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The Fourier Transform on Symmetric Spaces and Applications

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1. The Fourier Transform

The symmetric spaces the title refers to are the spaces $X = G/K$ where G is a connected semisimple Lie group with finite center and K is a maximal compact subgroup. The Fourier transform on X is defined by means of the Iwasawa decomposition $G = NAK$ of G where N is nilpotent and A abelian. Let $\mathfrak{g}, \mathfrak{n}, \mathfrak{a}, \mathfrak{k}$ denote the corresponding Lie algebras. We also need the group $M = K^A$, the centralizer of A in K. To define the *Fourier transform* we write for $g \in G$

$$g = n \exp A(g) k, \qquad n \in N,\ A(g) \in \mathfrak{a},\ k \in K$$

and define $A:\ G/K \times K/M \longrightarrow \mathfrak{a}$ by

$$A(gK, kM) = A(k^{-1}g).$$

If f is a function on X its *Fourier transform* $\tilde{f}$ is defined in [3a] by

$$\tilde{f}(\lambda, b) = \int_X f(x) e^{(-i\lambda+\rho)(A(x,b))} dx, \qquad b \in B, \tag{1}$$

for all $(\lambda, b) \in \mathfrak{a}_c^* \times B$ for which the integral exists. Here ρ is half the sum of the restricted roots. Definition (1) is the analog of the Euclidean Fourier transform

$$F(\lambda, \omega) = \int_{\boldsymbol{R}^n} F(x) e^{-i\lambda(x,\omega)} dx, \qquad |\omega| = 1. \tag{2}$$

With a *horocycle* in X being an orbit of a group gNg^{-1} the *Radon transform* $\hat{f}$ of f is defined by

$$\hat{f}(\xi) = \int_\xi f(x)\, dm(x), \qquad \xi \text{ a horocycle}, \tag{3}$$

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dm being the volume element on ξ. Then (1) can be written

$$\tilde{f}(\lambda, b) = \int\limits_A e^{(-i\lambda+\rho)(\log a)}\Bigg(\int\limits_{A(x,b)=\log a} f(x)\,dm(x)\Bigg)da.$$

With $x = gK$, $b = kM$ the inner integral runs over the set

$$\{gK :\ A(k^{-1}g) = \log a\} = ka\,N \cdot o,$$

where $o = \{eK\}$. Writing ξ_0 for the horocycle $N \cdot o$ this means

$$\tilde{f}(\lambda, kM) = \int\limits_A \hat{f}(ka \cdot \xi_0)e^{(-i\lambda+\rho)(\log a)}da. \tag{4}$$

This is the analog to the formula

$$\widetilde{F}(\lambda\omega) = \int\limits_{\boldsymbol{R}} \widehat{F}(\omega, p)e^{-ip\lambda}dp$$

connecting $\widetilde{F}$ with the Radon transform

$$\widehat{F}(\omega, p) = \int\limits_{(x,\omega)=p} F(x)\,dm(x)$$

of F over the hyperplane $(x, \omega) = p$ in $\boldsymbol{R}^n$.

The customary Fourier analysis theorems in $\boldsymbol{R}^n$ (inversion and Plancherel formula) have their analog for (1) as well; for applications it is more important to have range theorems, i.e. intrinsic descriptions of $\mathcal{F}(X)^{\sim}$ as $\mathcal{F}(X)$ is some naturally defined function space on X.

One basic result of this type is the following characterization of $\mathcal{D}(X)^{\sim}$ ($\mathcal{D}(X) = C_c^{\infty}(X)$) from [3b].

Theorem 1. *$\mathcal{D}(X)^{\sim}$ consists of the functions $(\lambda, b) \longrightarrow \phi(\lambda, b)$ on $\mathfrak{a}_{\mathfrak{c}}^* \times B$ which are of exponential type in λ (uniformly in $b \in B$) and satisfy*

$$\int\limits_B e^{(is\lambda+\rho)(A(x,b))}\phi(s\lambda, b) = \int\limits_B e^{(i\lambda+\rho)(A(x,b))}\phi(\lambda, b)db \tag{5}$$

for all $x \in X$ and all s in the Weyl group W.

2. Applications to Solvability Questions

The following result is a simple consequence of Theorem 1.

Corollary 2. *Let G/K have rank one and L the Laplace-Beltrami operator. Let $f \in \mathcal{D}(X)$. Then the equation*

$$Lu = f \tag{6}$$

has a solution $u \in \mathcal{D}(X)$ *if and only if*

$$\int_X f(x)dx = \int_X f(x)P(x,b)dx = 0, \qquad b \in B, \tag{7}$$

where P *is the Poisson kernel.*

In fact, if $h \in \mathcal{D}(X)$ then

$$(Lh)^{\sim}(\lambda, b) = -(\langle\lambda,\lambda\rangle + \langle\rho,\rho\rangle)\tilde{h}(\lambda, b), \tag{8}$$

where $\langle\ ,\ \rangle$ refers to the Killing form, and

$$P(x,b) = e^{2\rho(A(x,b))}.$$

Conditions (7) thus amount to $\tilde{f}(\pm i\rho, b) = 0$ which by (8) would be necessary. Also they amount to $\tilde{f}(\lambda, b)$ being divisible by $\langle\lambda,\lambda\rangle + \langle\rho,\rho\rangle$ so by Theorem 1, they are also sufficient.

Theorem 1 is also instrumental in proving

$$D\mathcal{E}(X) = \mathcal{E}(X) \qquad \text{where } \mathcal{E}(X) = C^\infty(X) \tag{9}$$

for X of any rank, $D \neq 0$ being any differential operator on X, invariant under G. Theorem 1 was extended in [2] to the space $\mathcal{S}(X)$ of rapidly decreasing functions. From this one can prove ([3b])

$$D\mathcal{S}'(X) = \mathcal{S}'(X) \tag{10}$$

in analogy with (9), $\mathcal{S}'(X)$ being the space of tempered distributions on X.

3. Wave Equations and Huygens' Principle

We shall now discuss applications of the Fourier transform on X to the question of the validity of Huygens' principle for wave equations on X.

Let d denote the distance function on X and $B_r(x)$ (resp. $S_r(x)$) the open ball (resp. sphere) with radius r and center x. Let L again be the Laplace-Beltrami operator on X and consider the Cauchy problem:

$$\frac{\partial^2}{\partial t^2} = (L + |\rho|^2)u, \qquad u(x,0) = 0,\ u_t(x,0) = f(x). \tag{11}$$

Huygens' principle is said to hold if for each $\epsilon > 0$, the solution $u(x,t)$ only depends on the restriction $f|V_{t,\epsilon}(x)$ where $V_{t,\epsilon}(x)$ denotes the shell $B_{t+\epsilon}(x) - B_{t-\epsilon}(x)$.

In the case of the wave equation on $\boldsymbol{R}^n$

$$\frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} = \frac{\partial^2 u}{\partial t^2}, \qquad u(x,0) = 0,\ u_t(x,0) = f(x) \tag{12}$$

the solution is given by the classical Poisson-Tedone formula

$$u(x,t) = \frac{1}{(n-2)!} \frac{\partial^{n-2}}{\partial t^{n-2}} \Big(\int_0^t (M^r f)(x) r (t^2 - r^2)^{\frac{1}{2}(n-3)} dr \Big), \tag{13}$$

where $(M^r f)(x)$ is the average of f on $S_r(x)$. If n is odd the integration can be obliterated and we obtain

$$u(x,t) = \sum_k c_k \, t^{k+1} \frac{d^k}{dt^k} (M^t f(x)), \tag{14}$$

from which the validity of Huygens' principle for (12) is obvious.

For (11) we consider in addition to the Radon transform $f \longrightarrow \hat{f}$ the dual transform $\phi \longrightarrow \check{\phi}$ which to a function ϕ on the space Ξ of horocycles associates the function $\check{\phi}$ on X given by

$$\check{\phi}(x) = \int_{\xi \ni x} \phi(\xi) d\mu(\xi). \tag{15}$$

the average over the set of horocycles ξ through x. Then $f \longrightarrow \hat{f}$, $\phi \longrightarrow \check{\phi}$ are adjoint operators, i.e.

$$\int_X f(x) \check{\phi}(x) dx = \int_\Xi \hat{f}(\xi) \phi(\xi) d\xi, \tag{16}$$

where dx and $d\xi$ are G-invariant measures. Thus we put for $S \in \mathcal{E}'(X)$ (space of compactly supported distributions)

$$\widehat{S}(\phi) = S(\check{\phi}) \qquad \phi \in \mathcal{E}(X). \tag{17}$$

Then we have the following support theorem proved in [3d] by the method of Theorem 1.

Theorem 3. *For $A > 0$ let $\beta_A = \{\xi \in \Xi : d(o,\xi) < A\}$. Suppose $T \in \mathcal{E}'(X)$ satisfies*

$$\operatorname{supp}(\widehat{T}) \subset \beta_A.$$

Then

$$\operatorname{supp}(T) \subset B_A(0).$$

Combining this with the inversion formula ([3a])

$$f = (\Lambda \hat{f})^\vee, \tag{18}$$

Λ being a certain pseudo-differential operator, we obtain a certain refinement of Theorem 3.

Theorem 4. *For $A > 0$ let $\sigma_A = \{\xi \in \Xi : \ d(o,\xi) = A\}$. Suppose G has all its Cartan subgroups conjugate. Suppose $T \in \mathcal{E}'(X)$ satisfies*

$$\operatorname{supp}(\widehat{T} \subset \sigma_A.$$

Then

$$\operatorname{supp}(T) \subset S_A(o).$$

The assumption on G implies that the operator Λ in (18) is a differential operator. These theorems are now used to prove the following result [3c,d]; different proofs are from [4] and [5].

Theorem 5. *Let X be odd-dimensional and that G has all its Cartan subgroups conjugate. Then the equation (11) satisfies Huygens' principle.*

To indicate the proof suppose f in (11) belongs to $\mathcal{D}(X)$ and take Fourier transforms. Then we find as in [1],

$$\tilde{u}(t,\lambda,b) = \tilde{f}(\lambda,b)\frac{\sin|\lambda|t}{|\lambda|}. \tag{19}$$

The holomorphic function $\lambda \longrightarrow |\lambda|^{-1}\sin|\lambda|t$ of exponential type is the Euclidean Fourier transform

$$\frac{\sin|\lambda|t}{|\lambda|} = \int_A e^{-i\lambda(\log a)}dT_t(a) \qquad T_t \in \mathcal{E}'(A) \tag{20}$$

and by the extension of Theorem 1 to distributions,

$$\frac{\sin|\lambda|t}{|\lambda|} = \int_X e^{(-i\lambda+\rho)(A(x,b))}d\tau_t(x) \tag{21}$$

where $\tau_t \in \mathcal{E}'(X)$.

Lemma 6. *In terms of the product representation $\Xi = (K/M) \times A$*

$$(\tau_t)^\wedge = 1 \otimes e^\rho T_t.$$

Now if $\dim X$ is odd and rank $X > 1$ then $\dim A$ is odd and > 1. Then it is known that $\operatorname{supp}(T_t) \subset S_t(e)$. Then by Lemma 6 and Theorem 4

$$\operatorname{supp}(\tau_t) \subset S_t(o). \tag{22}$$

Using (21) and the symmetry of τ_t we then deduce from (19),

$$u(g\cdot o,t) = \int_X f(g\cdot x)d\tau_t(x)$$

so the support property in (22) implies Huygens' principle.

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