ISOMORPHISM OF DE RHAM COHOMOLOGY AND RELATIVE HOCHSCHILD COHOMOLOGY OF DIFFERENTIAL OPERATORS

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Abstract. Any de Rham p-form α on a manifold M may be extended to become a Hochschild p-cochain α_S on the associative algebra $\mathcal{D}$ of differential operators on $C^\infty(M, \mathbb{R})$. The map $\alpha \mapsto \alpha_S$ depends on a choice of "allocation", S, which is a rule for filling in any $(p+1)$- tuple of sufficiently nearby points $x_0, \ldots, x_p$ of M with a p-simplex $S(x_0, \ldots, x_p)$ having these points as vertices. We show that $(d\alpha)_S = \delta(\alpha_S)$ so that the map $\alpha \mapsto \alpha_S$ passes to cohomology. We indicate the pattern of the proof that the map $[\alpha] \mapsto [\alpha_S]$ sending $H^p_{DR}(M, \mathbb{R}) \to H^p(\mathcal{D}, C^\infty(M, \mathbb{R}); \mathcal{D})$ is independent of S and is in fact an isomorphism. Here the latter group is the relative Hochschild cohomology group of $\mathcal{D}$ relative to $C^\infty(M, \mathbb{R})$, with coefficients in $\mathcal{D}$.

Keywords. De Rham cohomology, Hochschild cohomology, differential operators.

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1 Allocations: from de Rham to Alexander-Spanier

An allocation allows us to make an Alexander-Spanier (A-S) p-cochain ϕ_α on M [5] from a p-form α on M.

Let dim $M = n$. For $0 \le p \le n$, let $\Delta_p \subset \mathbb{R}^p$ be the standard p-simplex. Let $\mathcal{V}_p \subset M^{p+1}$ be a neighbourhood of the diagonal $M_D = \{(x, x, \ldots, x) \mid x \in M\} \subset M^{p+1}$ which can be retracted to M_D. An allocation is a collection of maps S_p,

$$S_p : \Delta_p \times \mathcal{V}_p \to M, \ 0 \le p \le n$$

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such that for each given $(x_0, \ldots, x_p) \in \mathcal{V}_p$ the image of Δ_p in M is a p-simplex having vertices $x_0, \ldots, x_p$. We denote this simplex $S(x_0, \ldots, x_p)$, and impose the condition that $S \circ \partial = \partial \circ S$, i.e.

$$\sum_{k=0}^{p+1} (-1)^k S(x_0, \ldots, \hat{x}_k, \ldots, x_{p+1}) = \partial S(x_0, \ldots, x_{p+1}). \tag{1}$$

We may now define the A-S cochain $\phi_\alpha^S : \mathcal{V}_p \to \mathbb{R}$ associated with a p-form α by

$$\phi_\alpha^S(x_0, \ldots, x_p) = \int_{S(x_0, \ldots, x_p)} \alpha. \tag{2}$$

Note that (1) implies that

$$\phi_{d\alpha}^S = \delta(\phi_\alpha^S),$$

where δ is the A-S coboundary.

Thus the map $\alpha \mapsto \phi_\alpha^S$ induces a map from de Rham cohomology to A-S cohomology. Any two allocations S_1, S_2 are homotopic. This implies that the induced map between cohomology groups does not depend on the choice of allocation S.

2 From Alexander-Spanier to Hochschild

To each A-S p-cochain $\phi : \mathcal{V}_p \to \mathbb{R}$ we may associate a *Hochschild* p-cochain $F_\phi \in C^p(\mathcal{D}; \mathcal{D})$ on the associative algebra $\mathcal{D}$ of differential operators on $C^\infty(M, \mathbb{R})$. Information about Hochschild cohomology may be found in [4, Chapter 10]. Let $H_j \in \mathcal{D}, j = 1, \ldots, p$. We shall use the following notation. If $f \in C^\infty(\mathcal{V}_1, \mathbb{R})$ and

$$H(x) = a^{ij}(x) \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} + a(x)$$

then

$$[H(y) f(x, y)]_{y=x} = a^{ij}(x) [\frac{\partial}{\partial y^i} \frac{\partial}{\partial y^j} f(x, y)]_{y=x} + a(x) f(x, x).$$

Define the differential operator $F_\phi(H_1, \ldots, H_p) \in \mathcal{D}$ as follows. For $\psi \in C^\infty(M, \mathbb{R})$,

$$(F_\phi(H_1, \ldots, H_p)\psi)(x_0) =$$

$$[H_1(x_1)[H_2(x_2)[\ldots H_p(x_p)[\phi(x_0, \ldots, x_p)\psi(x_p)]]_{x_p = x_{p-1}}]_{x_{p-1} = x_{p-2}} \cdots]_{x_1 = x_0}. \tag{3}$$

One may verify directly that

$$F_{\delta\phi} = \delta(F_\phi)$$

where the right hand side is the Hochschild codifferential.

For example, for an A-S 0-cochain $\phi : \mathcal{V}_0 \to \mathbb{R}$,

$$(F_{\delta\phi}(H)\psi)(x_0) = [H(x_1)[\delta\phi(x_0, x_1)\psi(x_1)]]_{x_1 = x_0}$$

$$= [H(x_1)[(\phi(x_1) - \phi(x_0))\psi(x_1)]]_{x_1 = x_0} = ([H, \phi]\psi)(x_0) = (\delta F_\phi(H))\psi(x_0)$$

where $F_\phi \in \mathcal{D}$ is the 0-cochain ϕ, regarded as a zero order differential operator on $C^\infty(M,\mathbb{R})$. Thus the map $\alpha \mapsto \phi^S_\alpha \mapsto F_{\phi^S_\alpha}$ is a cochain map, giving a map $\check{D} : H^p_{DR}(M,\mathbb{R}) \to H^p_{Hoch}(\mathcal{D},\mathcal{D})$. We write α_S for the Hochschild p-cochain $F_{\phi^S_\alpha}$ on $\mathcal{D}$ obtained by substituting (2) in (3). One can verify that α_S has the following properties.

(i) Regarding vector fields $X_1,\ldots,X_p$ as first order differential operators,

$$\alpha_S(X_1,\ldots,X_p) = \frac{1}{p!}\alpha(X_1,\ldots,X_p). \tag{4}$$

(ii) For $f_j \in C^\infty(M,\mathbb{R})$ and $H_j \in \mathcal{D}$,

$$\alpha_S(f_1\circ H_1,\ldots,H_{j-1},f_j\circ H_j,\ldots,H_p\circ f_p) = f_1\circ\alpha_S(H_1,\ldots,H_{j-1}\circ f_j,H_j,\ldots,H_p)\circ f_p.$$

(iii) $\alpha_S(H_1,\ldots,H_p) = 0$ if any H_j is a function.

(iv) The order of the operator $\alpha_S(H_1,\ldots,H_p)$ is $\sum_{j=1}^p(\mathrm{ord}(H_j)-1)$.

Thus $\alpha_S : \otimes^p_{\mathbb{R}}\mathcal{D} \to \mathcal{D}$ factors through $\otimes^p_{\mathcal{F}}(\mathcal{D}/\mathcal{F})$ where $\mathcal{F} = C^\infty(M,\mathbb{R})$, and is therefore an $\mathcal{F}$-relative p-cochain [3]. The Hochschild coboundary operation preserves $\mathcal{F}$-relativity and the cohomology of the $\mathcal{F}$-relative complex is denoted $H^p(\mathcal{D},\mathcal{F};\mathcal{D})$.

The map $\alpha \mapsto \alpha_S$ for $p=1$ also gives a construction of the Poincaré-Cartan form Θ for a class of Lagrangians in the multivariate higher order calculus of variations. The dependence of the map upon choice of allocation and the independence of $\check{D}$ relate to known properties of Θ, [1].

We end with a sketch of the proof that the de Rham cohomology of M and the relative Hochschild cohomology of $\mathcal{D}$ relative to $\mathcal{F}$ are isomorphic.

3 The isomorphism $H^p_{DR} \simeq H^p(\mathcal{D},\mathcal{F};\mathcal{D})$

The map $D_S : \Omega^p \to C^p_{Hoch}(\mathcal{D},\mathcal{F};\mathcal{D})$, $\alpha \mapsto \alpha_S$ defined above satisfies $D_S \circ d = \delta \circ D_S$ so passes to cohomology as the map $\check{D} : H^p_{DR} \to H^p(\mathcal{D},\mathcal{F};\mathcal{D})$. There is a map $\check{a}$ in the reverse direction. Define $a : C^p_{Hoch}(\mathcal{D},\mathcal{F};\mathcal{D}) \to \Omega^p$, $A \mapsto aA$, by

$$(aA)(X_1,\ldots,X_p) = \sum_{\sigma\in\mathcal{S}_p}(-1)^\sigma A(X_{\sigma_1},\ldots,X_{\sigma_p}).$$

Note that aA is indeed a de Rham p-form, and that (4) implies that

$$a\circ D_S = \mathrm{id}_{\Omega^p}.$$

One may check directly that $a\circ\delta = d\circ a$, so that a passes to cohomology,

$$\check{a} : H^p(\mathcal{D},\mathcal{F};\mathcal{D}) \to H^p_{DR}(M,\mathbb{R}),$$

and

$$\check{a}\circ\check{D} = \mathrm{id}_{H^p_{DR}}.$$

Thus $\check{D}$ is injective, ker $\check{D}$=0.

It remains to prove that $\check{a}$ is injective also. Let $A \in C^p_{Hoch}(\mathcal{D}, \mathcal{F}; \mathcal{D})$, $\delta A = 0$ and $aA = 0$. We seek $B \in C^{p-1}_{Hoch}(\mathcal{D}, \mathcal{F}; \mathcal{D})$ such that $A = \delta B$. We indicate the pattern of the argument by sketching it for the case p=3. We construct $B(H_1, H_2)$ for increasing order of the operator $H_1 H_2$. If either H_j is of zero order, then B=0. Suppose each $H_j = X_j$, a vector field. From $A = -\delta B_1, \mathrm{ord}(H_1 H_2 H_3) \leq 3$, (the minus sign is for convenience), we have

$$A(X_1, X_2, X_3) = -\delta B_1(X_1, X_2, X_3) =$$

$$= -X_1 B_1(X_2, X_3) + B_1(X_1 X_2, X_3) - B_1(X_1, X_2 X_3) + B_1(X_1, X_2) X_3. \qquad (5)$$

For fixed X_1, X_2, X_3 (5) together with 5 similar equations obtained by permuting X_1, X_2, X_3 are linear equations in the unknowns $B_1(X_i, X_j), B_1(X_i X_j, X_k), B_1(X_i, X_j X_k)$. Let us choose $B_1(X_i, X_j) = 0$, i.e. $B_1(H_1, H_2) = 0$, for $\mathrm{ord}(H_1 H_2) \leq 2$. Then

$$B_1(X_1 X_2, X_3) = B_1(X_2 X_1, X_3) \equiv B_1(\{1, 2\}, 3).$$

Write $A(X_1, X_2, X_3)$ as $A(1, 2, 3)$. Then equations (5) reduce to

$$A(1, 2, 3) = B_1(\{1, 2\}, 3) - B_1(1, \{2, 3\}) \qquad (6)$$

and its five permutations. The cochains in these equations may be regarded as functions of the ordered partitions of the set $\{1, 2, 3\}$ into 2 or 3 subsets. The condition $aA = 0$ is clearly necessary for a solution B_1 to exist. That it is also sufficient may be seen from figure 1 showing the barycentric decomposition of the circle.

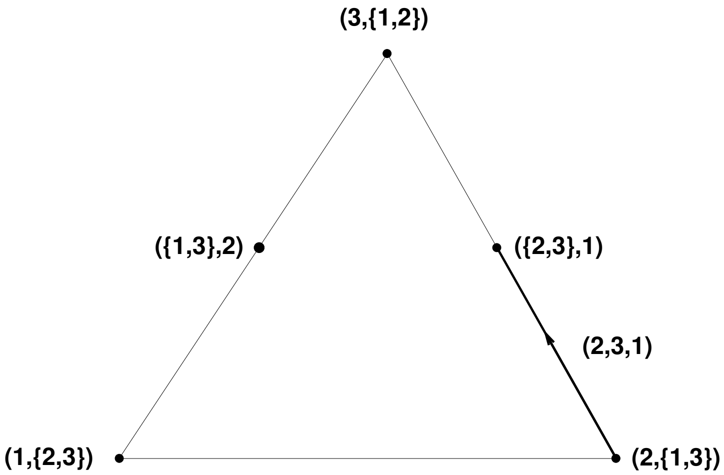

Figure 1: Barycentric decomposition of S^1

Label the vertices of the triangle $(1, \{2, 3\}), (2, \{1, 3\}), (3, \{1, 2\})$; label the midpoints of the sides 12, 23, 31 as $(\{1, 2\}, 3), (\{2, 3\}, 1), (\{3, 1\}, 2)$. Thus the points (vertices and barycentres) are ordered partitions of the set $\{1, 2, 3\}$ into 2 subsets. Label the half-side from the vertex $(2, \{1, 3\})$ to the midpoint $(\{1, 2\}, 3)$ as (2,1,3), and similarly label the other 5 half-sides. A now takes values on half-sides and B_1 on vertices and midpoints. The condition that $aA = 0$ means that the sum of the A-values taken round the triangle vanishes. This is the sufficient condition that there are B_1-values satisfying (6).

We may take

$$\begin{aligned} B_1(1,\{2,3\}) &= 0, \\ B_1(\{1,2\},3) &= A(1,2,3), \\ B_1(2,\{1,3\}) &= A(1,2,3) - A(2,1,3), \end{aligned}$$

etc. We thank Alastair King and Johan Dupont for this argument.

This defines $B_1(H_1, H_2)$ for $\mathrm{ord}(H_1 H_2) \le 3$. Extend the definition of B_1 to higher orders of $H_1 H_2$ arbitrarily, consistent with this. Now let

$$A_1 = A + \delta B_1.$$

Then $\delta A_1 = 0$ and $A_1(H_1, H_2, H_3) = 0$ for $\mathrm{ord}(H_1 H_2 H_3) \le 3$.

Fix vector fields X_1, X_2, X_3, X_4. We have

$$A_1(X_1X_2, X_3, X_4) = A_1(X_2X_1, X_3, X_4) \equiv A_1(\{1,2\},3,4).$$

Let $B_2(H_1, H_2) = 0$ for $\mathrm{ord}(H_1 H_2) \le 3$. Then

$$B_2(X_1X_2X_3, X_4) = B_2(X_{\sigma_1}X_{\sigma_2}X_{\sigma_3}, X_4) \equiv B_2(\{1,2,3\},4).$$

Similarly

$$B_2(X_1X_2, X_3X_4) \equiv B_2(\{1,2\},\{3,4\}).$$

We wish to solve the equations $A_1 = -\delta B_2$. These state

$$A_1(\{1,2\},3,4) = B_2(\{1,2,3\},4) - B_2(\{1,2\},\{3,4\}) \tag{7}$$

and 35 similar equations.

The appropriate diagram this time is the barycentric decomposition of S^2, (figure 2).

A_1-values are attached to line segments, which are ordered partitions of the set $\{1,2,3,4\}$ into 3 non-empty subsets, and B_2-values are attached to vertices and barycentres, which are ordered partitions into 2 non-empty subsets. The subfaces are permutations of (1,2,3,4). Again the equations (7) have the form $A_1(\sigma) = B_2(\partial\sigma)$ where σ is any line segment of the decomposition, and the "integrability conditions" for the existence of a solution B_2 have the form

$$\delta A_1(1,2,3,4) = -A_1(\{1,2\},3,4) + A_1(1,\{2,3\},4) - A_1(1,2,\{3,4\}) = 0.$$

We thus obtain $B_2(H_1, H_2)$ for $\mathrm{ord}(H_1H_2) \le 4$, and extend B_2 arbitrarily to higher orders of H_1H_2. Then

$$A(H_1, H_2, H_3) = -\delta(B_1 + B_2)(H_1, H_2, H_3)$$

for $\mathrm{ord}(H_1H_2H_3) \le 4$. Continuing in this way, one may define B_n for any n. The details are in a forthcoming paper [2]. Then for $\mathrm{ord}(H_1H_2H_3) = n$,

$$A(H_1, H_2, H_3) = -\delta B(H_1, H_2, H_3)$$

where $B(H_1, H_2) = \sum_{r=1}^{n-2} B_r(H_1, H_2)$.

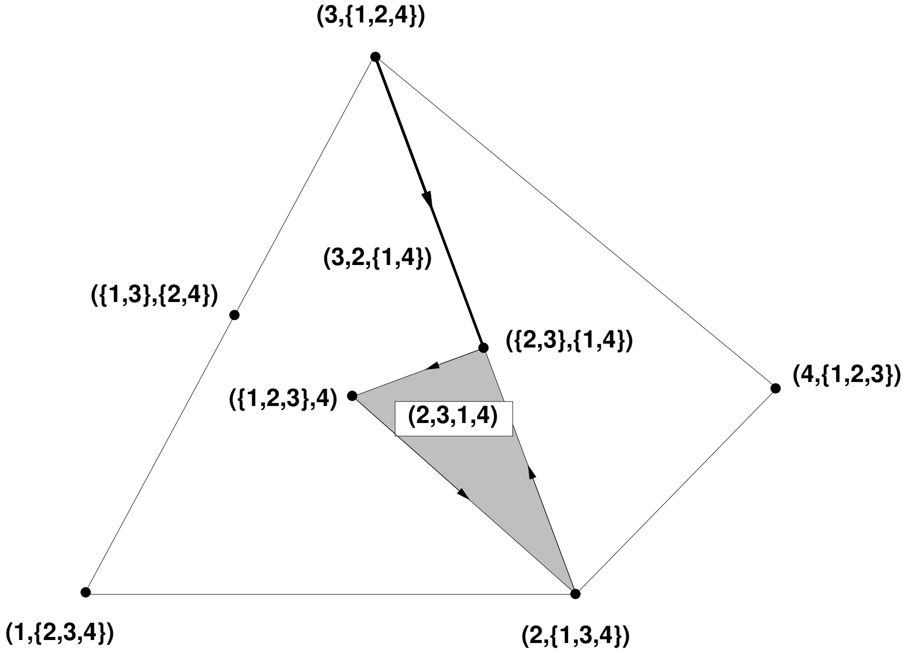

Figure 2: Barycentric decomposition of S^2

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