Solving vector-valued approximation problems by semi-infinite optimization : Numerical and genericity aspects

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Abstract

We consider the approximation of vector-valued functions $f \in C(I, X)$ by functions u from a given subset $U \subset C(I, X)$ in the norm $|||f - u|||_X = \max_{t \in I} \|f(t) - u(t)\|_X$, where $I = [0, 1]$, X a Banach space with norm $\| \cdot \|_X$. For numerical purposes we write the approximation problem as a semi-infinite optimization problem. We especially discuss the linear approximation problem which can be put in the form of either a convex problem or in the form of a linear (extended) semi-infinite problem. A comparison of the problems in different norms $\| \cdot \|_X$ leads to numerically relevant implications of strongly and strictly uniqueness. We give a theoretical and numerical comparison of two different approaches for solving the vector-valued approximation problems. One approach is based on nonlinear optimization the other on techniques from linear semi-infinite optimization. The last section is devoted to genericity results.

Keywords and phrases: vector-valued approximation, semi-infinite optimization, strong and strict best approximation
1991 Mathematics Subject Classification: 41A50, 41A63,90C30, 90C34

1 Introduction

Let X be a real Banach space of dimension d, $1 \le d \le \infty$, with norm $\|\cdot\|_X$. Setting $I = [0,1]$ let $C(I,X)$ denote the space of continuous functions $f : I \to X$ normed by

$$|||f|||_X = \max_{t \in I} \|f(t)\|_X \ , \qquad f \in C(I,X).$$

Given a map $u : \mathbb{R}^n \to C(I,X)$, $c \mapsto u(c,\cdot)$, we define the function set

$$U = \{u(c,\cdot) \in C(I,X) \mid c \in \mathbb{R}^n\}. \tag{1}$$

Then, we consider the following approximation problem

$\mathcal{A}$**:** For given $f \in C(I,X)$ (and U) determine an element $c^* \in \mathbb{R}^n$, such that

$$|||f - u(c^*,\cdot)|||_X = \rho_U(f) := \inf_{c \in \mathbb{R}^n} |||f - u(c,\cdot)|||_X. \tag{2}$$

Such a function $u(c^*,\cdot)$ is called a best approximation of f from U. Problem $(\mathcal{A})$ can be written as a semi-infinite optimization problem

$\mathcal{P}$: $\min c_{n+1}$ subject to

$$g(c, c_{n+1}; t) := \|f(t) - u(c,t)\|_X - c_{n+1} \le 0 \quad \text{for all } t \in I. \tag{3}$$

When the space U is a linear space,

$$U = U_n = \{u(c,\cdot) = \sum_{j=1}^{n} c_j u^j, \quad c = (c_1, \ldots, c_n) \in \mathbb{R}^n\}, \tag{4}$$

with given linearly independent functions $u^j \in C(I,X)$, $j = 1, \ldots, n$, then, $(\mathcal{A})$ represents a linear problem. In this case, the functions $g(c, c_{n+1}; t)$ in (3) are convex in $(c, c_{n+1}) \in \mathbb{R}^{n+1}$ for any fixed $t \in I$, i.e. $(\mathcal{P})$ is a convex semi-infinite optimization problem. If we choose $X = \mathbb{R}^d$ endowed with the common p-norms

$$\|x\|_p = \left(\sum_{m=1}^{d} |x_m|^p\right)^{1/p}, 1 \le p < \infty, \ \ \|x\|_\infty = \max_{1 \le m \le d} |x_m| \ ,$$

then, $(\mathcal{A})$ resp. $(\mathcal{P})$ will be denoted by $(\mathcal{A}_p)$ resp. $(\mathcal{P}_p)$ and $|||f|||_X$ by $|||f|||_p$.

The linear space U_n is called Haar-space if for any $f \in C(I,X)$ there is a unique best-approximation $u^* = u(c^*,\cdot) \in U_n$. If for any $f \in C(I,X)$ a best approximation $u^* \in U_n$ is strongly unique, i.e. with some neighborhood V of u^* and some $\kappa > 0$ we have

$$|||f-u|||_X \geq |||f-u^*|||_X + \kappa |||u-u^*|||_X^k \quad \text{for all } u \in V \cap U_n \tag{5}$$

with $k=1$, then U_n is called strong Haar-space. If the same holds for $k=2$, then, U_n is called a strict Haar-space. In [9] we dealt with the question of strict and strong (linear) Haar-spaces U_n. Characterizations of such Haar-spaces have been given for different norms $\|\cdot\|_X$. If for example, X is a Hilbert space, any Haar-space U_n is a strict Haar-space but never a strong Haar-space. The space U_n of Bernstein curves in $X = \mathbb{R}^d$, (endowed with $\|\cdot\|_p$, $1 \leq p \leq \infty$) for example,

$$U_n = \left\{ u(c,t) = \sum_{j=1}^{n} c_j b_{n,j}(t) \cdot e \right\}, \quad b_{n,j}(t) = \binom{n}{j} t^j (1-t)^{n-j}, \tag{6}$$

(where $e = (1,\ldots,1)^T \in \mathbb{R}^d$) cannot be a Haar-space for dimension $d \geq 2$ (cf. [9, Corollary 2 c]). However, if we consider the approximation of a given function $g \in C^{d-1}(I,\mathbb{R})$ simultaneously with its derivatives, i.e. we approximate $f = (g, g^{(1)}, \ldots, g^{(d-1)})$, by functions from

$$\widetilde{U}_n = \left\{ u(c,t) = \sum_{j=1}^{n} c_j \; (b_{n,j}(t), b_{n,j}^{(1)}(t), \ldots, b_{n,j}^{(d-1)}(t))^T \right\} \tag{7}$$

then, $\widetilde{U}_n$ is a strict Haar-space for $(\mathcal{A}_2)$ (i.e. $X = \mathbb{R}^d, \|\cdot\|_2$) (cf. [9, Corollary 2 d]) . For further examples and uniqueness results we refer to [12], [9] and the references there.

The classical complex linear Chebyshev approximation problem is a special case of problem $(\mathcal{A}_p)$ (or $(\mathcal{P}_p)$). As pointed out in [9, Sect. 5] by choosing $p = d = 2$, a complex Chebyshev approximation problem (on a region $G \subset \mathbb{C}$) with a complex Haar-space H_N of dimension N is equivalent to a $\mathbb{R}^2$-valued approximation problem in a corresponding $2N$-dimensional Haar-space U_{2N}. Consequently, the general theorems in [9] contain the result of Newman/Shapiro in [10], stating that a complex Haar-space is not a strong, but a strict Haar-space, as a special case. Note, that in [9, Theorem 6] it has been shown that when $d > 1$, $X = \mathbb{R}^d$ with $\|\cdot\|_p$, $1 < p < \infty$, no strong Haar-space U_n exist.

In this paper we are interested in numerical aspects of the approximation problems. Many relevant results from complex Chebyshev approximation (cf. eg. [11][17]) will be generalized to the vector-valued case. The paper is organized as follows. In Section 2 we discuss different equivalent formulations of the approximation problem and compare the problem for different norms $||\cdot||_X$. Section 3 is devoted to numerical methods and numerical examples. In Section 4 we consider genericity questions.

2 Numerical aspects of linear problems

When $U = U_n$ is a linear space (cf. (4)) problem $(\mathcal{A})$ or $(\mathcal{P})$ can be put in the form of a linear semi-infinite optimization problem as follows. Let X' denote the continuous dual of X and B' the unit ball of X', $B' = \{x' \in X' \mid ||x'||_{X'} = 1\}$. X being a Banach space, by the Theorem on Bounded Linear Functionals to any $x \in X$ there exists a $x' \in B'$ such that $x'(x) = ||x||_X$. Therefore, for any $\alpha \geq 0$ it follows

$$||x||_X \leq \alpha \Longleftrightarrow x'(x) \leq \alpha \qquad \text{for all } x' \in B'.$$

By using this relation in (3) problem $(\mathcal{P})$ (or $(\mathcal{A})$) becomes,

$\mathcal{PL}:\quad \min c_{n+1} \quad$ subject to $(c, c_{n+1}) \in Z$ where

$$Z = \{(c, c_{n+1}) \in I\!R^{n+1} \quad | \quad \sum_{j=1}^{n} c_j x'(u^j(t)) - c_{n+1} \leq x'(f(t)) \quad \text{for all } (t, x') \in I \times B'\}. \tag{8}$$

If we suppose, that $d = \dim X$ is finite, then $B' \subset I\!R^d$ is compact and $(\mathcal{PL})$ represents a common linear semi-infinite optimization problem (cf. [4] for a general introduction to semi-infinite problems). Now let be $X = I\!R^d$ with $||\cdot||_p$, $1 \leq p \leq \infty$. Then we have $B' = B_q = \{y \in I\!R^d \mid ||y||_q = 1\}$ where $\frac{1}{p} + \frac{1}{q} = 1$. Consequently, $(\mathcal{A}_p)$ or $(\mathcal{P}_p)$ reads

$\mathcal{PL}_p:\quad \min c_{n+1} \quad$ subject to $(c, c_{n+1}) \in Z_p$ where

$$Z_p = \{(c, c_{n+1}) \in I\!R^{n+1} \quad | \quad \sum_{j=1}^{n} c_j y^\top u^j(t) - c_{n+1} \leq y^\top f(t) \quad \text{for all } (t, y) \in I \times B_q\}. \tag{9}$$

For the cases $p = 1, \infty$ the inequalities in (9) need only to be satisfied for the finitely many extreme points (vertices) of B_∞, B_1. For $p = \infty$ for example we obtain

$$Z_\infty = \{(c, c_{n+1}) \in I\!R^{n+1} \quad | \quad \mp \sum_{j=0}^{n} c_j u_m^j(t) - c_{n+1} \leq \mp f_m(t) \quad \text{for all } t \in I,\ m = 1, \ldots, d\}. \tag{10}$$

Characterizations of strong Haar-spaces for $p = 1, \infty$ have been given in [9, Section 3].
Let be given a linear approximating space $U = U_n \subset C(I, \mathbb{R}^d)$ and a function $f \in C(I, \mathbb{R}^d)$. Let $E(c,t)$ denote the error function $E(c,t) = f(t) - \sum_{j=1}^{n} c_j u^j(t)$. Furthermore let $\rho_n(f,p)$ stand for the minimal deviation $\rho_{U_n}(f)$ (cf. (2)) of problem $(\mathcal{A}_p)$ or $(\mathcal{P}_p)$. We are interested in connections between the problems $(\mathcal{A}_p)$ for different values of p.
Let be given p, s, $1 \leq p < s \leq \infty$. Then, for any vector $E(c,t) \in \mathbb{R}^d$ (c, t fixed) the inequality

$$||E(c,t)||_s \leq ||E(c,t)||_p \leq d^{(\frac{1}{p}-\frac{1}{s})}||E(c,t)||_s \tag{11}$$

holds. By taking the maximum with respect to $t \in I$ and the minimum for c, we find

$$\rho_n(f,s) \leq \rho_n(f,p) \leq d^{(\frac{1}{p}-\frac{1}{s})}\rho_n(f,s). \tag{12}$$

In the context of complex Chebyshev approximation this relation has been given in [11]. Moreover, we can get estimates for the differences between the best approximations u_p^* of $(\mathcal{A}_p)$ for different values p. We need a 'global' version of the inequality in (5).

Lemma 1 *Suppose, (5) holds for $u^* = u_p^* \in U_n$ with some $\kappa > 0$ and $k = 1$ or $k = 2$. Then, there is a constant κ_0, $\kappa_0 \leq \kappa$, such that globally in case $k = 1$:*

$$|||f-u|||_p - |||f-u_p^*|||_p \geq \kappa_0 |||u-u_p^*|||_p \qquad \forall u \in U_n, \tag{13}$$

and in case $k = 2$:

$$|||f-u|||_p - |||f-u_p^*|||_p \geq \kappa_0 \min\{|||u-u_p^*|||_p^2, |||u-u_p^*|||_p\} \; \forall u \in U_n. \tag{14}$$

Proof: Define $B_\sigma(u_p^*) = \{u \in U_n \mid |||u-u_p^*|||_p < \sigma\}$ $(\sigma > 0)$ and suppose that for $\epsilon > 0$ we have $B_\epsilon(u_p^*) \subset V$ (with V the neighborhood of $u^* = u_p^*$ in (5)). Put $\delta = 4|||f-u_p^*|||_p$, $K = \overline{B_\delta(u_p^*) \backslash B_\varepsilon(u_p^*)}$ and

$$\gamma := \begin{cases} \inf\limits_{u \in K} \; |||f-u|||_p - |||f-u_p^*|||_p & \text{if } \delta > \varepsilon \\ \infty & \text{otherwise.} \end{cases}$$

We only consider the case $\delta > \varepsilon$. Then, since u_p^* is the unique best approximation and K is compact it follows $\gamma > 0$. Moreover for all $u \in K$ we find $|||f-u|||_p - |||f-u_p^*|||_p \geq \frac{\gamma}{\delta}\delta \geq \frac{\gamma}{\delta}|||u-u_p^*|||_p$, and for all $u \in U_n \backslash B_\delta(u_p^*)$

$$|||f-u|||_p - |||f-u_p^*|||_p \geq |||u-u_p^*|||_p - 2\,|||f-u_p^*|||_p \geq \frac{1}{2}|||u-u_p^*|||_p.$$

Hence, (13) is valid with $\kappa_0 = \min\{\kappa, \frac{\gamma}{\delta}, \frac{1}{2}\}$. In the same way we can prove (14). $\square$

Let us now suppose that u_p^* is strongly unique solution for $(\mathcal{A}_p)$, i.e. (13) holds with $\kappa_0 > 0$. Then, for any solution u_s of $(\mathcal{A}_s)$ with $s < p$ we find by using (11), (12)

$$|||f - u_s|||_p \;\leq\; |||f - u_s|||_s \;=\; \rho_n(f,s) \;\leq\; d^{|\frac{1}{p}-\frac{1}{s}|}\rho_n(f,p). \tag{15}$$

The same is true for $s > p$. Consequently, by putting $u = u_s$ in (13) we obtain

$$|||u_s - u_p^*|||_p \leq \frac{1}{\kappa_0}(d^{|\frac{1}{p}-\frac{1}{s}|} - 1)\rho_n(f,p). \tag{16}$$

If u_p^* is a strictly unique solution for $(\mathcal{A}_p)$, i.e. (5) is valid with $k = 2$, then by using (14) it follows

$$\min\left\{|||u_s - u_p^*|||_p^2, |||u_s - u_p^*|||_p\right\} \;\leq\; \frac{1}{\kappa_0}\left(d^{|\frac{1}{p}-\frac{1}{s}|} - 1\right)\rho_n(f,p) \tag{17}$$

for any solution u_s of $(\mathcal{A}_s)$, $1 \leq s \leq \infty$. Thus in view of (16) or (17), when $\rho_n(f,p)$ is small, roughly speaking, the difference between any solution u_s of $(\mathcal{A}_s)$ and the strongly or strictly unique solution u_p^* of $(\mathcal{A}_p)$ must be small too. A more precise investigation of the dependence of the solution u_p^* of $(\mathcal{A}_p)$ on p could be done by regarding $(\mathcal{A}_p)$ or $(\mathcal{P}_p)$ as a parametric optimization problem with parameter p.

The following useful observation is trivial. Let be given $s < p$ and a solution u_p^* of $(\mathcal{A}_p)$ (or $(\mathcal{P}_p)$). Now, if u_p^* is feasible for $(\mathcal{P}_s)$, i.e. with $c_{n+1} = \rho_n(f,p)$ we have

$$\max_{t\in I} \|f(t) - u_p^*(t)\|_s \leq c_{n+1},$$

then in view of (11) and (12), u_p^* is a solution of $(\mathcal{A}_r)$ for all r, $s \leq r \leq p$. A sufficient condition for such a situation is given in the following corollary. This corollary generalizes known results in complex Chebyshev approximation. Let us first define the condition

C1: Let be given $f \in C(I, I\!R^d)$, $U_n \subset C(I, I\!R^d)$ and $u^* \in U_n$, such that for $E^* = f - u^*$:

1. There exist $\rho \geq 0, N \in I\!N$, and a component m_0, $1 \leq m_0 \leq d$ such that with $2k = N + 1 \leq n + 1$ points $t_l^{\pm}$, $l = 1, \ldots, k$, $0 \leq t_1^+ < t_1^- < t_2^+ < \ldots < t_{k-1}^- < t_k^+ < t_k^- \leq 1$, and $\sigma = 1$ or $\sigma = -1$ we have for the ${m_0}^{th}$ component of E^*,
$$E_{m_0}^*(t_l^{\pm}) = \pm\sigma\rho, \qquad l = 1, \ldots, k.$$
2. For some p, $1 \leq p < \infty$, $\quad \|E^*(t)\|_p \leq \rho \qquad$ for all $t \in I$.

For $d = 2$ condition (C1) is satisfied with $p = 2$, if for example the error function $E^* = f - u^*$ is circular (around 0) with respect to $\|\cdot\|_2$ with winding number $\geq k$. Such circular error curves play a certain role in complex Chebyshev approximation (case $d = p = 2$) (cf. [18]).

Lemma 2 *Suppose, condition (C1) is satisfied with $U_n = span\{u^j, j = 1, \ldots, n\}$. Assume, that the linear space $S_{m_0} = \left\{\sum_{j=1}^n c_j u^j_{m_0}\right\} \subset C(I, \mathbb{R})$ is a Haar-space of dimension $\leq N$ ($N \leq n$), i.e. if $v \in S_{m_0}$ has more than $N-1$ distinct zeros on I then $v \equiv 0$. Then, $u^* \in U_n$ is a best approximation for $(\mathcal{A}_s)$ for all s, $p \leq s \leq \infty$.*
(Note, that this condition on S_{m_0} is satisfied for many spaces U_n.)

Proof: We can assume $\rho > 0$, i.e. $f \notin U_n$. Consider the problem $(\mathcal{A}_\infty)$ in the form $(\mathcal{PL}_\infty)$ (cf. (9), (10)). In view of $|||E^*|||_\infty \leq |||E^*|||_p \leq \rho = |E^*_{m_0}(t_l^\pm)|$, the function $u^* = \sum_{j=1}^n c_j^* u^j$ is feasible for $(\mathcal{PL}_\infty)$ with $c_{n+1} = \rho$. Let $(\widetilde{\mathcal{PL}_\infty})$ denote the problem corresponding to $(\mathcal{PL}_\infty)$ of minimizing c_{n+1} subject to $(c, c_{n+1}) \in \widetilde{Z}_\infty$ where $\widetilde{Z}_\infty$ denotes the larger set (cf. (10))

$$\widetilde{Z}_\infty = \Big\{(c, c_{n+1}) \in \mathbb{R}^{n+1} \;\Big|\; \pm\left(f_{m_0}(t) - \sum_{j=1}^n c_j u^j_{m_0}(t)\right) \leq c_{n+1} \quad \text{for } t = t_l^\pm, l = 1, \ldots, k\Big\}.$$

Then (c^*, ρ) is a solution of $(\widetilde{\mathcal{PL}_\infty})$. In fact, if we suppose that this is not true, then the so-called Kolmogoroff condition (cf. e.g. [8, p.28]) would imply the existence of $u \in U_n$ such that (cf. (C1, 1.))

$$(E^*(t_l^\pm))^T \cdot u(t_l^\pm) = \pm\sigma\rho u_{m_0}(t_l^\pm) > 0, \quad l = 1, \ldots, k.$$

Since $u_{m_0} \in S_{m_0}$ and $2k - 1 \geq N$, this contradicts the assumption on S_{m_0}. In view of $Z_\infty \subset \widetilde{Z}_\infty$, the function u^* is a solution of $(\mathcal{A}_\infty)$. Again by condition (C1, 2.) u^* is feasible for $(\mathcal{A}_p)$ and by the arguments above, u^* is a solution of $(\mathcal{A}_s)$, $p \leq s \leq \infty$. □

Now, let p, f, U_n be fixed, $1 < p < \infty, f \in C(I, \mathbb{R}^d), U_n \subset C(I, \mathbb{R}^d)$. We want to compare the different forms $(\mathcal{P}_p)$ and $(\mathcal{PL}_p)$ of the same approximation problem. To obtain $(\mathcal{PL}_p)$, any (convex) constraint $g(c, c_{n+1}; t) = \|f(t) - \sum_{j=1}^n c_j u^j(t)\|_p - c_{n+1} \leq 0$, $t \in I$ fixed, in $(\mathcal{P}_p)$ (cf. (3)) is replaced by infinitely many constraints

$$\widehat{g}(c, c_{n+1}; t, y) = \sum_{j=1}^n c_j \; y^T u^j(t) - c_{n+1} - y^T f(t) \leq 0 \qquad \text{for all } y \in B_q. \quad (18)$$

Note, that in contrast to the constraints g, the constraints $\widehat{g}$ are linear in (c, c_{n+1}) (for fixed t, y). With a discretization D of B_q ($D \neq \emptyset$) let $(\mathcal{PL}_p(D))$ denote the problem of minimizing c_{n+1} on the feasible set (cf. (9), (18))

$$Z_p(D) = \{(c, c_{n+1}) \in I\!R^{n+1} \mid \widehat{g}(c, c_{n+1}; t, y) \leq 0 \ \forall t \in I, \ y \in D\}. \qquad (19)$$

Let (c^D, c^D_{n+1}) denote a solution of $(\mathcal{PL}_p(D))$ with value

$$\rho^D := c^D_{n+1} = \min_{c \in I\!R^n} \max_{t \in I} \max_{y_D \in D} y_D^T (f(t) - u(c, t)). \qquad (20)$$

Suppose, ρ denotes the approximation error $\rho = \rho_n(f, p)$ of $(\mathcal{PL}_p)$. Then, as we shall see, ρ^D will converge to ρ quadratically with a 'mesh size' of D. A natural way to define the mesh size δ of D with respect to B_q is given by (remember that 1/p+1/q=1)

$$\cos\delta = \min_{x \in B_p} \max_{y_D \in D} y_D^T x \ , \qquad (0 \leq \delta \leq \pi). \qquad (21)$$

(Note, that for $D = B_q$ we have $\cos\delta = 1$, i.e. $\delta = 0$). If we choose for example $p = 2, d = 2$ and the discretization $D = \{y_D = (\cos\frac{2\pi\nu}{N}, \sin\frac{2\pi\nu}{N})^T \mid \nu = 0, \ldots, N-1\}$ $(N \in I\!N)$, then we have $\delta = \frac{\pi}{N}$. With these notations it follows that $\rho^D \leq \rho \leq \frac{\rho^D}{\cos\delta}$.
Here, the inequality $\rho^D \leq \rho$ is valid since $Z_p \subset Z_p(D)$. To prove $\rho \leq \rho^D / \cos\delta$ we note that $\frac{x}{\|x\|_p} \in B_p$ for $0 \neq x \in I\!R^d$. Hence by definition (21), $\cos\delta \leq \max_{y_D \in D} y_D^T \frac{x}{\|x\|_p}$ or $\|x\|_p \cos\delta \leq \max_{y_D \in D} y_D^T x$ for any $x \in I\!R^d$. Putting $x = f(t) - u(c, t)$ we get

$$\|f(t) - u(c, t)\|_p \leq \max_{y_D \in D} y_D^T (f(t) - u(c, t)) \frac{1}{\cos\delta}. \qquad (22)$$

Taking $\min_{c \in I\!R^n} \max_{t \in I}$ we have proved the desired inequality. A Taylor expansion of $1/\cos\delta$ leads to the relation

$$\rho - \rho^D \leq \rho \left(\frac{1}{\cos\delta} - 1\right) = \rho \left(\frac{\delta^2}{2} + \mathrm{O}(\delta^4)\right). \qquad (23)$$

For complex approximation, an inequality as in (23) has been obtained in [11]. In case, that (c^*, c^*_{n+1}) with corresponding $u^* = u(c^*, \cdot)$ is a strongly unique solution of $(\mathcal{PL}_p)$ (or strictly of order $k = 2$) we find for the solution $u^D = u(c^D, \cdot)$ of $(\mathcal{PL}_p(D))$ by (22) that $|||f - u^D|||_p \leq \rho^D / \cos\delta$ and in view of Lemma 1 for $k = 1$ (or $k = 2$),

$$|||u^* - u^D|||_p^k \leq \frac{1}{\kappa_0}(|||f - u^D|||_p - |||f - u^*|||_p) \leq \frac{\rho}{\kappa_0}\left(\frac{\delta^2}{2} + \mathrm{O}(\delta^4)\right).$$

3 Numerical methods and examples

Let us again assume that $X = \mathbb{R}^d$ with $\|\cdot\|_X = \|\cdot\|_p$, $1 \le p \le \infty$, and that $U = U_n \subset C(I, \mathbb{R}^d)$ is a linear space. To solve problems $(\mathcal{A}_p)$ or $(\mathcal{P}_p)$ we will consider two different methods.

Method I: (*Approach with techniques from nonlinear optimization applied to* $(\mathcal{P}_p)$.)

Step I$_1$: Choose a discretization $Q \subset I = [0,1]$ and solve instead of $(\mathcal{P}_p)$ (cf. (3)) the following finite convex problem

$$\mathcal{P}_p(Q): \quad \min c_{n+1} \quad \text{s.t.} \quad g(c, c_{n+1}; t) \le 0 \quad \text{for all } t \in Q.$$

To find a better approximation for a solution of $(\mathcal{P}_p)$ continue with Step I_2.

Step I$_2$: Take the solution $(\widehat{c}, \widehat{c}_{n+1})$ of $(\mathcal{P}_p(Q))$ calculated in Step I$_1$ together with appropriate active points $t_1, \ldots, t_k \in Q$, $0 \le t_1 < t_2 < \ldots < t_k \le 1$ and corresponding multipliers $\lambda_1, \ldots, \lambda_k$ as starting values for a Newton-method for solving the Kuhn-Tucker equations of $(\mathcal{P}_p)$: With $k,\ 1 \le k \le n+1,\ t_i \in I,\ \lambda_i > 0,\ i = 1, \ldots, k$,

$$\sum_{i=1}^{k} \lambda_i D_C g(c, c_{n+1}; t_i) = -e_{n+1} \tag{24}$$

$$g(c, c_{n+1}; t_i) = 0,\ i = 1, \ldots, k \tag{25}$$

$$D_t g(c, c_{n+1}; t_i) = 0,\ i = 1, \ldots, k,\ 0 < t_i < 1. \tag{26}$$

Here, $D_C g$, $D_t g$ denote the partial derivatives with respect to $(c, c_{n+1}), t$ resp. and e_{n+1} is the unit vector $e_{n+1} = (0, \cdots, 0, 1)^T \in \mathbb{R}^{n+1}$. Note that the equations (24),(25) represent the Karush-Kuhn-Tucker conditions whereas (26) is simply the necessary condition for an inner maximum of g (w.r.t. t). To solve $(\mathcal{P}_p(Q))$ we have used a routine (VF13) from the 'Harwell Library' by M.J.D. Powell. This routine is an efficient implementation of a Quasi-Newton-SQP-method (cf. [14] for further information). To use this routine all functions involved must be of type C^1.

Method II: (*Approach from linear semi-infinite optimization applied to* $(\mathcal{PL}_p)$.) To solve $(\mathcal{PL}_p)$ calculate a solution of the problem: $\min\ c_{n+1}$ on the (larger) feasible set

$$Z_p(Q) = \left\{ (c, c_{n+1}) \in \mathbb{R}^{n+1} \,\middle|\, \sum_{j=1}^{n} c_j y^T u^j(t) - c_{n+1} \le y^T f(t) \quad \forall\ (t, y) \in Q \right\}$$

where Q is a discretization of $[0,1] \times B_q$. For Method II we have used a routine (DISKR) by R. Hettich. This routine is an efficient implementation of a Remez-type-Simplex algorithm. See [3] for a description.

The problems $(\mathcal{P}_\infty)$ have been solved by

Method III: Apply as in Method II the routine (DISKR) to $(\mathcal{PL}_\infty)$ (cf. (9)) restricted to a discretization Q of I in (10).

Note, that also the Methods II, III could be completed by a step corresponding to step I_2. We remark however, that the routine (DISKR) can handle a very fine discretization such that an extra "Newton step" could be needless. Moreover, by the remarks in Section 2 concerning the discretization D of B_q, this discretization D can be chosen coarser than the discretization of I. In the following we will compare approximate solutions

$$u(c^*, t) = \sum_{j=1}^{n} c_j^* u^j(t) \qquad * = \mathrm{I,II,III} \tag{27}$$

of $(\mathcal{P}_2)$, $(\mathcal{P}_\infty)$ calculated by the methods I, II, III. Let ρ_2^c, ρ_∞^c denote the approximation error of the computed best approximations of $(\mathcal{P}_2)$, $(\mathcal{P}_\infty)$. By ρ_2^d, ρ_∞^d we mean the corresponding errors of the solutions $c^{\mathrm{I}}, c^{\mathrm{II}}, c^{\mathrm{III}}$ of $(\mathcal{P}_2)$, $(\mathcal{P}_\infty)$ (cf. (27)) on the discretization Q used in I_1, II, III. In view of the remarks in Section 2 we have

$$\rho_2^d \le \rho_2 \le \rho_2^c, \qquad \rho_\infty^d \le \rho_\infty \le \rho_\infty^c.$$

Hence, the difference $\rho_2^c - \rho_2^d$, $\rho_\infty^c - \rho_\infty^d$ gives us an indication about the difference between the exact and the computed solutions. Note, that by (12) we have $\rho_\infty \le \rho_2 \le \sqrt{d}\,\rho_\infty$. All computations have been performed in (FORTRAN) double precision.
As a first test problem we have considered a problem $(\mathcal{P}_2)$ which corresponds to the complex Chebyshev approximation problem of the approximation of $f(z) = 1/(z-\alpha)$ by polynomials $p(z) = \sum_{j=0}^{K-1} c_j z^j$ on the unit circle $|z| = 1$. Here, the best approximation can be given explicitly. The error curve is always circular with winding number K (cf. e.g. [7]) such that Lemma 2 applies. The best approximations are always strongly unique. If we choose $\alpha = 2 + i$ and $K = 8$ $(n = 16)$ the coefficients c_j of the best approximation are:

$c_0 = -.4 + i\,.2$	$c_1 = -.12 + i\,.16$	$c_2 = -.016 + i\,.088$
$c_3 = .0112 + i\,.0384$	$c_4 = .01216 + i\,.01312$	$c_5 = .007488 + i\,.002816$
$c_6 = .0035584 - i\,.0003712$	$c_7 = .0016864 - i\,.0010752$	

The coefficients calculated by I_1 coincided with these values up to the first 10 digits. Note that since the error function $E(\overline{c}, \cdot)$ is circular, the solution of $(\mathcal{P}_2(Q))$ in I_1 with $|Q| \geq 2K+1$ coincides with the solution of our problem $(\mathcal{P}_2)$. Thus in this example, step I_2 is not necessary (and not possible since the extremal points are not unique (any different $2K+1$ points would do)). In methods II and III we have chosen discretizations leading to approximations which were exact up to the 6-th digit. The errors obtained are given in Table 1. In the following tables a factor 10^d is denoted simply by d, e.g. .12345 2 means $.12345 \cdot 10^2$.

Table 1:

exact solution:	$\rho_2 = \rho_\infty = .894427191\ {-3}$	
I_1:	$\rho_2^d = .894427019..\ {-3}$	$\rho_2^c = .894427730..\ {-3}$
II:	$\rho_2^d = .894417761..\ {-3}$	$\rho_2^c = .894544448..\ {-3}$
III:	$\rho_\infty^d = .893846764..\ {-3}$	$\rho_\infty^c = .895908135..\ {-3}$

In the next example we consider the approximation of $f(t) = \begin{pmatrix} e^t \\ 1/(t-1.1) \end{pmatrix}$ by functions of coordinate type (cf. [9, Section 5]):

$$u_n(c,t) = \sum_{j=0}^{n_1-1} c_j \begin{pmatrix} t^j \\ 0 \end{pmatrix} + \sum_{j=0}^{n_2-1} c_{n_1+j} \begin{pmatrix} 0 \\ t^j \end{pmatrix}. \tag{28}$$

By [9, Corollary 2] the corresponding spaces U_n are strict Haar-spaces for $p=2$ iff $n_1 = n_2 \pm 1$, but never Haar-spaces for $p=\infty$. Note, that in the case $p=\infty$ the approximations in the two components are completely decoupled. So, uniqueness can only occur if the error of the Chebyshev best approximation of $f_1(t) = e^t$ by polynomials from Π_{n_1-1} coincides with the error of the approximation of $f_2(t) = 1/(t-1.1)$ in Π_{n_2-1}. Table 2 (Table 3) contains the numerical results for $n_1 = 2$, $n_2 = 6$ ($n_1 = 5, n_2 = 6$).

Table 2: (results for $n_1 = 2$, $n_2 = 6$) || **Table 3:** (results for $n_1 = 5$, $n_2 = 6$)

	Table 2: I_1	Table 2: II	Table 2: III	Table 3: I_1	Table 3: II	Table 3: III
c_0	.85970 0	.85973 0	.86929 0	.10001 1	.10088 1	.81116 1
c_1	.17896 1	.17945 1	.19208 1	.99833 0	.67080 1	.81907 0
c_2	-.76454 0	-.74872 0	-.70677 0	.51200 0	.21567 1	.14081 1
c_3	-.11129 2	-.11249 2	-.12202 2	.13663 0	-.24741 1	-.18605 1
c_4	.94702 2	.95855 2	.10164 2	.71261 -1	.13649 1	.17427 1
c_5	-.30551 3	-.30943 3	-.32411 3	-.71163 0	-.70695 0	-.70674 0
c_6	.39779 3	.40263 3	.41865 3	-.11974 2	-.12190 2	-.12201 2
c_7	-.18489 3	-.18685 3	-.19307 3	.99875 2	.10154 3	.10163 3
c_8				-.31914 3	-.32383 3	-.32408 3
c_9				.41292 3	.41834 3	.41862 3
c_{10}				-.19077 3	-.19295 3	-.19306 3

The errors of the approximations in Table 2 (with the used discretization) are: For I_1, $\rho_2^d = .2014$, $\rho_2^c = .2272$. For II, $\rho_2^d = .2130$, $\rho_2^c = .2146$ and for III, $\rho_\infty^d = .2023$, $\rho_\infty^c = .2024$.
The corresponding errors in Table 3 are: For I_1, $\rho_2^d = .1974$, $\rho_2^c = .2119$, for II, $\rho_2^d = .2021$, $\rho_2^c = .2027$ and for III, $\rho_\infty^d = .2023$, $\rho_\infty^c = .2024$. The error of c^{III} in the $\|\cdot\|_2$-norm is $\widehat{\rho} = .2861$. In all cases, I_2 could not be performed, even in the case $n_1 = 5$, $n_2 = 6$ where theoretically the best approximation must be strictly unique. The difference in some coefficients of the approximations c^I and c^{II} with nearly the same error indicates that the problem is very ill-conditioned.

As a last example we consider simultaneous approximations with spaces as in (7). We approximate

$$f(t) = \begin{pmatrix} \sin & 2\pi t \\ 2\pi\cos & 2\pi t \end{pmatrix} \qquad \text{by} \qquad p_n(t) = \sum_{j=0}^{n-1} c_j \begin{pmatrix} t^j \\ jt^{j-1} \end{pmatrix}.$$

By Corollary 2 in [9] the corresponding spaces $\widetilde{U}_n$ are strict Haar-spaces for $p = 2$. Table 4 and Table 5 contain the results for $n = 5$ and $n = 10$.

Table 4: (results for $n = 5$)

	I_1+I_2	II	III
c_0	.87070 -1	.83825 -1	.17898 1
c_1	.82021 1	.82031 1	.82030 1
c_2	-.25128 2	-.25133 2	-.25132 2
c_3	.16752 2	.16756 2	.16755 2
c_4	-.62559 -16	-.10889 -2	-.49960 -13

For I the computed errors are $\rho_2^d = \rho_2^c = 1.92091$ for II, $\rho_2^d = 1.9199$, $\rho_2^c = 1.9217$ and for III, $\rho_\infty^d = 1.9198$, $\rho_\infty^c = 1.9208$. The error of c^{III} in the $\|\cdot\|_2$-norm is $\widehat{\rho} = 2.6247$. In I_2 there have been chosen 5 extremal points. So by theoretical results, c^I is not strongly unique (e.g. [9]).

Table 5: (results for $n = 10$)

	I_1	II	III
c_0	.28770 -4	.76597 -5	-.24131 -3
c_1	.62872 1	.62829 1	.62829 1
c_2	-.25782 0	.24965 -1	.24873 -1
c_3	-.37994 2	-.41881 2	-.41879 2
c_4	-.17991 2	.50647 1	.50569 1
c_5	.12807 3	.56062 2	.56092 2
c_6	-.51799 2	.75768 2	.75727 2
c_7	-.83884 2	-.21288 3	-.21283 3
c_8	.74013 2	.14342 3	.14340 3
c_9	-.16447 -2	-.31872 2	-.31867 2

For I we obtained $\rho_2^d = 4.081\cdot 10^{-3}$, $\rho_2^c = 4.522\cdot 10^{-3}$, for II, $\rho_2^d = 2.517\cdot 10^{-4}$, $\rho_2^c = 2.561 \cdot 10^{-4}$ and for III, $\rho_\infty^d = 2.525 \cdot 10^{-4}$, $\rho_\infty^c = 2.527 \cdot 10^{-4}$. The error of c^{III} in the $\|\cdot\|_2$-norm is $\hat{\rho} = 3.493 \cdot 10^{-4}$. Here, I_1 failed to find a nearly best approximation. Since the best approximations for $p = 2$ must be strictly unique, by our remarks above (cf. (17)), the difference between the approximations c^{II} for $p = 2$ and c^{III} for $p = \infty$ must decrease with the error ρ. This can be seen in this example (compare the first components c_0).

In nearly all computed examples, the step from I_1 to I_2 has been difficult or not possible at all due to non-uniqueness or "nearly non-uniqueness". In this respect the approaches II and III are more robust and more precise. Clearly, to avoid I_2, method I_1 could be combined with techniques of the routine (DISKR) (cf. [2]). Generally, I_1 performed very "rapidly" especially for small problems. Note that in method I all functions are supposed to be continuously differentiable whereas methods II, III only assume continuity.

4 Genericity aspects

This section is devoted to genericity results for problems $(\mathcal{A})$ or $(\mathcal{P})$ (cf. (3)) for the case that $U = U_n$ is a linear space. Roughly speaking, genericity results are statements which assert that there exists a dense and open subset $\mathcal{F}_+$ (the 'nice' set) of $C(I, X)$ consisting of functions f having desirable properties. For example, functions f with (strongly/strictly) unique best approximation. Problem $(\mathcal{P})$ is a semi-infinite optimization problem with a special structure. Since a genericity analysis is performed with respect to a parametrization of the problem set which employs this special structure

the genericity results for general semi-infinite optimization (cf. [5], [6], [15]) are not directly applicable.
For certain types of such problems only 'negative' genericity results hold (negative in the sense, that the complement of the 'nice set' is open and dense). Consider for example the approximation of $f = (f_1, f_2) \in C(I, \mathbb{R}^2)$ by functions (of coordinate type) in (28) (n_1, n_2 fixed) in the norm $|||\cdot|||_\infty$. As remarked above, a best approximation can only be unique if for the errors $\rho^1(f_1) = \min_{p\in\Pi_{n_1-1}} |||f_1-p|||_\infty$ in the first and $\rho^2(f_2) = \min_{p\in\Pi_{n_2-1}} |||f_2-p|||_\infty$ in the second component we have $\rho^1(f_1) = \rho^2(f_2)$. By noticing that the errors ρ^1, ρ^2 depend continuously on f_1, f_2, the following 'negative' genericity result holds for this special problem: Define the 'bad' set $\mathcal{F}_- = \{f = (f_1, f_2) \in C(I, \mathbb{R}^2) \mid f$ does not have a unique best approximation$\}$. Then, the set $\mathcal{F}_-$ is dense and open in $C(I, \mathbb{R}^2)$ w.r.t. $|||\cdot|||_\infty$.
We will now consider genericity results for the following class of problems. Let in the sequel $U_n \subset C(I, \mathbb{R}^d)$, $q, d \in \mathbb{N}$, p, $1 < p < \infty$, be fixed. Let $C^q(I, \mathbb{R}^d)$ be endowed with $|||\cdot|||_p$. Suppose, U_n is a Haar-space (i.e. U_n satisfies the condition (C_2) in [9, Theorem 7]). It has been shown in [9] that U_n cannot be a strong Haar-space, i.e. there are always functions $f \in C^q(I, \mathbb{R}^d)$ such that the best approximation is not strongly unique. Then, we can define the 'nice' set

$$\mathcal{F}_+^q = \{f \in C^q(I, \mathbb{R}^d) | f \text{ has a strongly unique best approximation in } U_n\}.$$

Theorem 1 *The set $\mathcal{F}_+^q$ is open and dense in $C^q(I, \mathbb{R}^d)$ (with $|||\cdot|||_p$).*

For the complex Chebyshev approximation problem (i.e. $p = d = 2$) such a result has been proved by Blatt (for $q = 0$) in [1] and by Spagl (in the set of analytic functions f (cf. [16])). We omit the technical proof which can be done by modifying the construction in [1]. Roughly speaking, this construction is as follows. Let be given a function $f \in C^q(I, \mathbb{R}^d)$ such that the conditions (24), (25), (26) for a solution of $(\mathcal{P}_p)$ are fulfilled with $k < n+1$ (i.e. the solution is not strongly unique). Then to any $\epsilon > 0$ we construct a perturbation $\widetilde{f}$ of f, $|||f - \widetilde{f}|||_p < \epsilon$, such that the corresponding optimality conditions are (uniquely) fulfilled with $k = n+1$ (i.e. $\widetilde{f} \in \mathcal{F}_+^q$). However, this construction has a special structure. So, for an interpretation of this genericity result we must be very careful. An open and dense subset of a topological space can be very 'thin'. This means that Theorem 1 does not assure that 'in general' a best approximation is strongly unique. This becomes more evident if we consider the spaces $C^q(I, \mathbb{R}^d)$, $q \geq 2$ in the norm

$$|||f|||_{p,q} = \max_{0\leq j\leq 2}\{|||f^{(j)}|||_p\}$$

where $f^{(j)} = (f_1^{(j)}, \ldots, f_d^{(j)})$. Note, that in contrast to the norm $|||\cdot|||_p$ this norm measures also the perturbations in the derivatives up to order 2. A main step in the construction of the perturbation $\widetilde{f} \in \mathcal{F}_+^q$ from f could roughly be compared with the following perturbation: Perturb $g(t) = t^2$ (with one local minimum $t_1 = 0$ as indicated in Figure 1) in such a way that the perturbation $\widetilde{g}$ has two local minima $\widetilde{t}_0, \widetilde{t}_1$ near t_1. Here the error

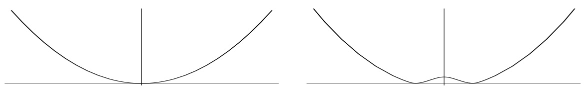

Figure 1:

$|||g - \widetilde{g}||| = \max_{t \in I} |g(t) - \widetilde{g}(t)|$ can be chosen arbitrarily small. However, $g''(t) = 2$ for all t and $\widetilde{g}''$ has (at least two) turning points $\widetilde{t}$ between $\widetilde{t}_0, \widetilde{t}_1$ (i.e. $\widetilde{g}''(\widetilde{t}) = 0$) implying $|||g'' - \widetilde{g}''||| \geq 2$. This means that in the above construction the error $|||f - \widetilde{f}|||_{p,2}$ is large. Consequently, the genericity result of Theorem 1 is not true if we consider the space $f \in C^q(I, \mathbb{R}^d)$ in the norm $|||\cdot|||_{p,2}$.
It is possible to give genericity results of more practical relevance. By using the techniques developed in [5] based on multi-jet transversality theorems it can for example be proven:

Generically the equations (24),(25),(26) defining a best approximation, are non-degenerate in the sense that the Jacobian of the system (24),(25),(26) is non-singular. This implies that the Newton method applied to this system is quadratically convergent.

Acknowledgement. The author is indebted to an anonymous referee for his careful reading.

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