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On zero-curvature representations of partial differential equations

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Abstract. This paper concerns the problem of finding all possible zero-curvature representations of a given system of partial differential equations. A new computational procedure is proposed, designed to give complete answers. The procedure is applicable to non-overdetermined systems when the number m of independent variables is two, while for $m > 3$ we prove a theorem of non-existence. The approach, inspired by A.M. Vinogradov's $\mathcal{C}$-spectral sequence, is based on finding representatives of gauge equivalence classes.

Keywords. Nonlinear partial differential equation, zero-curvature representation, linear pseudopotential, linear covering, AKNS formulation, diffiety, $\mathcal{C}$-spectral sequence, gauge complex, gauge equivalence, jet comonad.

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Introduction

Consider a system of partial differential equations

$$f^l(x^i, u^k, u^k_I) = 0, \tag{0.1}$$

where x^i, $i = 1, \dots, m$ are independent variables, u^k's are unknown functions, and $u^k_I = \partial^s u^k / \partial x^{i_1} \dots \partial x^{i_s}$ are their derivatives, $I = i_1 \dots i_s$ being a multiindex. Suppose further a matrix Lie algebra $\mathfrak{g}$ to be given. Our problem is to find all $\mathfrak{g}$-matrices A_i, $i = 1, \dots, m$, depending on the x^i, u^k, u^k_I's, such that conditions

$$\mathcal{D}_{x^j} A_i - \mathcal{D}_{x^i} A_j + [A_i, A_j] = 0, \quad i < j \tag{0.2}$$

hold as a consequence of the system (0.1). Here $\mathcal{D}_{x^i}$'s are total derivatives, restricted to the equation, see Sect. 1.2. In this paper such matrices are said to constitute a $\mathfrak{g}$-*valued zero-curvature representation* of the given system (0.1) (following the observation that (0.2) coincides with the Maurer-Cartan condition on an appropriate connection to be flat, see [13]). Then a *linear pseudopotential* is defined as a solution Z of the overdetermined linear system

$$\mathcal{D}_{x^i} Z = A_i Z, \tag{0.3}$$

of which (0.2) is the integrability condition.

A very important case from the point of view of applications is that of $m = 2$. An intensive interest in this setting emerged in connection with the inverse scattering transform (AKNS formalism [1]). Various methods (see, e.g., [2,13] and references therein)

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have been developed to generate integrable differential equations or even to check a given system for integrability, which implies the existence of a zero-curvature representation, although special, depending in an appropriate way on what is called a spectral parameter.

Up to now the most widely recognized method of solving our problem in its full generality is that of Wahlquist and Estabrook [17]. Appropriate modifications of their "prolongation procedure" proved themselves to be powerful enough in finding general (even nonlinear) pseudopotentials (moreover, Bäcklund transformations).

Later on I.S. Krasilshchik and A.M. Vinogradov in [7] introduced *coverings* of differential equations as "fibrations" in the category of partial differential equations, thereby providing in a sense ultimate generalization of the notion of a pseudopotential. The authors also suggested a computational procedure designed to give complete answers, some results having been collected in [8]. Their procedure exploited the fact that coverings usually come in whole families of mutually equivalent (isomorphic) instances. Picking up one representative from every equivalence class was a key to efficient computing.

In this paper we profit from the existence of the explicit *gauge equivalence*

$$A_i \sim A'_i = \mathcal{D}_{x^i} h \cdot h^{-1} + h \cdot A \cdot h^{-1},$$

induced by an equivalence of pseudopotentials $Z \sim Z' = hZ$. Here h is an invertible matrix, depending on the x^i, u^k, u^k_I's, lying in the corresponding matrix Lie group G.

This is similar situation as with cohomological (differing by trivial) conservation laws (and, incidentally, for $m = 2$ and $n = 1$ the two concepts coincide). But the cohomology group of conservation laws was identified by A.M. Vinogradov with the term $E_1^{0,m-1}$ of his $\mathcal{C}$-spectral sequence [15], and thereby related to another, efficiently computable term $E_1^{1,m-1}$. The aim of this paper was to find similar construction for zero-curvature representations. However, it should be stressed that we do *not* obtain a new spectral sequence.

As a result, we propose a new computational procedure for solving (0.2). It is based on formula (2.11) and Proposition 3.9 below. In the practically meaningful case $m = 2$, $\mathfrak{g} = \mathfrak{sl}(n)$ and, for simplicity, a single equation (0.1), its core may be formulated as follows:

Procedure. *Replace in the coordinate description of the formal adjoint of the operator of universal linearization* (2.12) *every occurrence of* $\mathcal{D}_{x^i}$ *with* $\mathcal{D}_{x^i} - [A_i, -]$. *So arising operator, when applied to an appropriate* (*a priori unknown*) $\mathfrak{sl}(n)$-*matrix* χ, *gives zero. However, when passing to gauge equivalent matrices* A'_i, *then* χ *changes to a conjugate matrix, and therefore we may suppose, without loss, it to be in the Jordan normal form. Thereby we obtain an additional condition helping us to solve the* (*otherwise underdetermined and hence difficult*) *system* (0.2).

For $m > 2$ (three and more independent variables) a principial problem arises, which may be explained as follows: For $n = 1$ and general m the zero-curvature representation turns out to be what is called an *abelian covering* [6], but the term $E_1^{0,1}$ of the $\mathcal{C}$-spectral sequence responsible for abelian coverings is different from $E_1^{0,m-1}$ for $m > 2$. Moreover, under certain conditions satisfied, in particular, by all non-overdetermined systems (0.1), $E_1^{0,1}$ vanishes for $m > 2$, and therefore no abelian coverings exist at all.

In this paper similar statement (Corollary 4.4.) is proved for the zero-curvature representations. For $m > 2$ there can be non-trivial solutions to (0.2) only when (0.1) is overdetermined. On this basis we are naturally led to study this problem for overdetermined systems.

It should be stressed that there are important generalizations of (0.2) to $m > 2$ and non-overdetermined systems (see [2] and references therein), but out of the scope of this paper. Their treatment should be based on the "generalized" $\mathcal{C}$-spectral sequence introduced in [11].

The author is aware of the fact that the formalism used throughout this paper is far from being traditional and hopes that the example placed in the last section will be sufficiently instructive to allow everybody to apply the method to other examples.

1. Preliminaries

For our purposes it is desirable to represent the differential equation as a geometric object, namely as a *diffiety* [15]. This is an ∞-dimensional manifold capable of carrying the $\mathcal{C}$-differential calculus (the calculus of total derivatives). We recall necessary facts about diffieties here, for general picture and more details we refer to [15] (from where the reader may get acquainted with the necessary background in the $\mathcal{C}$Diff formalism) and [14].

However, here we use slightly different approach based on the notion of a *comonad.* The corresponding references are [10,11] and in general also [9]. The motivation for doing so originates from our intention to apply the Tsujishita technique without having a spectral sequence.

We consider ∞-dimensional manifolds (and fibre bundles) as smooth in the sense of [3]. Algebraic constructions (Hom, $\otimes$, etc.) are always supposed to be induced from finite-dimensional ones via limits. Also, we adopt here an agreement that we make no difference between vector bundles over a manifold X and projective $\mathcal{F}$-modules of their sections, where $\mathcal{F}$ is the algebra of smooth functions on X.

1.1. Jet comonad and differential operators. Consider a fibre bundle $Y \to M$, M being an m-dimensional manifold, $m < \infty$. By $j^r Y \to M$ we denote the fibred manifold of r-jets $j^r_x\gamma$ of local sections of the projection $Y \to M$. For a morphism, say $f : Y \to Z$, of fibre bundles we denote by $j^r f : j^r Y \to j^r Z$, $j^r_x\gamma \mapsto j^r_x(f \circ \gamma)$, its *r-jet prolongation.* Then j^r may be viewed as an endofunctor of the category $\mathcal{M}_M$ of fibred manifolds over M.

Projections $\pi^r Y : j^r Y \to Y$, $j^r_x\gamma \mapsto \gamma(x)$ and $\pi^r_s Y : j^r Y \to j^s Y$, $j^r_x\gamma \mapsto j^s_x\gamma$ for $s \leqslant r$ are natural transformations, and so are inclusions $\iota^{r,s} Y : j^{r+s} Y \to j^r j^s Y$, $j^{r+s}_x\gamma \mapsto j^r_x j^s\gamma$. Both are subject to certain obvious commutativity conditions, namely $\pi^s_t \circ \pi^r_s = \pi^r_t$, $j^r\pi^s_t \circ \iota^{r,s} = \iota^{r,t} \circ \pi^{r+s}_{r+t}$, $\pi^r_s j^t \circ \iota^{r,t} = \iota^{s,t} \circ \pi^{r+t}_{s+t}$, $\iota^{r,s} j^t \circ \iota^{r+s,t} = j^r \iota^{s,t} \circ \iota^{r,s+t}$. This gives, as a consequence, for $\pi = \pi^\infty_0$, $\iota = \iota^{\infty,\infty}$,

$$j^\infty\pi \circ \iota = \mathrm{id}, \quad \pi j^\infty \circ \iota = \mathrm{id}, \quad \iota j^\infty \circ \iota = j^\infty \iota \circ \iota,$$

which amounts to saying that the triple (j^∞, π, ι) is a *comonad* [9] in the category $\mathcal{M}_M$. We call it a *jet comonad.*

An rth order differential operator $\Delta : P \rightharpoonup Q$ between two fibre bundles P, Q over M is defined as a bundle morphism $j^r P \to Q$ (we distinguish operators from bundle morphisms by the arrow tip). As a map of sections, the operator Δ maps a section $\gamma : M \to P$ to a section $\Delta\gamma = \Delta \circ j^r\gamma : M \to Q$. Being given another, sth order differential operator $\nabla : Q \to R$, the composition $\nabla \circ \Delta$ (as maps of sections) is, in terms of bundle morphisms,

$$j^{r+s}P \xrightarrow{\iota^{s,r}P} j^s j^r P \xrightarrow{j^s\Delta} j^r Q \xrightarrow{\nabla} R,$$

i.e., the *Kleisli composition* [11], for general aspects see [9].

An sth *prolongation* of the operator Δ is defined as $\Delta^{(s)} = j^s\Delta \circ \iota^{s,r}P : j^{r+s}P \to j^sQ$. Then the Kleisli composition may be written as $\nabla \circ \Delta^{(s)}$.

1.2. Diffieties and coalgebras. On $j^\infty Y$ there exists a *Cartan connection* $\mathcal{C}$, consisting of tangent planes to local sections of the form $j^\infty\gamma$. Another description is as the map $\iota^{1,\infty} : j^\infty Y \to j^1 j^\infty Y$. It is locally spanned by vector fields $\mathcal{D}_i$, lifts of coordinate vector fields $d_i = \partial/\partial x^i$ on M. Operators $\mathcal{D}_i$ commute since d_i's do (the Cartan connection is flat).

Let u^k denote local coordinates in the fibres of $Y \to M$, and let u^k_I, where I is a multiindex in the i's, denote the standard jet coordinates in the fibres of $j^\infty Y \to M$. Then

$$\mathcal{D}_i = \frac{\partial}{\partial x^i} + \sum_{k,I} u^k_{Ii} \frac{\partial}{\partial u^k_I},$$

Let an equation (0.1) be given. We regard it as a submanifold of $j^\infty Y$, namely, as the inverse image under an appropriate differential operator $f : j^\infty Y \to Z$, of a single section of Z. As in [15], the corresponding *diffiety* $\mathcal{X}$ is obtained when considering the ∞-prolongation $f^{(\infty)} : j^\infty Y \to j^\infty Z$ in this construction. In terms of equations, this is as if (0.1) were considered together with all its differential consequences $\mathcal{D}_I f^l = 0$, where $\mathcal{D}_I$ denotes the composition of commuting operators $\mathcal{D}_i$, indicated by the multiindex I, f^l being the components of the operator f.

It follows that the operators $\mathcal{D}_i$, as well as any $\mathcal{C}$-differential operator (see below), and the Cartan connection, admit a restriction on $\mathcal{X}$.

We use the interpretation [10,11] of diffieties as *coalgebras* [9] of the jet comonad. Namely, under certain regularity conditions (see [11, Definition 0.1]), there is a bundle morphism $\xi : \mathcal{X} \to j^\infty\mathcal{X}$, subject to commutativity conditions $\pi \circ \xi = \mathrm{id}$, $\iota \circ \xi = j^\infty\xi \circ \xi$. The map ξ is called a *coalgebra map* and provides a canonical representation of the equation as a submanifold in a jet space.

Together with ξ we have its projections $\xi^r = \pi^\infty_r \circ \xi : \mathcal{X} \to j^r\mathcal{X}$. The Cartan connection on $\mathcal{X}$ is then described alternatively as $\xi^1 : \mathcal{X} \to j^1\mathcal{X}$.

Let $\mathcal{X}, \mathcal{Y}$ be two coalgebras with coalgebra maps ξ, η, respectively. Then a map $p : \mathcal{Y} \to \mathcal{X}$ is called a *coalgebra homomorphism*, if $j^\infty p \circ \eta = \xi \circ \pi$. Coalgebra homomorphisms are precisely the maps preserving the Cartan connection [10, Proposition 3.1], i.e., are morphisms in the category of diffieties [15]. On this basis we do not distinguish between diffieties and coalgebras in the sequel.

1.3. Covering comonad and $\mathcal{C}$-differential operators. Given a fibred manifold $p : P \to \mathcal{X}$, denote by $j^r_{\mathcal{C}}P$ the pullback of $j^r p : j^r P \to j^r\mathcal{X}$ along the coalgebra map ξ. Then $j^r_{\mathcal{C}}P$ is a fibred manifold over $\mathcal{X}$ and $j^r_{\mathcal{C}}$ can be regarded as an endofunctor in the category $\mathcal{M}_{\mathcal{X}}$ of fibred manifolds over $\mathcal{X}$ (and coincides with $j_{\mathcal{X}}$ of [11] and J^H of [14]).

To a section, say $\gamma : \mathcal{X} \to P$, a section $j^r_{\mathcal{C}}\gamma : \mathcal{X} \to j^r_{\mathcal{C}}P$, induced by $j^r\gamma \circ \xi^r : \mathcal{X} \to j^r P$, is assigned. Projections $\pi_{\mathcal{C}}{}^r_s$ and injections $\iota_{\mathcal{C}}{}^{r,s}$ are then introduced in exactly the same way as in Section 1.1, giving rise to a comonad $(j^\infty_\mathcal{C}, \pi_\mathcal{C}, \iota_\mathcal{C})$ in $\mathcal{M}_\mathcal{X}$. We call it a *covering comonad* in this paper. More details have been given in [11].

A motivation for this new terminology is the following: In [7] a covering of a diffiety $\mathcal{X}$ has been introduced as a morphism $\mathcal{Y} \to \mathcal{X}$, which is simultaneously a fibred manifold, i.e., an $\mathcal{M}_\mathcal{X}$-object. When rewritten in terms of coalgebra homomorphisms, this leads to an existence of a map $\tilde\eta : \mathcal{Y} \to j^\infty_\mathcal{C}\mathcal{Y}$, naturally interpreted as a coalgebra map with respect to the comonad $(j^\infty_\mathcal{C}, \pi_\mathcal{C}, \iota_\mathcal{C})$.

$\mathcal{C}$-differential operators [15] can be introduced via the covering comonad, in analogy with Sect. 1.2, as follows: Given fibre bundles P, Q over $\mathcal{X}$, an rth order *$\mathcal{C}$-differential operator* $P \rightharpoonup Q$ on $\mathcal{X}$ is defined as a bundle morphism $j^r_\mathcal{C} P \to Q$. Natural transformations $\pi_\mathcal{C}$, $\iota_\mathcal{C}$ then give the Kleisli composition.

$\mathcal{C}$-differential operators over $j^\infty Y$ are described in local coordinates as operators in total derivatives $\mathcal{D}_i$, while $\mathcal{C}$-differential operators over $\mathcal{X}$ are described via the restriction of the operators $\mathcal{D}_i$ on the submanifold $\mathcal{X} \subset j^\infty Y$; we do not distinguish them in notation.

For vector bundles P, Q over M we denote by $\mathcal{C}\mathrm{Diff}(P, Q)$ the $\mathcal{F}$-module (or vector bundle) of all *linear* $\mathcal{C}$-differential operators $P \rightharpoonup Q$, i.e., vector bundle morphisms $j^\infty_\mathcal{C} P \to Q$.

For a fibre bundle $P \to M$ we denote by $\overline{P} \to \mathcal{X}$ the pullback of P along the projection $\mathcal{X} \to M$. An isomorphism $j^r_\mathcal{C}\overline{P} \cong \overline{j^r P}$ is then easily established. Given a differential operator $\Delta : P \rightharpoonup Q$ between two fibre bundles P, Q over M, consider a $\mathcal{C}$-differential operator $\overline{\Delta} : \overline{P} \rightharpoonup \overline{Q}$, defined as the composition $j^r_\mathcal{C}\overline{P} \cong \overline{j^r P} \to \overline{Q}$ of the above isomorphism and the pullback of $\Delta : j^r P \to Q$. So defined assignment $\Delta \mapsto \overline{\Delta}$ is obviously functorial. A coordinate description of $\overline{\Delta}$ is obtained from that of Δ by replacing each occurrence of the partial derivative $d_i = \partial/\partial x^i$ with the total derivative $\mathcal{D}_i = \bar{d}_i$.

1.4. Linear coverings and universal linearization. A covering $\mathcal{E} \to \mathcal{X}$ is called *linear* if it is a linear bundle over $\mathcal{X}$, and the coalgebra map $\mathcal{E} \to j^\infty_\mathcal{C}\mathcal{E}$ is a linear map. An example of great importance is delivered by the universal linearization [15]. In the simplest way it is introduced as the vertical vector bundle $V\mathcal{X}$ under the coalgebra map

$$V\mathcal{X} \xrightarrow{V\xi} Vj^\infty\mathcal{X} \cong j^\infty V\mathcal{X}.$$

Given two linear coverings $\mathcal{E}_1$ and $\mathcal{E}_2$ over $\mathcal{X}$, with coalgebra maps ε_1 and ε_2, their *tensor product* is defined as $\mathcal{E}_1 \otimes \mathcal{E}_2$ over $\mathcal{X}$, under the coalgebra map

$$\mathcal{E}_1 \otimes \mathcal{E}_2 \xrightarrow{\varepsilon_1 \otimes \varepsilon_2} j^\infty_\mathcal{C}\mathcal{E}_1 \otimes j^\infty_\mathcal{C}\mathcal{E}_2 \to j^\infty_\mathcal{C}(\mathcal{E}_1 \otimes \mathcal{E}_2),$$

where the right-hand arrow sends a section $j^\infty_\mathcal{C}\gamma_1 \otimes j^\infty_\mathcal{C}\gamma_2$ to $j^\infty_\mathcal{C}(\gamma_1 \otimes \gamma_2)$. (This coalgebra may be infinitary in the sense that it cannot be regarded as a finite system of differential equations.)

Similarly, a *wedge product* $\bigwedge^p \mathcal{E}$ of equal copies of a covering $\mathcal{E}$ can be introduced.

By an easy diagram chasing it can be proved:

1.5. Lemma. *Let $h : \mathcal{E}_1 \to \mathcal{E}_2$ be a linear covering morphism, let $\mathcal{E}$ be another linear covering over $\mathcal{X}$. Then $\mathcal{E} \otimes h : \mathcal{E} \otimes \mathcal{E}_1 \to \mathcal{E} \otimes \mathcal{E}_2$ is a linear coalgebra homomorphism.*

Given a linear covering $\mathcal{E}$ over $\mathcal{X}$ and a linear $\mathcal{C}$-differential operator $\Delta : P \rightharpoonup Q$ between linear bundles P, Q over $\mathcal{X}$, we define $\mathrm{Hom}(\mathcal{E}, \Delta) : \mathrm{Hom}(\mathcal{E}, P) \rightharpoonup \mathrm{Hom}(\mathcal{E}, Q)$ as an assignment to a linear bundle morphism $\omega : \mathcal{E} \to P$ of the linear bundle morphism

$$\mathcal{E} \xrightarrow{\varepsilon} j^\infty_{\mathcal{C}} \mathcal{E} \xrightarrow{j^\infty_{\mathcal{C}} \omega} j^\infty_{\mathcal{C}} P \xrightarrow{\Delta} Q, \tag{1.1}$$

where ε is the coalgebra map of $\mathcal{E}$.

The functoriality in the second argument, i.e., the formula

$$\mathrm{Hom}(\mathcal{E}, \Delta \circ \nabla) = \mathrm{Hom}(\mathcal{E}, \Delta) \circ \mathrm{Hom}(\mathcal{E}, \nabla)$$

is easily checked, as well as the fact that any linear covering morphism $h : \mathcal{E}_1 \to \mathcal{E}_2$ induces a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(\mathcal{E}_2, P) & \xrightarrow{\mathrm{Hom}(\mathcal{E}_2,\Delta)} & \mathrm{Hom}(\mathcal{E}_2, Q) \\ {\scriptstyle \mathrm{Hom}(h,P)}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}(h,Q)} \\ \mathrm{Hom}(\mathcal{E}_1, P) & \xrightarrow{\mathrm{Hom}(\mathcal{E}_1,\Delta)} & \mathrm{Hom}(\mathcal{E}_1, Q) \end{array} \tag{1.2}$$

An important example of this construction arises when $\mathcal{E} = j^\infty_{\mathcal{C}} E$ for some linear bundle E over $\mathcal{X}$. Then, naturally, $\mathrm{Hom}(\mathcal{E}, P) \cong \mathcal{C}\mathrm{Diff}(E, P)$ and $\mathrm{Hom}(\mathcal{E}, \Delta)$ coincides with $\mathcal{C}\mathrm{Diff}(E, \Delta) : \mathcal{C}\mathrm{Diff}(E, P) \to \mathcal{C}\mathrm{Diff}(E, Q)$, acting by composition from the left: $\nabla : E \to P$ sends to $\Delta \circ \nabla : E \to Q$.

1.6. Variational bicomplex [15]. The decomposition of the tangent bundle $T\mathcal{X}$ into its direct summands $V\mathcal{X}$ (vertical bundle) and $\mathcal{C}$ (Cartan connection) induces a decomposition of the vector bundles $T^*\mathcal{X}$ and $\Lambda^r\mathcal{X} = \bigwedge^r T^*\mathcal{X}$. The latter decomposes into $r + 1$ direct summands

$$\Lambda^{p,q}\mathcal{X} = \underbrace{\mathcal{C}\Lambda^1\mathcal{X} \wedge \cdots \wedge \mathcal{C}\Lambda^1\mathcal{X}}_{p} \wedge \underbrace{\bar\Lambda^1\mathcal{X} \wedge \cdots \wedge \bar\Lambda^1\mathcal{X}}_{q},$$

$p+q = r$, where $\mathcal{C}\Lambda^1\mathcal{X} = \mathrm{Ann}\,\mathcal{C}$, and $\bar\Lambda^1 = \mathrm{Ann}\,V\mathcal{X} = \overline{\Lambda^1}\mathcal{X}$ in the above sense, $p+q = r$ is a *full degree*. Then

$$\begin{aligned} \Lambda^{p,q}\mathcal{X} &\cong \mathrm{Hom}(\underbrace{V\mathcal{X} \wedge \cdots \wedge V\mathcal{X}}_{p} \otimes \underbrace{\mathcal{C} \wedge \cdots \wedge \mathcal{C}}_{q}, \mathcal{F}) \\ &\cong \mathrm{Hom}(\textstyle\bigwedge^p V\mathcal{X}, \bar\Lambda^q\mathcal{X}). \end{aligned} \tag{1.3}$$

Consider the exterior differential $d : \Lambda^r\mathcal{X} \to \Lambda^{r+1}\mathcal{X}$. Since $d\Lambda^{p,q}\mathcal{X} \subset \Lambda^{p+1,q}\mathcal{X} \oplus \Lambda^{p,q+1}\mathcal{X}$, d decomposes into its *horizontal* part $\bar d : \Lambda^{p,q}\mathcal{X} \to \Lambda^{p,q+1}\mathcal{X}$ and *contact* part $\ell : \Lambda^{p,q}\mathcal{X} \to \Lambda^{p+1,q}\mathcal{X}$. The two differentials satisfy $\bar d \circ \ell + \ell \circ \bar d = 0$, making $\Lambda^{p,q}\mathcal{X}$ into so called *variational bicomplex*.

The operators $\ell : \Lambda^q\mathcal{X} \to \Lambda^{q+1}\mathcal{X}$, being M-homomorphisms, admit a restriction onto fibers $\mathcal{X}_x$ of $\mathcal{X} \to M$. This restriction coincides with the operator $d : \Lambda^q\mathcal{X}_x \to \Lambda^{q+1}\mathcal{X}_x$ of exterior differentiation.

On the other hand, $\bar d : \Lambda^{0,q}\mathcal{X} \to \Lambda^{0,q+1}\mathcal{X}$ is $\bar d : \bar\Lambda^q\mathcal{X} \to \bar\Lambda^{q+1}\mathcal{X}$ in the above sense, while $\bar d : \Lambda^{p,q}\mathcal{X} \to \Lambda^{p,q+1}\mathcal{X}$, $p \geqslant 1$ admits a description in terms of the maps $\mathrm{Hom}(\mathcal{E}, P)$ above. Indeed, by (1.3), $\Lambda^{p,q}\mathcal{X} \cong \mathrm{Hom}(\bigwedge^p V\mathcal{X}, \bar\Lambda^q\mathcal{X})$, and $\mathrm{Hom}(\bigwedge^p V\mathcal{X}, \bar d) : \mathrm{Hom}(\bigwedge^p V\mathcal{X}, \bar\Lambda^q\mathcal{X}) \to \mathrm{Hom}(\bigwedge^p V\mathcal{X}, \bar\Lambda^{q+1}\mathcal{X})$ can be identified with

$(-1)^p\bar d : \Lambda^{p,q}\mathcal X \to \Lambda^{p,q+1}\mathcal X$. The last identification has been established in [15, Lemma and Proposition 9.3] for $\mathcal X = j^\infty Y$, wherefore the general case follows by restriction to a subcoalgebra $\mathcal X \subset j^\infty Y$. The fact that operators $\bar d$ commute with the restriction $|_{\mathcal X}$ is well-known while similar property for $\mathrm{Hom}(\bigwedge^p V\mathcal X, \bar d)$ amounts to saying that the diagram

$$\begin{array}{ccccccc}
\bigwedge^p V\mathcal X & \xrightarrow{\text{coalg}} & j^\infty \bigwedge^p V\mathcal X & \xrightarrow{j^\infty\omega|_{\mathcal X}} & j^\infty\Lambda^q M & \xrightarrow{d} & \Lambda^{q+1}M \\
\downarrow & & \downarrow & & \| & & \| \\
\bigwedge^p V j^\infty Y & \xrightarrow[\text{coalg}]{} & j^\infty \bigwedge^p V j^\infty Y & \xrightarrow[j^\infty\omega]{} & j^\infty\Lambda^q M & \xrightarrow[d]{} & \Lambda^{q+1}M
\end{array}$$

is commutative, which is easily checked. Here the rows represent another description of the homomorphism $\mathrm{Hom}(\bigwedge^p V\mathcal X, \bar d)$, and ω is to be understood as a fibrewise linear map. The downward arrows are inclusions.

1.7. Formal adjointness. We recall basic facts, as explained in [15, Sections 1.1–1.4, 7.2–7.3] (with a change in notation—we maintain * for the algebraic dual).

For a vector bundle P over $\mathcal X$ let $\widehat P$ be defined as $\mathrm{Hom}(P, \bar\Lambda^m\mathcal X)$, where $m = \dim M$. Since $\bar\Lambda^m\mathcal X$ is of fibre dimension 1, there is a canonical isomorphism $\widehat{\widehat P} \cong P$. Then the formal adjointness, as a mapping (in fact, a $\mathcal C$-differential operator) $\mathcal C\mathrm{Diff}(P,Q) \to \mathcal C\mathrm{Diff}(\widehat Q, \widehat P)$, $\Delta \mapsto \widehat\Delta$, satisfies the following rules:

1° Additivity: $\widehat{\nabla \pm \Delta} = \widehat\nabla \pm \widehat\Delta$;
2° Contravariance: $\widehat{\nabla \circ \Delta} = \widehat\Delta \circ \widehat\nabla$;
3° For a vector field $\eta \in \mathcal C\mathrm{Diff}(\mathcal F, \mathcal F)$, $\widehat\eta \in \mathcal C\mathrm{Diff}(\bar\Lambda^m, \bar\Lambda^m)$ acts as $\widehat\eta(\omega) = -\bar d i_\eta\omega$;
4° $\widehat h = h$ for $h \in \mathrm{Hom}(P,Q)$, after canonical identification $\mathrm{Hom}(\widehat Q, \widehat P) = \mathrm{Hom}(P,Q)$.
5° $\widehat{\widehat\Delta} = \Delta$, after canonical identification $\widehat{\widehat P} = P$.

By 5°, $\Delta \leftrightarrow \widehat\Delta$ is a bijection $\mathcal C\mathrm{Diff}(P,Q) \rightleftharpoons \mathcal C\mathrm{Diff}(\widehat Q, \widehat P)$, which is by 1° an isomorphism of abelian groups but by 2° not of $\mathcal F$-modules.

The coordinate expression is as follows: Let a^i, b^j be local coordinates in fibres of vector bundles P, Q, respectively, let $\widehat a_i, \widehat b_j$ be the coordinates in fibres of $\widehat P, \widehat Q$, determined by the condition $\widehat a_i(a^j) = \delta^i_j dx^1 \wedge \cdots \wedge dx^m$. Then the formal adjoint of a $\mathcal C$-differential operator $b^j = f^{j,I}_i \mathcal D_I a^i$ is given by $\widehat a_i = (-\mathcal D)_I(f^{j,I}_i \widehat b_j)$.

When considering an operator $\varphi : \bar\Lambda^p\mathcal X \to \bar\Lambda^q\mathcal X$ then, using Poincaré duality $\delta : \bar\Lambda^{m-q}\mathcal X \to \widehat{\bar\Lambda^q}\mathcal X$, $\chi \mapsto \chi \wedge (-)$, the formally adjoint operator $\widehat\varphi : \widehat{\bar\Lambda^q}\mathcal X \to \widehat{\bar\Lambda^p}\mathcal X$ can be viewed as an operator $\bar\Lambda^{m-q}\mathcal X \to \bar\Lambda^{m-p}\mathcal X$. We denote it by $\varphi^\dagger$. In particular, when φ is an $\mathcal F$-homomorphism, then 4° gives $\chi \wedge \varphi(\sigma) = \varphi^\dagger(\chi) \wedge \sigma$. This gives $\varphi^\dagger(\chi) = \chi \wedge \rho$ for $\varphi(\sigma) = \rho \wedge \sigma$, and $i^\dagger_\eta\sigma = (-1)^{s+1} i_\eta\sigma$, for $\sigma \in \bar\Lambda^s\mathcal X$, and any $\mathcal C$-vector field η. From here and 3° it follows that $\bar d^\dagger\sigma = (-1)^{s+1}\bar d\sigma$, for $\sigma \in \bar\Lambda^s\mathcal X$. (These formulas differ in signs from those of [15, Section 2.3] because of different definitions. To convert them properly it has to be taken into account that the Poincaré duality $\delta : \chi \mapsto \chi \wedge (-)$ itself has its formal adjoint $\widehat\delta : \widehat{\bar\Lambda^q}\mathcal X \to \widehat{\bar\Lambda^{m-q}}\mathcal X$, acting as $\omega \mapsto (-) \wedge \omega$, i.e., $\widehat\delta = (-1)^{q(m-q)}\delta$.)

2. Linear gauge complexes

Fix a finite-dimensional connected Lie group G, let $\mathfrak g$ be the corresponding finite-dimensional Lie algebra. Define a differential graded algebra $\Lambda(\mathcal X; \mathfrak g)$ of horizontal exterior forms with values in $\mathfrak g$ as the tensor product of the Lie algebra $\mathfrak g$ and the associative algebra $\Lambda\mathcal X$ (see [5]). Then, $\Lambda^s(\mathcal X; \mathfrak g) = \mathfrak g \otimes_{\mathbb R} \Lambda^s\mathcal X$, and the multiplication in $\Lambda(\mathcal X; \mathfrak g)$, which we denote by $[-,-]$, is given on generators $a\otimes\rho \in \mathfrak{g}\otimes\Lambda^r(\mathcal{X})$, and $b\otimes\sigma \in \mathfrak{g}\otimes\Lambda^s(\mathcal{X})$, by $[a\otimes\rho, b\otimes\sigma] = [a,b]\otimes\rho\wedge\sigma \in \mathfrak{g}\otimes\Lambda^{r+s}(\mathcal{X})$. Next, the operator d of exterior differentiation, being $\mathbb{R}$-linear, admits tensoring by $\mathfrak{g}$, giving $d : \Lambda^r(\mathcal{X};\mathfrak{g}) \to \Lambda^{r+1}(\mathcal{X};\mathfrak{g})$, $a\otimes\rho \mapsto a\otimes d\rho$.

For $\alpha \in \Lambda^r(\mathcal{X};\mathfrak{g})$, $\beta \in \Lambda^s(\mathcal{X};\mathfrak{g})$, and $\gamma \in \Lambda^t(\mathcal{X};\mathfrak{g})$ we then have standard formulas

$$\begin{aligned}&[\alpha,\beta] = (-1)^{rs+1}[\beta,\alpha],\\ &(-1)^{rt}[\alpha,[\beta,\gamma]] + (-1)^{rs}[\beta,[\gamma,\alpha]] + (-1)^{st}[\gamma,[\alpha,\beta]],\end{aligned} \tag{2.1}$$

and

$$d[\alpha,\beta] = [d\alpha,\beta] + (-1)^r[\alpha,d\beta]. \tag{2.2}$$

Here it is possible to replace all occurrences of Λ^r with $\Lambda^{p,q}$ (or $\bar\Lambda^q$, for $p = 0$) and all d's with either $\bar d$'s or ℓ's, obtaining for $\alpha \in \Lambda^{p,q}(\mathcal{X};\mathfrak{g})$ and $\beta \in \Lambda^{p,q}(\mathcal{X};\mathfrak{g})$ as a result $[\alpha,\beta] \in \Lambda^{p+s,q+t}(\mathcal{X};\mathfrak{g})$, $\bar d\alpha \in \Lambda^{p,q+1}(\mathcal{X};\mathfrak{g})$, and $\ell\alpha \in \Lambda^{p+1,q}(\mathcal{X};\mathfrak{g})$, satisfying the same rules (2.1), and obvious analogues $\bar d[\alpha,\beta] = [\bar d\alpha,\beta] + (-1)^r[\alpha,\bar d\beta]$, $\ell[\alpha,\beta] = [\ell\alpha,\beta] + (-1)^r[\alpha,\ell\beta]$ of (2.2), with signs determined by full degrees now.

For $\alpha \in \bar\Lambda^r(\mathcal{X};\mathfrak{g})$ define an $\mathcal{F}$-homomorphism $\mathrm{ad}_\alpha : \Lambda^{p,s}(\mathcal{X};\mathfrak{g}) \to \Lambda^{p,r+s}(\mathcal{X};\mathfrak{g})$ by $\mathrm{ad}_\alpha\beta = [\alpha,\beta]$. Furthermore, define its "dual" $\mathrm{ad}^*_\alpha : \Lambda^{p,s}(\mathcal{X};\mathfrak{g}^*) \to \Lambda^{p,r+s}(\mathcal{X};\mathfrak{g}^*)$ as follows: $\Lambda^{p,s}(\mathcal{X};\mathfrak{g}^*) = \Lambda^{p,s}\otimes\mathfrak{g}^*$, $\mathfrak{g}^* = \mathrm{Hom}(\mathfrak{g},\mathbb{R})$, and $\mathrm{ad}^*_{a\otimes\rho} = k([a,-])\otimes\sigma\wedge\rho$ for generators $a\otimes\rho \in \mathfrak{g}\otimes\bar\Lambda^r\mathcal{X}$, and $k\otimes\sigma \in \mathfrak{g}^*\otimes\Lambda^{p,s}\mathcal{X}$.

2.1. Definition. A horizontal 1-form $\alpha \in \bar\Lambda^1(\mathcal{X};\mathfrak{g})$ is said to be a ($\mathfrak{g}$-*valued*) *zero-curvature representation* of the equation $\mathcal{X}$, or simply a ($\mathfrak{g}$-*valued*) *structure form* on $\mathcal{X}$, whenever it satisfies the *horizontal Maurer–Cartan condition*

$$\bar d\alpha = \tfrac{1}{2}[\alpha,\alpha]. \tag{2.3}$$

For $\alpha = A_i dx^i$ the condition (2.3) is just (0.2) of the Introduction. In this section we assume some structure form α to be fixed, and define $\mathcal{C}$-differential operators $\bar\partial_\alpha : \Lambda^{p,q}(\mathcal{X};\mathfrak{g}) \to \Lambda^{p,q+1}(\mathcal{X};\mathfrak{g})$ by

$$\bar\partial_\alpha = \bar d - \mathrm{ad}_\alpha\,.$$

2.2. Proposition 1. *It holds $\bar\partial_\alpha\circ\bar\partial_\alpha = 0$, i.e., for a fixed p, the $\mathcal{F}$-modules $\Lambda^{p,q}(\mathcal{X};\mathfrak{g})$ together with the $\mathcal{C}$-differential operators $\bar\partial_\alpha$ form a complex.*

Proof. For $\rho \in \Lambda^{p,q}(\mathcal{X};\mathfrak{g})$ we have, as a consequence of (2.1) and (2.3), $\bar\partial_\alpha\bar\partial_\alpha\rho = -[\bar d\alpha,\rho] + [\alpha,[\alpha,\rho]] = [-\bar d\alpha + \frac{1}{2}[\alpha,\alpha],\rho] = 0$. □

The complex $(\Lambda^{p,-}\mathcal{X},\bar\partial_\alpha)$ of Proposition 1 will be called a pth (*linear*) *gauge complex*. This terminology is due to J.-F. Pommaret and comes from comparison with Spencer sequences [12]. We use the linear gauge complexes as substitutes for the complexes $(E_0^{p,q}, d_0)$ of the $\mathcal{C}$-spectral sequence.

2.3. Proposition. *The element $\ell\alpha \in \Lambda^{1,1}(\mathcal{X};\mathfrak{g})$ satisfies a condition $\bar\partial_\alpha\ell\alpha = 0$, i.e., is closed with respect to $\bar\partial_\alpha$.*

Proof. $\bar{d}\ell\alpha = -\ell\bar{d}\alpha = -\frac{1}{2}\ell[\alpha,\alpha] = [\alpha,\ell\alpha] = \mathrm{ad}_\alpha\,\ell\alpha$.

Consequently, there exists a well-defined cohomology class $[\ell\alpha]$ of the element $\ell\alpha \in \Lambda^{1,1}(\mathcal{X};\mathfrak{g})$, crucial in our future considerations. Therefore it is very important that the homology of the first linear gauge complex is computable with the aid of the techniques developed by Vinogradov [15] and Tsujishita [14].

Our starting point will be the *Janet sequence* [14, Section 5.2, 5.3] of $\mathcal{C}$-differential operators

$$P_0 \xrightarrow{\varphi_1} P_1 \xrightarrow{\varphi_2} P_2 \to \dots, \tag{2.4}$$

which exists under very general assumptions (involutiveness) and is characterized by the property that the induced sequence of prolongations

$$j^\infty_{\mathcal{C}} P_0 \xrightarrow{\varphi_1^{(\infty)}} j^\infty_{\mathcal{C}} P_1 \xrightarrow{\varphi_2^{(\infty)}} j^\infty_{\mathcal{C}} P_2 \to \dots, \tag{2.5}$$

is an exact sequence of $\mathcal{F}$-homomorphisms such that the kernel of $\varphi_1^{(\infty)}$ is $V\mathcal{X}$. We call (2.5) a *Janet resolution.* It is known (see [12]) that generally it holds $P_{m+1} = P_{m+2} = \dots = 0$ while for non-overdetermined systems even $P_2 = P_3 = \dots = 0$.

We remark that the operator φ_1 was introduced in [15] under the name of a *universal linearization.* In local coordinates it is described, for the system (0.1), by a formula

$$F^l = \frac{\partial f^l}{\partial u^k_I}\mathcal{D}_I U^k, \tag{2.6}$$

where U^k, F^l are coordinates in fibres of vector bundles P_0, P_1, respectively, the fibre dimensions of the bundles P_0, P_1 being equal to the number of unknowns and to the number of equations, respectively.

When combined with the first linear gauge complex, written via the isomorphism (1.3) as

$$\to \mathrm{Hom}(V\mathcal{X}, \bar{\Lambda}^q(\mathcal{X};\mathfrak{g})) \to \mathrm{Hom}(V\mathcal{X}, \bar{\Lambda}^{q+1}(\mathcal{X};\mathfrak{g})) \to$$

the Janet sequence gives rise to a bicomplex $\mathrm{Hom}(j^\infty_{\mathcal{C}} P_i, \bar{\Lambda}^q(\mathcal{X};\mathfrak{g})) \cong \mathcal{C}\mathrm{Diff}(P_i, \bar{\Lambda}^q(\mathcal{X};\mathfrak{g}))$:

$$\begin{array}{ccc}
\downarrow & & \downarrow \\
\to \mathcal{C}\mathrm{Diff}(P_i, \bar{\Lambda}^q(\mathcal{X};\mathfrak{g})) & \xrightarrow{\mathcal{C}\mathrm{Diff}(P_i, \bar{\partial}_\alpha)} & \mathcal{C}\mathrm{Diff}(P_i, \bar{\Lambda}^{q+1}(\mathcal{X};\mathfrak{g})) \to \\
{\scriptstyle \mathcal{C}\mathrm{Diff}(\varphi_i, \bar{\Lambda}^q(\mathcal{X};\mathfrak{g}))}\downarrow & & \downarrow{\scriptstyle \mathcal{C}\mathrm{Diff}(\varphi_i, \bar{\Lambda}^q(\mathcal{X};\mathfrak{g}))} \\
\to \mathcal{C}\mathrm{Diff}(P_{i+1}, \bar{\Lambda}^q(\mathcal{X};\mathfrak{g})) & \xrightarrow[\mathcal{C}\mathrm{Diff}(P_{i+1}, \bar{\partial}_\alpha)]{} & \mathcal{C}\mathrm{Diff}(P_{i+1}, \bar{\Lambda}^{q+1}(\mathcal{X};\mathfrak{g})) \to \\
\downarrow & & \downarrow
\end{array} \tag{2.7}$$

Here horizontal arrows are operators given by composition from the left with operators $\bar{\partial}_\alpha$, while vertical arrows coincide with $\mathcal{F}$-homomorphisms given by composition from the right with the operators φ's. Hence, the horizontal and vertical arrows of the bicomplex commute.

2.4. Proposition. *The columns of this bicomplex are exact with the only exception of lowermost arrows, which have cokernels* $\mathrm{Hom}(V\mathcal{X}, \bar{\Lambda}^q(\mathcal{X};\mathfrak{g}))$. *The induced complex of cokernels coincides with the first linear gauge complex.*

Proof. The first statement is derived from the Janet resolution by virtue of the projectivity of the modules $\bar\Lambda^q(\mathcal{X};\mathfrak{g})$.

The second statement is equivalent to the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Hom}(j^\infty_{\mathcal{C}} P_0, \bar\Lambda^q(\mathcal{X};\mathfrak{g})) & \xrightarrow{\mathrm{Hom}(j^\infty_{\mathcal{C}} P_0, \bar\partial_\alpha)} & \mathrm{Hom}(j^\infty_{\mathcal{C}} P_0, \bar\Lambda^{q+1}(\mathcal{X};\mathfrak{g})) \\ {\scriptstyle |_{V\mathcal{X}}}\downarrow & & \downarrow{\scriptstyle |_{V\mathcal{X}}} \\ \mathrm{Hom}(V\mathcal{X}, \bar\Lambda^q(\mathcal{X};\mathfrak{g})) & \xrightarrow[\bar\partial_\alpha]{} & \mathrm{Hom}(V\mathcal{X}, \bar\Lambda^q(\mathcal{X};\mathfrak{g})) \end{array}$$

which follows by formula (1.2).

The rows have similar property, but before proving this we first need to know the formal adjoint of the operators $\bar\partial_\alpha : \bar\Lambda^q(\mathcal{X};\mathfrak{g}) \to \bar\Lambda^{q+1}(\mathcal{X};\mathfrak{g})$. We proceed as follows: Generally, for operators $\bar\Lambda^p(\mathcal{X};\mathfrak{g}) \to \bar\Lambda^q(\mathcal{X};\mathfrak{g})$ between $\mathfrak{g}$-valued forms, by the above Poincaré duality we obtain $\widehat{\bar\Lambda^p(\mathcal{X};\mathfrak{g})} \cong \mathfrak{g}^* \otimes \widehat{\bar\Lambda^p\mathcal{X}} \cong \bar\Lambda^{m-p}(\mathcal{X};\mathfrak{g}^*)$, so that formal adjoints are maps $\bar\Lambda^{m-q}(\mathcal{X};\mathfrak{g}^*) \to \bar\Lambda^{m-p}(\mathcal{X};\mathfrak{g}^*)$. While $\bar d^\dagger : \bar\Lambda^{m-q-1}(\mathcal{X};\mathfrak{g}^*) \to \bar\Lambda^{m-q}(\mathcal{X};\mathfrak{g}^*)$ is simply $g^* \otimes \bar d^\dagger$, for ad_α we have $\mathrm{ad}^\dagger_\alpha = \mathrm{ad}^*_\alpha$. Indeed, this follows from the commutativity of the diagram

$$\begin{array}{ccc} \mathfrak{g}^* \otimes \bar\Lambda^{m-s-1}\mathcal{X} & \xrightarrow{\mathrm{ad}^*_\alpha} & \mathfrak{g}^* \otimes \bar\Lambda^{m-s}\mathcal{X} \\ {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle \delta} \\ \mathrm{Hom}(\mathfrak{g} \otimes \bar\Lambda^{s+1}\mathcal{X}; \bar\Lambda^m\mathcal{X}) & \xrightarrow[\mathrm{Hom}(\mathrm{ad}_\alpha, \bar\Lambda^m\mathcal{X})]{} & \mathrm{Hom}(\mathfrak{g} \otimes \bar\Lambda^{s}\mathcal{X}; \bar\Lambda^m\mathcal{X}) \end{array}$$

which, for generators $\alpha = a \otimes \rho \in \mathfrak{g} \otimes \bar\Lambda^r\mathcal{X}$, $k \otimes \sigma \in \mathfrak{g}^* \otimes \bar\Lambda^s\mathcal{X}$, leads to the formula $\mathrm{ad}^*_{a\otimes\rho}(k \otimes \sigma) = k([a,-]) \otimes \sigma \wedge \rho$. Consequently,

$$\bar\partial^\dagger_\alpha = (-1)^{s+1}\bar d - \mathrm{ad}^*_\alpha .$$

Now, we return to the bicomplex (2.7) on $\mathcal{X}$. Applying the formal adjointness isomorphism of abelian groups $\mathcal{C}\mathrm{Diff}(P_i, \bar\Lambda^q(\mathcal{X};\mathfrak{g})) \cong \mathcal{C}\mathrm{Diff}(\bar\Lambda^{m-q}(\mathcal{X};\mathfrak{g}^*), \widehat{P}_i)$ we get a new description of the rows:

$$\cdots \to \mathcal{C}\mathrm{Diff}(\bar\Lambda^{m-q}(\mathcal{X};\mathfrak{g}^*), \widehat{P}_i) \to \mathcal{C}\mathrm{Diff}(\bar\Lambda^{m-q-1}(\mathcal{X};\mathfrak{g}^*), \widehat{P}_i) \to \dots$$

where arrows act by composition with the operators $\bar\partial^\dagger_\alpha$ from the right now, i.e., are $\mathcal{F}$-homomorphisms. The columns become

$$\dots \to \mathcal{C}\mathrm{Diff}(\bar\Lambda^{m-q}(\mathcal{X};\mathfrak{g}^*), \widehat{P}_{i+1}) \to \mathcal{C}\mathrm{Diff}(\bar\Lambda^{m-q}(\mathcal{X};\mathfrak{g}^*), \widehat{P}_i) \to \dots,$$

where arrows act by the composition with the operators $\widehat\varphi$ of (2.5) from the left, i.e., are operators now.

However, the rows can be written as

$$\cdots \to \mathrm{Hom}(j^\infty_{\mathcal{C}}\bar\Lambda^{m-q}(\mathcal{X};\mathfrak{g}^*), \widehat{P}_i) \to \mathrm{Hom}(j^\infty_{\mathcal{C}}\bar\Lambda^{m-q-1}(\mathcal{X};\mathfrak{g}^*), \widehat{P}_i) \to \dots,$$

where the arrows coincide with $\mathrm{Hom}(\bar\partial^{\dagger(\infty)}_\alpha, \widehat{P}_i)$. The homology of this complex obviously depends on the homology of the complex

$$\cdots \to j^\infty_{\mathcal{C}}\bar\Lambda^{m-q}(\mathcal{X};\mathfrak{g}^*) \xrightarrow{\bar\partial^{\dagger(\infty)}_\alpha} j^\infty_{\mathcal{C}}\bar\Lambda^{m-q-1}(\mathcal{X};\mathfrak{g}^*) \to \dots. \tag{2.8}$$

2.5. Lemma. *The complex* (2.8) *is a resolution of the* $\mathcal{F}$*-module* $\mathcal{F}\otimes\mathfrak{g}^*$.

Proof. Consider a system of complexes of $\mathcal{F}$-homomorphisms

$$0\to\mathfrak{g}^*\otimes\mathcal{F}\xrightarrow{\ \ker\ } j^r_{\mathcal{C}}(\mathfrak{g}^*\otimes\mathcal{F})\xrightarrow{\bar\partial^{\dagger\,(r-1)}_\alpha}\dots\xrightarrow{\bar\partial^{\dagger\,(r-s)}_\alpha} j^{r-s}_{\mathcal{C}}(\mathfrak{g}^*\otimes\bar\Lambda^s\mathcal{X})\to\dots$$

with arrows induced by the operators $\bar\partial^\dagger_\alpha$. We prove the exactness of all these sequences by induction on r, starting with $r=0$, i.e., with $0\to\mathfrak{g}^*\otimes\mathcal{F}=\mathfrak{g}^*\otimes\mathcal{F}\to 0$, whose exactness is self-evident. The induction step uses the following short exact sequence of complexes of $\mathcal{F}$-homomorphisms induced by operators $\bar\partial^\dagger_\alpha$ and their symbols:

$$\begin{array}{ccccccc}
0 & \to & \mathfrak{g}^*\otimes S^rT^*M\otimes\mathcal{F} & \to\cdots\to & \mathfrak{g}^*\otimes S^{r-s}T^*M\otimes\bar\Lambda^s\mathcal{X} & \to \\
\downarrow & & \downarrow & & \downarrow & \\
\mathfrak{g}^*\otimes\mathcal{F} & \to & j^r_{\mathcal{C}}(\mathfrak{g}^*\otimes\mathcal{F}) & \to\cdots\to & j^{r-s}_{\mathcal{C}}(\mathfrak{g}^*\otimes\bar\Lambda^s\mathcal{X}) & \to \\
\| & & \downarrow & & \downarrow & \\
\mathfrak{g}^*\otimes\mathcal{F} & \to & j^{r-1}_{\mathcal{C}}(\mathfrak{g}^*\otimes\mathcal{F}) & \to\cdots\to & j^{r-s-1}_{\mathcal{C}}(\mathfrak{g}^*\otimes\bar\Lambda^s\mathcal{X}) & \to
\end{array}$$

Here the lower row is exact by induction assumption, the upper row is exact since the symbol of $\bar\partial^\dagger_\alpha$ coincides with that of $\bar d^\dagger$, for which the upper row is exact by [15], and the exactness of the middle row follows. Finally, the limit $r=\infty$ is reached by standard projectivity arguments.

2.6. Corollary. *The rows of the bicomplex* (2.7) *are exact with the only exception of rightmost arrows which have cokernels* $\operatorname{Hom}(\mathcal{F}\otimes\mathfrak{g}^*,\widehat P_i)\cong\mathfrak{g}\otimes\widehat P_i$.

Denote by $[\widehat\varphi_i]_\alpha$ the operators induced between the cokernels. Thus, the induced complex of cokernels is

$$\dots\to\mathfrak{g}\otimes\widehat P_2\xrightarrow{[\widehat\varphi_2]_\alpha}\mathfrak{g}\otimes\widehat P_1\xrightarrow{[\widehat\varphi_1]_\alpha}\mathfrak{g}\otimes\widehat P_0. \tag{2.9}$$

Proof. This follows from the last lemma, and the isomorphism $\operatorname{Hom}(\mathcal{F}\otimes\mathfrak{g}^*,\widehat P_i)\cong\mathfrak{g}\otimes\widehat P_i$.

2.7. Proposition. *The coordinate expression of the operators* $[\widehat\varphi_i]_\alpha$ *in* (2.9) *is obtained when replacing every occurrence of* $\mathfrak{g}\otimes\mathcal{D}_i$ *in the coordinate description of the operator* $\mathfrak{g}\otimes\widehat\varphi_i$ *with* $\mathfrak{g}\otimes\mathcal{D}_i-\operatorname{ad}_{A_i}$. (*Recall that* $A_i dx^i$ *is the coordinate description of the* $\mathfrak{g}$*-valued form* α).

Proof. For $\Delta:\widehat P\to\widehat Q$ consider the defining diagram with exact rows:

$$\begin{array}{ccccc}
\mathcal{C}\mathrm{Diff}(\bar\Lambda^1(\mathcal{X};\mathfrak{g}^*),\widehat P) & \to & \mathcal{C}\mathrm{Diff}(\mathcal{F}\otimes\mathfrak{g}^*,\widehat P) & \to & \operatorname{Hom}(\mathcal{F}\otimes\mathfrak{g}^*,\widehat P) \\
\Big\downarrow{\scriptstyle \mathcal{C}\mathrm{Diff}(\bar\Lambda^1(\mathcal{X};\mathfrak{g}^*),\Delta)} & & \Big\downarrow{\scriptstyle \mathcal{C}\mathrm{Diff}(\mathcal{F}\otimes\mathfrak{g}^*,\Delta)} & & \Big\downarrow{\scriptstyle [\Delta]_\alpha} \\
\mathcal{C}\mathrm{Diff}(\bar\Lambda^1(\mathcal{X};\mathfrak{g}^*),\widehat Q) & \to & \mathcal{C}\mathrm{Diff}(\mathcal{F}\otimes\mathfrak{g}^*,\widehat Q) & \to & \operatorname{Hom}(\mathcal{F}\otimes\mathfrak{g}^*,\widehat Q).
\end{array}$$

Obviously, $[\Delta]_\alpha\circ[\nabla]_\alpha=[\Delta\circ\nabla]_\alpha$ and therefore it is sufficient to consider the following two cases:

1° Δ is an $\mathcal{F}$-homomorphism. Then $[\Delta]_\alpha=\mathfrak{g}\otimes\Delta$ since it makes the right-hand square commutative.

2° $\Delta=\mathcal{D}_i:\mathcal{F}\to\mathcal{F}$. Fix a basis v_n of the Lie algebra $\mathfrak{g}$, let C^n_{kl} be the corresponding structure constants. Then define A^k_i from $A_i=A^k_i v_k$, so that $\operatorname{ad}_{A_i}v_l=[A^k_i v_k,v_l]=$

$A_i^k C_{kl}^n v_n$. Next consider a basis f_n in the free $\mathcal{F}$-module $\mathcal{F} \otimes g^* \cong \mathrm{Hom}(\mathfrak{g}, \mathcal{F})$, defined by $f_n v^j = \delta_n^j$. Then for the operator $\bar{\partial}_\alpha^\dagger = -\bar{d} - \mathrm{ad}_\alpha^* : \mathcal{F} \otimes \mathfrak{g} \to \bar{\Lambda}^1 \mathcal{X} \otimes \mathfrak{g}^*$ we have $\bar{\partial}_\alpha^\dagger(f_n) = -(\mathcal{D}_i f^n + A_i^k C_{kl}^n f^l) dx^i$. To describe the action of the operator $[\mathcal{D}_i]_\alpha$ on a homomorphism, say $z = F_l f^l : \mathcal{F} \otimes \mathfrak{g}^* \to \mathcal{F}$, where z is the fibre coordinate of the bundle $\mathcal{F}$, consider the following computations modulo $\mathcal{C}\mathrm{Diff}(\bar{\Lambda}^1(\mathcal{X}; \mathfrak{g}^*), \mathcal{F})$:

$$\mathcal{D}_i z = \mathcal{D}_i(F_l f^l) = \mathcal{D}_i F_l \cdot f^l + F_l \cdot \mathcal{D}_i f^l = \mathcal{D}_i F_l \cdot f^l - F_n A_i^k C_{kl}^n f^l,$$

since $F_l \cdot \mathcal{D}_i f^l - F_n A_i^k C_{kl}^n f^l = F_n \cdot (\mathcal{D}_i f^n - A_i^k C_{kl}^n f^l)$ lies in the kernel. But this means that

$$[\mathcal{D}_i]_\alpha : F_l \mapsto (\mathcal{D}_i - \mathrm{ad}_{A_i}) F_l. \quad \square$$

Therefore, we have got a bicomplex of abelian groups, with exact rows and columns, supplied with nontrivial complexes of cokernels on the right-hand side and below:

$$\begin{array}{ccccc}
 & \downarrow & & \downarrow & & \downarrow{\scriptstyle [\varphi_2]_\alpha} \\
\to & \bullet_1^{m-1} & \longrightarrow & \bullet_1^m & \longrightarrow & \mathfrak{g} \otimes \widehat{P}_1 \\
 & \downarrow & & \downarrow & & \downarrow{\scriptstyle [\varphi_1]_\alpha} \\
\to & \bullet_0^{m-1} & \longrightarrow & \bullet_0^m & \longrightarrow & \mathfrak{g} \otimes \widehat{P}_0 \\
 & \downarrow & & \downarrow & & \downarrow \\
\to & \Lambda^{1,m-1}(\mathcal{X}; \mathfrak{g}) & \underset{\bar{\partial}_\alpha}{\longrightarrow} & \Lambda^{1,m}(\mathcal{X}; \mathfrak{g}) & \longrightarrow & 0,
\end{array} \tag{2.10}$$

$\bullet_i^q$ denoting the abelian group $\mathcal{C}\mathrm{Diff}(P_i, \bar{\Lambda}^q(\mathcal{X}; \mathfrak{g})) \cong \mathcal{C}\mathrm{Diff}(\bar{\Lambda}^{m-q}(\mathcal{X}; \mathfrak{g}^*), \widehat{P}_i)$. In this situation it is well-known that the corresponding homology groups of both cokernel complexes are isomorphic. Consequently, the homology of the first gauge complex is isomorphic to the homology of the complex (2.9).

2.8. Definition. Let $[\chi_\alpha]$, $\chi_\alpha \in \mathfrak{g} \otimes \widehat{P}_{m-1}$, denote the image of the homology class $[\ell\alpha]$, $\ell\alpha \in \Lambda^{1,1}(\mathcal{X}; \mathfrak{g})$ under the above isomorphism of homology. Then χ_α is called a *characteristic element* of the structure form α. Obviously, χ_α is determined uniquely up to a summand lying in $\mathrm{Im}[\widehat{\varphi}_m]_\alpha$.

It follows directly from the definition that

$$[\widehat{\varphi}_{m-1}]_\alpha(\chi_\alpha) = 0. \tag{2.11}$$

The characteristic element is an analogue of the generating function of a conservation law [15,16], and coincides with it for $m = 2$ and $\mathfrak{g} = \mathbb{R}$. The condition (2.10) will finally help us to solve the equation (2.3), despite of the fact that both χ_α and $[\widehat{\varphi}_{m-1}]_\alpha$ depend on the unknown structure form α.

To apply condition (2.11) it is necessary to know a coordinate description of $[\widehat{\varphi}_{m-1}]_\alpha$. The answer, given in Proposition 2.7, depends on the coordinate description of $\widehat{\varphi}_{m-1}$. In the case of two independent variables this is $\widehat{\varphi}_1$. Since φ_1 is given by formula (2.6), we have for $\widehat{\varphi}_1$

$$\widehat{U}_k = (-\mathcal{D})_I \Big(\frac{\partial f^l}{\partial u_I^k} \widehat{F}_l \Big). \tag{2.12}$$

3. The gauge equivalence

In this section we confine ourself to the case of a matrix Lie group, i.e., a closed subgroup G of $GL(n)$, and its matrix Lie algebra, i.e., a subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(n)$. This not only simplifies all proofs but also anticipates the widely applied choice $G = SL(n)$ and $\mathfrak{g} = \mathfrak{sl}(n)$. (It is known that the search for zero-curvature representations may be, without loss, confined to traceless matrices A_i, i.e., to $\mathfrak{sl}(n)$-valued structure forms. The reason is that $\operatorname{tr}\alpha$ comes out to be a conservation law and as such may be found by known method—see [15,16]).

Consider the multiplication $\cdot$ of $\mathfrak{gl}(n)$-valued forms based on the associative algebra structure of $\mathfrak{gl}(n)$. We have formulas

$$\begin{aligned} &\rho\cdot(\sigma\cdot\tau) = (\rho\cdot\sigma)\cdot\tau, \\ &[\rho,\sigma] = \rho\cdot\sigma - (-1)^{rs}\sigma\cdot\rho, \end{aligned} \tag{3.1}$$

and

$$d(\rho\cdot\sigma) = d\rho\cdot\sigma + (-1)^r\rho\cdot d\sigma, \tag{3.2}$$

as well as its $\bar d$ and ℓ analogues.

Recall that the $\mathfrak{g}$-valued *canonical* 1-*form* θ on G is defined as the map $TG \to \mathfrak{g}$ such that $\theta_g : T_gG \to \mathfrak{g}$ coincides with the composition of the tangent map $TR_g^{-1} : T_gG \to T_eG$ to the inverse of the right translation $R_g : G \to G$, with the isomorphism $T_eG \cong \mathfrak{g}$. When considering G as a subset of $\mathfrak{gl}(n)$ then an alternative description is $\theta = dg\cdot g^{-1}$, well suited for coordinate calculations.

Denote by $G\mathcal{X}$ the group of all smooth maps $h : \mathcal{X} \to G$ with the group structure derived from G. Consider the map $G\mathcal{X} \to \Lambda^1(\mathcal{X};\mathfrak{g})$, assigning to a smooth map h the induced $\mathfrak{g}$-valued 1-form $\theta_h = h^*\theta$. When viewing elements of $G\mathcal{X}$ as $\mathcal{F}$-valued matrices then $\theta_h = dh\cdot h^{-1}$. Similarly, the horizontal and contact parts are $\bar\theta_h = \bar d h\cdot h^{-1}$ and $\Theta_h = \ell h\cdot h^{-1}$, respectively.

The group $G\mathcal{X}$ acts on $\Lambda^q(\mathcal{X};\mathfrak{g})$ from the left by conjugation $\operatorname{Ad}_h : \rho \mapsto h\cdot\rho\cdot h^{-1}$. The induced left action on the $\mathcal{F}$-modules $\Lambda^{1,q}(\mathcal{X};\mathfrak{g})$ and $\mathcal{C}\mathrm{Diff}(F,\bar\Lambda^q(\mathcal{X};\mathfrak{g}))$ is denoted by the same symbol Ad.

3.1. Lemma. *For $h \in G\mathcal{X}$, $\rho \in \Lambda^r(\mathcal{X};\mathfrak{g})$ we have formulas*

$$d(h^{-1}) = -h^{-1}\cdot dh\cdot h^{-1}, \tag{3.3}$$

$$d\operatorname{Ad}_h\rho = \operatorname{Ad}_h d\rho + [\theta_h, \operatorname{Ad}_h\rho]. \tag{3.4}$$

and their horizontal and contact variants.

Proof. The first equation immediately follows from the fact that $h\cdot h^{-1}$ = unit matrix. Then using (3.3) and (3.2) we obtain $d\operatorname{Ad}_h\rho = dh\cdot\rho\cdot h^{-1} + h\cdot d\rho\cdot h^{-1} - (-1)^r h\cdot\rho\cdot h^{-1}\cdot dh\cdot h^{-1} = \theta_h\cdot\operatorname{Ad}_h\rho + \operatorname{Ad}_h d\rho - (-1)^r\operatorname{Ad}_h\rho\cdot\theta_h$, which reduces to (3.4) by (3.1). The variants are proved in an analogous way.

Notice also that (2.3) may be written as $\bar d\alpha = \alpha\cdot\alpha$. Then $\bar\theta_h$ is easily seen to be a structure form for arbitrary h.

3.2. Definition. Structure forms $\bar\theta_h$, $h \in \mathcal{F}$ are called *trivial*.

One of the pseudopotentials of the trivial structure form $\bar\theta_h$ is simply h and the others are $h\cdot C$, where C is a constant matrix. This justifies the terminology.

The following proposition is well-known in the context of gauge theories:

3.3. Proposition. *Let a* 1*-form* $\alpha \in \overline{\Lambda}^1(\mathcal{X};\mathfrak{g})$ *satisfy the horizontal Maurer–Cartan condition* (2.3). *Then*

$$\alpha'_h = \bar\theta_h + \mathrm{Ad}_h\,\alpha = \bar d h \cdot h^{-1} + h \cdot \alpha \cdot h^{-1} \tag{3.5}$$

satisfies (2.3) *as well.*

Proof. $\bar d\alpha'_h = \bar d\bar\theta_h + \bar d\,\mathrm{Ad}_h\,\alpha$ while $\frac12[\alpha'_h,\alpha'_h] = \frac12[\bar\theta_h,\bar\theta_h] + [\bar\theta_h, \mathrm{Ad}_h\,\alpha] + \frac12\,\mathrm{Ad}_h[\alpha,\alpha]$. Now the assertion follows from the fact that both $\bar\theta_h$ and α satisfy (2.3), and the horizontal variant $\bar d\,\mathrm{Ad}_h\,\rho = \mathrm{Ad}_h\,\bar d\rho + [\bar\theta_h, \mathrm{Ad}_h\,\rho]$ of (3.4).

It is easily seen that the formula (3.5) describes a left action of $G\mathcal{X}$ on $\overline{\Lambda}^1(\mathcal{X};\mathfrak{g})$.

3.4. Definition. Structure forms α,β, lying on the same orbit of the left action of $G\mathcal{X}$ on $\overline{\Lambda}^1(\mathcal{X};\mathfrak{g})$, i.e., such that $\beta = \alpha'_h$ for a suitable $h \in G\mathcal{X}$, are called *(gauge) equivalent.*

The trivial structure forms can be characterized as equivalent to the zero structure form. On the other hand, every structure form equivalent to a trivial form is itself trivial, since $(\bar\theta_z)'_h = \bar\theta_{hz}$.

The left action Ad of $G\mathcal{X}$ on $\Lambda^{p,q}(\mathcal{X};\mathfrak{g})$ is compatible with the horizontal differentials:

3.5. Proposition. *The diagram*

$$\begin{array}{ccc}
\Lambda^{p,q}(\mathcal{X};\mathfrak{g}) & \xrightarrow{\bar\partial_\alpha} & \Lambda^{p,q+1}(\mathcal{X};\mathfrak{g}) \\
{\scriptstyle \mathrm{Ad}_h}\downarrow & & \downarrow{\scriptstyle \mathrm{Ad}_h} \\
\Lambda^{p,q}(\mathcal{X};\mathfrak{g}) & \xrightarrow[\bar\partial_{\alpha'_h}]{} & \Lambda^{p,q+1}(\mathcal{X};\mathfrak{g})
\end{array}$$

is commutative.

Proof. The commutativity of the diagram is checked by the following computations:

$$\begin{aligned}
&\bar d\,\mathrm{Ad}_h\,\rho - [\alpha'_h, \mathrm{Ad}_h\,\rho] \\
&\qquad = \mathrm{Ad}_h\,\bar d\rho + [\bar\theta_h, \mathrm{Ad}_h\,\rho] - [\alpha'_h, \mathrm{Ad}_h\,\rho] \\
&\qquad = \mathrm{Ad}_h\,\bar d\rho - [\mathrm{Ad}_h\,\alpha, \mathrm{Ad}_h\,\rho] \\
&\qquad = \mathrm{Ad}_h(\bar d\rho - [\alpha,\rho])
\end{aligned}$$

for arbitrary $\rho \in \Lambda^{p,q}(\mathcal{X};\mathfrak{g})$.

3.6. Proposition. *The induced action* Ad *of* $G\mathcal{X}$ *on* $\mathcal{C}\mathrm{Diff}(P_i, \overline{\Lambda}^q(\mathcal{X};\mathfrak{g}))$ *is compatible with the differentials of the bicomplex* (2.7):

Proof. The commutativity of diagrams

$$\begin{array}{ccc}
\mathcal{C}\mathrm{Diff}(P_i, \overline{\Lambda}^q(\mathcal{X};\mathfrak{g})) & \xrightarrow{\mathcal{C}\mathrm{Diff}(P_i,\bar\partial_\alpha)} & \mathcal{C}\mathrm{Diff}(P_i, \overline{\Lambda}^{q+1}(\mathcal{X};\mathfrak{g})) \\
{\scriptstyle \mathrm{Ad}_h}\downarrow & & \downarrow{\scriptstyle \mathrm{Ad}_h} \\
\mathcal{C}\mathrm{Diff}(P_i, \overline{\Lambda}^q(\mathcal{X};\mathfrak{g})) & \xrightarrow[\mathcal{C}\mathrm{Diff}(P_i,\bar\partial_{\alpha'_h})]{} & \mathcal{C}\mathrm{Diff}(P_i, \overline{\Lambda}^{q+1}(\mathcal{X};\mathfrak{g}))
\end{array}$$

follows from the case $p = 0$ of the previous proposition, while the commutativity of diagrams

$$\begin{array}{ccc} \mathcal{C}\mathrm{Diff}(P_i, \bar\Lambda^q(\mathcal{X};\mathfrak{g})) & \xrightarrow{\mathrm{Ad}_h} & \mathcal{C}\mathrm{Diff}(P_i, \bar\Lambda^q(\mathcal{X};\mathfrak{g})) \\ {\scriptstyle \mathcal{C}\mathrm{Diff}(\varphi_i, \bar\Lambda^q(\mathcal{X};\mathfrak{g}))}\downarrow & & \downarrow{\scriptstyle \mathcal{C}\mathrm{Diff}(\varphi_i, \bar\Lambda^q(\mathcal{X};\mathfrak{g}))} \\ \mathcal{C}\mathrm{Diff}(P_{i-1}, \bar\Lambda^q(\mathcal{X};\mathfrak{g})) & \xrightarrow[\mathrm{Ad}_h]{} & \mathcal{C}\mathrm{Diff}(P_{i-1}, \bar\Lambda^q(\mathcal{X};\mathfrak{g})) \end{array}$$

is obvious.

Finally, from the universal property of the cokernels we deduce

3.7. Proposition. *The induced action* Ad *of* $G\mathcal{X}$ *on the bottom row and the rightmost column of the bicomplex* (2.11) *is compatible with the arrows.*

On the other side, we have

3.8. Proposition. *Let* α, α'_h *be two equivalent structure forms. Then the forms* $\mathrm{Ad}_h \ell\alpha$ *and* $\ell\alpha'_h$ *are homological with respect to the differential* $\bar\partial_{\alpha'_h}$.

Proof. Recall that we denoted by Θ_h the contact, $\mathfrak{g}$-valued form $\ell h \cdot h^{-1} \in \Lambda^{1,0}(\mathcal{X};\mathfrak{g})$. The desired fact will be proved if we check the formula

$$\mathrm{Ad}_h \ell\alpha - \ell\alpha'_h = \bar\partial_{\alpha'_h} \Theta_h. \tag{3.6}$$

However, by the contact variant

$$\ell \,\mathrm{Ad}_h \alpha = \mathrm{Ad}_h \ell\alpha + [\mathrm{Ad}_h \alpha, \Theta_h] \tag{3.7}$$

of (3.4) the left-hand side is easily seen to be $-\ell\bar\theta_h - [\Theta_h, \mathrm{Ad}_h \alpha]$, while the right-hand side is computed as

$$\begin{aligned} \bar\partial_{\alpha'_h} \Theta_h &= \bar d\Theta_h - [\alpha'_h, \Theta_h] \\ &= \bar d\Theta_h - [\bar d\theta'_h, \Theta_h] - [\mathrm{Ad}_h \alpha, \Theta_h] \\ &= -\ell\Theta_h - [\mathrm{Ad}_h \alpha, \Theta_h], \end{aligned}$$

where we used the formula

$$\ell\bar\theta_h + \bar d\Theta_h = [\bar\theta_h, \Theta_h], \tag{3.8}$$

which is easily verified using the horizontal and contact variant of (3.3).

3.9. Proposition. *Let* α, α'_h *be two equivalent structure forms. Then there exist conjugate characteristic elements* $\chi_\alpha, \chi_{\alpha'_h} \in \mathfrak{g} \otimes \widehat{P}_{m-1}$.

Proof. It suffices to show that $[\chi_{\alpha'_h}]$ equals $\mathrm{Ad}_h[\chi_\alpha]$, but this follows from the fact that the homology classes of $\ell\alpha'_h$ and of $\mathrm{Ad}_h \ell\alpha$ are the same by Proposition 3.8, and from the compatibility of Ad_h with the isomorphisms of homology, which follows immediately from Propositions 3.5–3.7.

3.10. Remark. It may so happen that h lies in the stabilizer of χ_α, and then $[\chi_{\alpha'_h}] = [\mathrm{Ad}_h \chi_\alpha] = [\chi_\alpha]$. It follows that our procedure then fails to distinguish α and α'_h. However, this failure can be, at least for $\mathfrak{g} = \mathfrak{sl}$, eliminated by quite elementary means, see the computational example below.

4. Trivial structure forms

In this section we under very general assumptions characterize locally trivial structure forms as those whose characteristic element is homological to zero.

We start with the following technical definition:

4.1. Definition. *Base forms* on $\mathcal{X}$ are defined as pullbacks of ordinary forms on the base manifold M along the projection $\mathcal{X} \to M$.

Obviously, base forms are horizontal and among horizontal forms they are characterized by the property that their coordinate description does not involve fibre coordinates (independent variables). From the geometric meaning of the operator $\ell : \bar{\Lambda}^q \to \Lambda^{1,q}$ it follows that base forms constitute the kernel of ℓ.

By [4, Ch. 3, Sect. 6.7, Proposition 15], every base structure form on M locally equals $\bar{\theta}_z$ for some $z \in \mathfrak{g} \otimes C^\infty M$. Hence every base structure form on $\mathcal{X}$ is locally trivial.

4.2. Proposition. *Let α be a structure form on $\mathcal{X}$. Assume $\bar{\partial}_\alpha : \Lambda^{2,0}(\mathcal{X};\mathfrak{g}) \to \Lambda^{2,1}(\mathcal{X};\mathfrak{g})$ being injective. Then α is locally trivial if and only if χ_α is homological to zero, i.e., $\chi_\alpha \in \operatorname{Im}[\widehat{\varphi}_m]_\alpha$.*

Proof. By Definition 2.8, $\chi_\alpha \in \operatorname{Im}[\widehat{\varphi}_m]_\alpha$ iff $\ell\alpha \in \operatorname{Im}\bar{\partial}_\alpha$, i.e., iff $\ell\alpha = \bar{\partial}_\alpha\beta$ for some $\beta \in \Lambda^{1,0}(\mathcal{X};\mathfrak{g})$. Let us show that the last condition holds if and only if α is locally equivalent to a base structure form.

To prove the 'if' part we start with formula (3.6). Assuming that α is a base form, i.e., $\ell\alpha = 0$, we get immediately $\ell\alpha'_h = -\bar{\partial}_{\alpha'_h}\Theta_h$, wherefore it follows that to every equivalent structure form α'_h we may take $-\Theta_h$ as the corresponding β.

To prove the 'only if' part we suppose that $\ell\alpha = \bar{\partial}_\alpha\beta$ and deduce that α is equivalent to a base form. We have $0 = \ell\ell\alpha = \ell\bar{\partial}_\alpha\beta = \ell\bar{d}\beta - [\ell\alpha,\beta] + [\alpha,\ell\beta] = -\bar{\partial}_\alpha\ell\beta - [\ell\alpha,\beta] = -\bar{\partial}_\alpha\ell\beta - [\bar{\partial}_\alpha\beta,\beta] = -\bar{\partial}_\alpha(\ell\beta + \frac{1}{2}[\beta,\beta])$. Since $\bar{\partial}_\alpha$ is injective by assumption, we have $\ell\beta + \frac{1}{2}[\beta,\beta] = 0$, and therefore locally $\beta = -\ell h \cdot h^{-1} = -\Theta_h$ by a parametric version of [4, Ch. 3, Sect. 6.7, Proposition 15]. Then using (3.8) we compute $0 = \ell\alpha - \bar{\partial}_\alpha\beta = \ell\alpha + \bar{d}\Theta_h - [\alpha,\Theta_h] = \ell\alpha - \ell\bar{\theta}_h + [\bar{\theta}_h,\Theta_h] - [\alpha,\Theta_h] = \ell\zeta - [\zeta,\Theta_h]$, where we denoted $\zeta = \alpha - \bar{\theta}_h$. Applying $\mathrm{Ad}_{h^{-1}}$ on the last expression we get $0 = \mathrm{Ad}_{h^{-1}}\ell\zeta - [\mathrm{Ad}_{h^{-1}}\zeta, \mathrm{Ad}_{h^{-1}}\Theta_h] = \mathrm{Ad}_{h^{-1}}\ell\zeta + [\mathrm{Ad}_{h^{-1}}\zeta, \Theta_{h^{-1}}] = \ell\,\mathrm{Ad}_{h^{-1}}\zeta$ by (3.7). Hence, $\mathrm{Ad}_{h^{-1}}\zeta$ is a base form, denote it by γ. Then we see that $\zeta = \mathrm{Ad}_h\gamma$ and therefore $\alpha = \bar{\theta}_h + \zeta = \bar{\theta}_h + \mathrm{Ad}_h\gamma = \gamma'_h$, i.e., α is equivalent to the base form γ.

The assumption that $\bar{\partial}_\alpha : \Lambda^{2,0}(\mathcal{X};\mathfrak{g}) \to \Lambda^{2,1}(\mathcal{X};\mathfrak{g})$ is injective is, in fact, not very restrictive. Below we show that this is so whenever the length of the Janet sequence of $V\mathcal{X}$ is less than the number m of the independent variables (the equation is not very overdetermined). This replaces the analysis done in [15, Sect. 10.6–10.12].

4.3. Proposition. *Let in the Janet sequence* (2.4) $P_{m-i} = \cdots = P_m = 0$. *Then complexes $\Lambda^{1,q}$, $\Lambda^{2,q}$ are acyclic for $q = 0,\dots,i$.*

Proof. The statement for $\Lambda^{1,q}$ follows directly from the properties of (2.10), hence we concentrate on the case $\Lambda^{2,q}$. Throughout this proof we use a simplified notation: V denotes $V\mathcal{X}$, while $\Lambda^{p,q}$ denotes $\Lambda^{p,q}(\mathcal{X};\mathfrak{g})$, and similarly for $\bar{\Lambda}^q$. Then, by (1.3) we have $\Lambda^{2,q} \cong \operatorname{Hom}(V \wedge V, \bar{\Lambda}^q)$, and the differential $\bar{\partial}_\alpha : \Lambda^{2,q} \to \Lambda^{2,q+1}$ coincides with $\operatorname{Hom}(V \wedge V, \bar{\partial}_\alpha) : \operatorname{Hom}(V \wedge V, \bar{\Lambda}^q) \to \operatorname{Hom}(V \wedge V, \bar{\Lambda}^{q+1})$, see Section 1.6. However, $\operatorname{Hom}(V \wedge V, \bar{\Lambda}^q) \cong \operatorname{Alt}\operatorname{Hom}(V \otimes V, \bar{\Lambda}^q)$ as a complex and therefore it is sufficient to prove the acyclicity of $\mathrm{Hom}(V\otimes V,\bar\Lambda^q)$. We start with the following bicomplex, and aim at proving the exactness of its lowermost row:

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] \\
{} \arrow[r] & \mathrm{Hom}(V,\mathcal{C}\mathrm{Diff}(P_1,\bar\Lambda^{m-1})) \arrow[r] \arrow[d] & \mathrm{Hom}(V,\mathcal{C}\mathrm{Diff}(P_1,\bar\Lambda^{m})) \arrow[r] \arrow[d] & \mathrm{Hom}(V,\widehat{P}_1)\otimes\mathfrak{g} \arrow[d] \\
{} \arrow[r] & \mathrm{Hom}(V,\mathcal{C}\mathrm{Diff}(P_0,\bar\Lambda^{m-1})) \arrow[r] \arrow[d] & \mathrm{Hom}(V,\mathcal{C}\mathrm{Diff}(P_0,\bar\Lambda^{m})) \arrow[r] \arrow[d] & \mathrm{Hom}(V,\widehat{P}_0)\otimes\mathfrak{g} \arrow[d] \\
{} \arrow[r] & \mathrm{Hom}(V\otimes V,\bar\Lambda^{m-1}) \arrow[r] & \mathrm{Hom}(V\otimes V,\bar\Lambda^{m}) \arrow[r] & 0.
\end{tikzcd}$$

Here arrows are $\mathrm{Hom}(V,\mathcal{C}\mathrm{Diff}(P_i,\bar\partial_\alpha))\cong\mathrm{Hom}(V\otimes j^\infty_{\mathcal{C}}P_i,\bar\partial_\alpha))$, and $\mathrm{Hom}(V\otimes V,\bar\partial_\alpha)$, introduced by formula (1.1) above. We see that the commutativity of the squares follows from Lemma 1.5 and formula (1.2) above, with the exception of the rightmost squares, which are commutative by definition.

Our statement will follow from the fact that the homology of the lowermost row coincides with the homology of the rightmost column, provided other rows and columns are exact. However, the exactness of the other columns follows from the exactness of Janet resolution (2.5) by standard projectivity arguments, while the exactness of the other rows can be proved on the basis of the following diagram:

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] \\
{} \arrow[r] & \mathcal{C}\mathrm{Diff}(P_1,\mathcal{C}\mathrm{Diff}(P_i,\bar\Lambda^{m-1})) \arrow[r] \arrow[d] & \mathcal{C}\mathrm{Diff}(P_1,\mathcal{C}\mathrm{Diff}(P_i,\bar\Lambda^{m})) \arrow[r] \arrow[d] & \mathcal{C}\mathrm{Diff}(P_1,\widehat{P}_i)\otimes\mathfrak{g} \arrow[d] \\
{} \arrow[r] & \mathcal{C}\mathrm{Diff}(P_0,\mathcal{C}\mathrm{Diff}(P_i,\bar\Lambda^{m-1})) \arrow[r] \arrow[d] & \mathcal{C}\mathrm{Diff}(P_0,\mathcal{C}\mathrm{Diff}(P_i,\bar\Lambda^{m})) \arrow[r] \arrow[d] & \mathcal{C}\mathrm{Diff}(P_0,\widehat{P}_i)\otimes\mathfrak{g} \arrow[d] \\
{} \arrow[r] & \mathrm{Hom}(V,\mathcal{C}\mathrm{Diff}(P_i,\bar\Lambda^{m-1})) \arrow[r] & \mathrm{Hom}(V,\mathcal{C}\mathrm{Diff}(P_i,\bar\Lambda^{m})) \arrow[r] & \mathrm{Hom}(V,\widehat{P}_i)\otimes\mathfrak{g}
\end{tikzcd}$$

Again, all squares are commutative as a consequence of (1.2), and all columns are exact via the Janet resolution (2.5). Therefore, the exactness of the lowermost row follows from the fact that the other rows are exact by [15, Corollary 2.2 (Theorem 3.1) and Section 7.6].

4.4. Corollary. *Let in the Janet sequence* (2.4) $P_m=0$. *Then all zero-curvature representations are trivial.*

5. An example

Now we are in a position to discuss an example. The nonlinear Klein-Gordon equation $u_{xy}=g(u)$ is one of those for which the answer is well known. The corresponding diffiety $\mathcal{X}$ is an ∞-dimensional manifold with natural coordinates $x,y,u,u_x,u_{xx},\dots,u_y,u_{yy},\dots$ and the Cartan distribution is spanned by total derivatives

$$\mathcal{D}_x=\frac{\partial}{\partial x}+u_x\frac{\partial}{\partial u}+u_{xx}\frac{\partial}{\partial u_x}+u_{xxx}\frac{\partial}{\partial u_{xx}}+\dots+g\frac{\partial}{\partial u_y}+g'u_y\frac{\partial}{\partial u_{yy}}+\dots$$
$$\mathcal{D}_y=\frac{\partial}{\partial y}+u_y\frac{\partial}{\partial u}+g\frac{\partial}{\partial u_x}+g'u_x\frac{\partial}{\partial u_{xx}}+\dots+u_{yy}\frac{\partial}{\partial u_y}+u_{yyy}\frac{\partial}{\partial u_{yy}}+\dots.$$

The operator of universal linearization $\mathcal{D}_{xy} - g'$ is self-adjoint. Here and henceforth the prime $'$ denotes the derivative with respect to u.

We shall compute $\mathfrak{sl}(2)$-valued zero-curvature representations

$$\alpha = \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11} \end{pmatrix}}_{A} dx + \underbrace{\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & -b_{11} \end{pmatrix}}_{B} dy.$$

The Maurer-Cartan conditions $\mathcal{D}_y A - \mathcal{D}_x B + [A, B] = 0$ are represented by three equations

$$\mathcal{D}_y a_{11} - \mathcal{D}_x b_{11} + a_{12} b_{21} - a_{21} b_{12} = 0, \tag{5.1}$$

$$\mathcal{D}_y a_{12} - \mathcal{D}_x b_{12} + 2a_{11} b_{12} - 2a_{12} b_{11} = 0, \tag{5.2}$$

$$\mathcal{D}_y a_{21} - \mathcal{D}_x b_{21} + 2a_{21} b_{11} - 2a_{11} b_{21} = 0, \tag{5.3}$$

on six unknowns $a_{11}, a_{12}, a_{21}, b_{11}, b_{12}, b_{21}$. Three more equations are obtained from the condition (2.11), i.e.,

$$[\mathcal{D}_x]_\alpha [\mathcal{D}_y]_\alpha \chi - g'\chi = 0,$$

where $[\mathcal{D}_x]_\alpha = \mathcal{D}_x - [A, -]$, $[\mathcal{D}_y]_\alpha = \mathcal{D}_y - [B, -]$ (the two operators $[\mathcal{D}_x]_\alpha, [\mathcal{D}_y]_\alpha$ commute once the Maurer–Cartan conditions hold).

The matrix $\chi \in \mathfrak{sl}(2)$ is unknown but may be taken in the Jordan form. In the case of a "general position" it is a diagonal $\mathfrak{sl}(2)$-matrix

$$\chi = \begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix}$$

(this is valid on an open subset of $\mathcal{X}$). Then the additional equations assume the form

$$\mathcal{D}_{xy} r + 2b_{12} a_{21} r + 2b_{21} a_{12} r - g' r = 0, \tag{5.4}$$

$$2a_{12} \mathcal{D}_y r + 2b_{12} \mathcal{D}_x r + 2r \mathcal{D}_y a_{12} - 4r b_{11} a_{12} = 0, \tag{5.5}$$

$$-2a_{21} \mathcal{D}_y r - 2b_{21} \mathcal{D}_x r - 2r \mathcal{D}_y a_{21} - 4r b_{11} a_{21} = 0. \tag{5.6}$$

Now we have altogether six equations on seven unknowns. To obtain a seventh equation we must know the stabilizer subgroup of χ, i.e., the group of all gauge $SL(2)$-matrices H commuting with χ. Recall that by Remark 3.10 these matrices lead to the same conditions (5.1–5.6). By an elementary calculation,

$$H = \begin{pmatrix} h & 0 \\ 0 & 1/h \end{pmatrix}, \qquad h \neq 0,$$

and then

$$A'_H = \begin{pmatrix} h_x/h + a_{11} & h^2 a_{12} \\ a_{21}/h^2 & -h_x/h - a_{11} \end{pmatrix}, \quad B'_H = \begin{pmatrix} h_y/h + b_{11} & h^2 b_{12} \\ b_{21}/h^2 & -h_y/h - b_{11} \end{pmatrix}.$$

If all of the values $a_{12}, a_{21}, b_{12}, b_{21}$ were zero, then our $\mathfrak{sl}(2)$-problem would split into two separate $\mathfrak{gl}(1)$-problems (conservation laws). If, on the contrary, at least one of the values is non-zero then it can be normalized to 1 by an appropriate choice of h. Assuming, without loss in generality, that b_{21} is the non-zero value, we may take

$$b_{21} = 1 \tag{5.7}$$

as the seventh equation. Substituting (5.7) into (5.1–5.6) we get finally:

$$\begin{aligned}
&\mathcal{D}_y a_{11} - \mathcal{D}_x b_{11} + a_{12} - a_{21} b_{12} = 0,\\
&\mathcal{D}_y a_{12} - \mathcal{D}_x b_{12} + 2a_{11} b_{12} - 2a_{12} b_{11} = 0,\\
&\mathcal{D}_y a_{21} + 2a_{21} b_{11} - 2a_{11} = 0,\\
&\mathcal{D}_{xy} r + 2b_{12} a_{21} r - g' r = 0,\\
&2a_{12}\mathcal{D}_y r + 2b_{12}\mathcal{D}_x r + 2r\mathcal{D}_y a_{12} - 4r b_{11} a_{12} = 0,\\
&-2a_{21}\mathcal{D}_y r - 2r\mathcal{D}_y a_{21} - 4r b_{11} a_{21} = 0.
\end{aligned}$$

These equations can be solved, as usual, under appropriate restrictions on the jet order. The standard method of splitting (differentiating individual equations with respect to higher order jet coordinates) gives the desired result in a few steps. Throughout the computations the condition $r \neq 0$, which follows from Proposition 4.2, is of frequent use.

E.g., for $a_{11}, a_{12}, a_{21}, b_{11}, b_{12}, r$ depending solely on u, u_x, u_y we get the final result that, in order to admit zero-curvature representations, g, when nonlinear, must be of the form

$$g = \frac{2}{c}(M e^{cu} - N e^{-cu}),$$

while the full set of zero-curvature representations (1-parametric with respect to K) is given by formulas

$$A = \begin{pmatrix} 0 & N e^{-cu} \\ K e^{cu} & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} \frac{1}{2} c u_y & M/K \\ 1 & -\frac{1}{2} c u_y \end{pmatrix},$$

where $M, N, K, c = \mathrm{const}$. This is exactly the known result [2, Section 3.2.d].

At present more intriguing computations are in progress and their results will be published elsewhere, presumably together with a more traditional exposition of the material of this paper.

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