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Heat Content Asymptotics of Riemannian Manifolds

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Let M be a compact Riemannian manifold with smooth boundary ∂M, let V be a smooth vector bundle over M, let ∇ be a connection on V, and let $E \in C^\infty(\text{End}(V))$ be an auxiliary endomorphism of V. Let

$$D = -(g^{ij}\nabla_i\nabla_j + E)$$

be an operator of Laplace type on V. Let $S \in C^\infty(\text{End}(V))$ and let ∇_m be covariant differentiation with respect to the inward pointing unit normal. Let $\mathcal{B}_S^+ f = (\nabla_m^D f + Sf)|_{\partial M}$ and $\mathcal{B}^- f = f|_{\partial M}$ be Neumann and Dirichlet boundary conditions. Let $h(x,t) = e^{-tD_{\mathcal{B}}} f_1(x)$ for $\mathcal{B} \in \{\mathcal{B}_S^+, \mathcal{B}^-\}$ be the solution for $t > 0$ to the equations:

$$\partial_t h(x,t) = -Dh(x,t),\ \mathcal{B}h(x,t) = 0, \text{ and } \lim_{t\to 0^+} h(x,t) = f_1(x).$$

Denote the natural pairing between V and the dual bundle V^* by $\langle\cdot,\cdot\rangle$. Let dx and dy be the Riemannian measures on M and ∂M. To study the short time behavior of h, introduce an auxiliary smooth test function $f_2 \in C^\infty(V^*)$ and define:

$$\begin{aligned}\beta(f_1, f_2, D, B)(t) &= \int_M \langle e^{-tD_{\mathcal{B}}} f_1, f_2\rangle dx\\ &\simeq \sum_{n=0}^{\infty} \beta_n(f_1, f_2, D, \mathcal{B}) t^{n/2}.\end{aligned}$$

There exist local invariants $\beta_n^{int}(f_1, f_2, D)$ and $\beta_n^{bd}(f_1, f_2, D, \mathcal{B})$ so

$$\beta_n(f_1, f_2, D, \mathcal{B}) = \int_M \beta_n^{int}(f_1, f_2, D)dx + \int_{\partial M} \beta_n^{bd}(f_1, f_2, D, \mathcal{B})dy.$$

Let indices a, b range from 1 through $m-1$ and index a local orthonormal frame for the tangent bundle of the boundary; adopt the Einstein convention and sum over repeated indices. Let R_{ijkl} be the curvature tensor of the Levi-Civita connection ∇^M of M; with our sign convention $R_{1212} = -1$ for the standard sphere in $\mathbf{R}^3$. Let $\tau = R_{ijji}$ and $\rho_{ij} = R_{ikkj}$ be the scalar curvature and the Ricci tensor. Let ";" and ":" be multiple covariant differentiation with respect to ∇^M and $\nabla^{\partial M}$. When sections of bundles built from V are involved, ";" and ":" will mean

$$\nabla^M \otimes 1 + 1 \otimes \nabla^D \text{ and } \nabla^{\partial M} \otimes 1 + 1 \otimes \nabla^D.$$

Let $L_{ab} = (e_m, \nabla^M_{e_a} e_b)$ be the second fundamental form and let $\tilde{D}$ be the formal adjoint on $C^\infty(V^*)$; let $\tilde{\mathcal{B}}$ be the adjoint boundary condition.

Theorem 1: (see [3, 5])

(a) $\beta_0(f_1, f_2, D, \mathcal{B}^-) = \int_M \langle f_1, f_2\rangle dx$.

(b) $\beta_1(f_1, f_2, D, \mathcal{B}^-) = -2\pi^{-1/2} \int_{\partial M} \langle f_1, f_2\rangle dy$.

(c) $\beta_2(f_1, f_2, D, \mathcal{B}^-) = -\int_M \langle Df_1, f_2\rangle dx + \int_{\partial M} \{\langle \frac{1}{2} L_{aa} f_1, f_2\rangle - \langle f_1, f_{2;m}\rangle\} dy$.

(d) $\beta_3(f_1, f_2, D, \mathcal{B}^-) = -2\pi^{-1/2} \int_{\partial M} \{\frac{2}{3}\langle f_{1;mm}, f_2\rangle + \frac{2}{3}\langle f_1, f_{2;mm}\rangle - \langle f_{1:a}, f_{2:a}\rangle$
$+\langle Ef_1, f_2\rangle - \frac{2}{3} L_{aa}\langle f_{1;m}, f_2\rangle - \frac{2}{3} L_{aa}\langle f_1, f_{2;m}\rangle + \langle(\frac{1}{12} L_{aa}L_{bb} - \frac{1}{6} L_{ab}L_{ab}$
$+\frac{1}{6} R_{amam}) f_1, f_2\rangle\} dy$.

(e) $\beta_4(f_1, f_2, D, \mathcal{B}^-) = \frac{1}{2}\int_M \langle Df_1, \tilde{D}f_2\rangle + \int_{\partial M} \{\frac{1}{2}\langle (Df_1)_{;m}, f_2\rangle + \frac{1}{2}\langle f_1, (\tilde{D}f_2)_{;m}\rangle$
$-\frac{1}{4}\langle L_{aa} Df_1, f_2\rangle - \frac{1}{4}\langle L_{aa} f_1, \tilde{D}f_2\rangle + \langle(\frac{1}{8} E_{;m} - \frac{1}{16} L_{ab}L_{ab}L_{cc} + \frac{1}{8} L_{ab}L_{ac}L_{bc}$
$-\frac{1}{16} R_{ambm}L_{ab} + \frac{1}{16} R_{abcb}L_{ac} + \frac{1}{32}\tau_{;m} + \frac{1}{16} L_{ab:ab}) f_1, f_2\rangle - \frac{1}{4} L_{ab}\langle f_{1:a}, f_{2:b}\rangle$
$-\frac{1}{8}\langle \Omega_{am} f_{1:a}, f_2\rangle + \frac{1}{8}\langle \Omega_{am} f_1, f_{2:a}\rangle\} dy$.

Theorem 2: (see [3, 7])

(a) $\beta_0(f_1, f_2, D, \mathcal{B}_S^+) = \int_M \langle f_1, f_2\rangle dx$.

(b) $\beta_1(f_1, f_2, D, \mathcal{B}_S^+) = 0$.

(c) $\beta_2(f_1, f_2, D, \mathcal{B}_S^+) = -\int_M \langle Df_1, f_2\rangle dx + \int_{\partial M} \langle \mathcal{B}_S^+ f_1, f_2\rangle dy$.

(d) $\beta_3(f_1, f_2, D, \mathcal{B}_S^+) = \frac{2}{3}\cdot 2\pi^{-1/2} \int_{\partial M} \langle \mathcal{B}_S^+ f_1, \tilde{\mathcal{B}}_S^+ f_2\rangle dy$.

(e) $\beta_4(f_1, f_2, D, \mathcal{B}_S^+) = \frac{1}{2}\int_M \langle Df_1, \tilde{D}f_2\rangle dx + \int_{\partial M} \{-\frac{1}{2}\langle \mathcal{B}_S^+ f_1, \tilde{D}f_2\rangle$
$-\frac{1}{2}\langle Df_1, \tilde{\mathcal{B}}_S^+ f_2\rangle + \langle(\frac{1}{2} S + \frac{1}{4} L_{aa})\mathcal{B}_S^+ f_1, \tilde{\mathcal{B}}_S^+ f_2\rangle dy$.

(f) $\beta_5(f_1, f_2, D, \mathcal{B}_S^+) = 2\pi^{-1/2} \int_{\partial M} \{-\frac{4}{15}\langle \mathcal{B}_S^+ Df_1, \tilde{\mathcal{B}}_S^+ f_2\rangle - \frac{4}{15}\langle \mathcal{B}_S^+ f_1, \tilde{\mathcal{B}}_S^+ \tilde{D}f_2\rangle$
$-\frac{2}{15}\langle (\mathcal{B}_S^+ f_1)_{:a}, (\tilde{\mathcal{B}}_S^+ f_2)_{:a}\rangle + \langle(\frac{2}{15} E + \frac{4}{15} S^2 + \frac{4}{15} S L_{aa} + \frac{1}{30} L_{aa}L_{bb} + \frac{1}{15} L_{ab}L_{ab}$
$-\frac{1}{15} R_{amam})\mathcal{B}_S^+ f_1, \tilde{\mathcal{B}}_S^+ f_2\rangle\} dy$.

(g) $\beta_6(f_1, f_2, D, \mathcal{B}_S^+) = -\frac{1}{6}\int_M \langle D^2 f_1, \tilde{D}f_2\rangle + \int_{\partial M} \{\frac{1}{6}\langle \mathcal{B}_S^+ Df_1, \tilde{D}f_2\rangle$
$+\frac{1}{6}\langle D^2 f_1, \tilde{\mathcal{B}}_S^+ f_2\rangle + \frac{1}{6}\langle \mathcal{B}_S^+ f_1, \tilde{D}^2 f_2\rangle - \frac{1}{6}\langle S\mathcal{B}_S^+ Df_1, \tilde{\mathcal{B}}_S^+ f_2\rangle - \frac{1}{6}\langle S\mathcal{B}_S^+ f_1, \tilde{\mathcal{B}}_S^+ \tilde{D}f_2\rangle$
$-\frac{1}{12}\langle L_{aa}\mathcal{B}_S^+ Df_1, \tilde{\mathcal{B}}_S^+ f_2\rangle - \frac{1}{12}\langle L_{aa}\mathcal{B}_S^+ f_1, \tilde{\mathcal{B}}_S^+ \tilde{D}f_2\rangle + \langle(\frac{1}{24} E_{;m} + \frac{1}{12} E L_{aa}$
$+\frac{1}{48} L_{ab}L_{ab}L_{cc} + \frac{1}{24} L_{ab}L_{ac}L_{bc} - \frac{1}{48} R_{ambm}L_{ab} + \frac{1}{48} R_{abcb}L_{ac} - \frac{1}{24} R_{amam}L_{bb}$
$+\frac{1}{96}\tau_{;m} + \frac{1}{48} L_{ab:ab} + \frac{1}{12} S L_{aa}L_{bb} + \frac{1}{12} S L_{ab}L_{ab} - \frac{1}{12} S R_{amam}$
$+\frac{1}{12}(SE + ES) + \frac{1}{4} S^2 L_{aa} + \frac{1}{6} S^3 + \frac{1}{6} S_{:aa})\mathcal{B}_S^+ f_1, \tilde{\mathcal{B}}_S^+ f_2\rangle$
$-\frac{1}{12} L_{aa}\langle (\mathcal{B}_S^+ f_1)_{:b}, (\tilde{\mathcal{B}}_S^+ f_2)_{:b}\rangle - \frac{1}{12} L_{ab}\langle (\mathcal{B}_S^+ f_1)_{:a}, (\tilde{\mathcal{B}}_S^+ f_2)_{:b}\rangle$
$-\frac{1}{6}\langle S(\mathcal{B}_S^+ f_1)_{:a}, (\tilde{\mathcal{B}}_S^+ f_2)_{:a}\rangle - \frac{1}{24}\langle \Omega_{am}(\mathcal{B}_S^+ f_1)_{:a}, \tilde{\mathcal{B}}_S^+ f_2\rangle$
$+\frac{1}{24}\langle \Omega_{am}\mathcal{B}_S^+ f_1, (\tilde{\mathcal{B}}_S^+ f_2)_{:a}\rangle\} dy$.

One can mix Neumann and Dirichlet boundary conditions. Near ∂M, let $\chi \in C^\infty(\mathrm{End}(V))$ satisfy $\chi^2 = 1$ where χ is self-adjoint. Split $V = V^+ \oplus V^-$ into the relevant eigenspaces; the trivial splitting $V = V^+$ or $V = V^-$ is permitted. Let $\Pi^\pm$ be the associated projections on $V^{\pm}$. Let $S \in C^{\infty}(\mathrm{End}(V^{+}|_{\partial M}))$. Define mixed boundary conditions

$$\mathcal{B} = \mathcal{B}(\chi, S, \nabla^{D})f = \{\Pi^{+}(\nabla_{m}^{D} + S)\Pi^{+}f\}|_{\partial M} \oplus \Pi^{-}f|_{\partial M}.$$

Absolute and relative boundary conditions for the DeRham complex are such boundary conditions; these are important in index theory. Let $f^{\pm} = \Pi^{\pm}f$.

Theorem 3: (see [7])

(a) $\beta_0(f_1, f_2, D, \mathcal{B}) = \int_M \langle f_1, f_2\rangle dx$.

(b) $\beta_1(f_1, f_2, D, \mathcal{B}) = -2\pi^{-1/2}\int_{\partial M}\langle f_1^-, f_2^-\rangle dy$.

(c) $\beta_2(f_1, f_2, D, \mathcal{B}) = -\int_M\langle Df_1, f_2\rangle dx + \int_{\partial M}\{\langle f_{1;m}^{+} + Sf_1^{+}, f_2^{+}\rangle + \langle \frac{1}{2}L_{aa}f_1^-, f_2^-\rangle$
$-\langle f_1^-, f_{2;m}^-\rangle\}dy$.

(d) $\beta_3(f_1, f_2, D, \mathcal{B}) = 2\pi^{-1/2}\int_{\partial M}\{-\frac{2}{3}\langle f_{1;mm}^-, f_2^-\rangle - \frac{2}{3}\langle f_1^-, f_{2;mm}^-\rangle$
$+\frac{2}{3}L_{aa}\langle f_{1;m}^-, f_2^-\rangle + \frac{2}{3}L_{aa}\langle f_1^-, f_{2;m}^-\rangle + \langle(-\frac{1}{12}L_{aa}L_{bb} + \frac{1}{6}L_{ab}L_{ab}$
$-\frac{1}{6}R_{amam})f_1^-, f_2^-\rangle + \frac{2}{3}\langle f_{1;m}^{+} + Sf_1^{+}, f_{2;m}^{+} + Sf_2^{+}\rangle - \langle Ef_1^-, f_2^-\rangle$
$-\frac{2}{3}\langle Ef_1^-, f_2^{+}\rangle - \frac{2}{3}\langle Ef_1^{+}, f_2^-\rangle + \langle f_1^-{}_{:a}, f_2^-{}_{:a}\rangle + \frac{2}{3}\langle f_1^{+}{}_{:a}, f_2^-{}_{:a}\rangle$
$+\frac{2}{3}\langle f_1^-{}_{:a}, f_2^{+}{}_{:a}\rangle\}dy$.

The study of heat energy content asymptotics is relatively recent. If we set $f_1 = f_2 = 1$ and take $D = \Delta_0$ to be the scalar Laplacian, then

$$\beta(t) = \beta(1, 1, \Delta_0, \mathcal{B}^-)(t) \simeq \sum_{n=0}^{\infty}\beta_n(M)t^{n/2}$$

gives the asymptotic expansion of the total heat content of M with an initial temperature one. $\beta_0(M)$ and $\beta_1(M)$ for domains in Euclidean space were first computed by van den Berg and Davies [2]; $\beta_2(M)$ was computed for domains in Euclidean space by van den Berg and Le Gall [4]; $\beta_0(M)$, $\beta_1(M)$, and $\beta_2(M)$ were computed by van den Berg [1] for the upper hemisphere of a sphere; these results follow from Theorem 1. The case of polygonal domains in the plane was considered by van den Berg and Srisatkunarajah [6]; the analysis is quite different if the boundary is not smooth and these results do not follow from Theorem 1. We also refer to related work by Phillips and Jansons [9]. McAvity [8] first considered mixed boundary conditions and proved Theorem 3 using a modified DeWitt ansatz; we subsequently gave an independent derivation. Our results agree modulo notational changes.

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