there is a linear connection Γ on X such that $W(X)$ is a set of its geodesics. The set $\bar{W}(X) \supset W(X)$ of all geodesics of the linear connection Γ satisfies, from Lemma 1 and Lemma 2, the conditions a), b). From the maximality property $W(X) = \bar{W}(X)$ follows, and 1) holds. This completes the proof of the Theorem.

Remark: Let us suppose that the assertion 2) of the Theorem is satisfied. From the condition b) we have that the monoid of all mappings $\theta \in P$ acts on $W(X)$ from right in the following way $W(X) \ni \gamma \to \gamma\circ\theta \in W(X)$. The condition a) makes one possible to introduce a natural structure of a manifold on $W(X)$, with respect to which the mentioned monoid is the Lie monoid.

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