Stochastic programming: Approximation via scenarios

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Abstract

We shall survey and classify various ways of approximation of the true probability distribution by a discrete distribution concentrated in a finite number of atoms, called scenarios, which occur with preassigned probabilities. Resistance of the numerical results with respect to the choice of scenarios and/or of their probabilities and also the relation between results obtained for the selected scenarios and those valid for the true underlying distribution are of a great practical interest. We shall discuss these questions bearing in mind the diverse origin of scenarios that puts limitations both on the choice of suitable approaches to the analysis and on the obtained conclusions.

Keywords and phrases: Stochastic programs with recourse, generation of scenarios, postoptimality and stability.

1 Introduction

Stochastic programming deals with optimization problems in which uncertainties are modeled via explcitly spelled out random parameters. It originated in the late fifties, cf. seminal papers by Beale [1], Dantzig [16] and

Charnes and Cooper [12]; some low dimensional examples of stochastic programming can be traced even in earlier papers on inventory, maintenance, etc.

A frequent requirement is to find a decision *before* the values of realizations of the random parameters can be observed, i. e., using only the available information about their probability distribution. The general formulation of such stochastic programming problems, cf. [98], reads:

$$\text{Minimize} \quad Ef_0(\boldsymbol{x}, \omega) \tag{1}$$

$$\begin{aligned} \text{subject to} \quad & E\{f_i(\boldsymbol{x}, \omega)\} \le 0, \quad i = 1, \ldots, k \\ & E\{f_i(\boldsymbol{x}, \omega)\} = 0, \quad i = k+1, \ldots, k+r \\ & \boldsymbol{x} \in \mathcal{X} \end{aligned} \tag{2}$$

where

ω is a random parameter with support Ω, probability distribution P_0 and the corresponding expectation $E = E_{P_0}$,

$\mathcal{X}$ is a given nonempty closed set,

$f_0 : R^n \times \Omega \to R^1 \cup \{+\infty\}, \quad f_i : R^n \times \Omega \to R^1$ are given functions.

A typical, even though not quite realistic assumption is that the probability distribution P_0 is known (decision making under risk) and independent of the decision $\boldsymbol{x}$. The existence of expectations in (1), (2) for all feasible decisions $\boldsymbol{x} \in \mathcal{X}$ is guaranteed by special assumptions from case to case; see also [98] for general properties of expectation functionals.

The above formulation looks like an ordinary static deterministic program whose objective and constraints are of the form of expectations. The main stumbling block for its algorithmic solution is the necessity to compute repeatedly at least the values of the multidimensional integrals of the involved functions, which themselves need not be defined explicitly. To overcome this problem various approximation schemes, both stochastic and deterministic ones, were designed; see for instance [5, 58, 59, 82, 97] and the references therein. One can replace the functions f by simpler ones or to approximate the true assumed probability distribution P_0 or both. The goal is to get a numerically tractable optimization problem, or a sequence of such problems, whose solution would be acceptable as an approximation of the solution of the true underlying problem.

We will not discuss the ways how to approximate the functions f, cf. for instance [4, 6, 94]. For approximation of the true probability distribution P_0, one should certainly use the structure of the problem and also the available information about P_0 that comes from theory, historical data and experience.

There are relatively many prospects if the approximation of P_0 reduces into the approximation of one-dimensional probability distributions: Besides of an approximation by a discrete distribution, one can use piecewise uniform distributions [1, 96], or approximate the density by kernel estimates [42, 43], etc. In the multi-dimensional case, approximation by discrete distributions is the prevailing approach. It means that the true distribution P_0 is replaced by a *discrete* distribution P concentrated in a finite number of points, say, $\omega_1, \dots, \omega_S$ with probabilities $p_s > 0 \quad \forall s, \quad \sum_{s=1}^S p_s = 1$ that enter the coefficients and/or the functions in a known way. The atoms ω_s of the distribution P are called *scenarios*.
The origin of scenarios can be very diverse; they can come from a truly discrete known distribution, can be obtained in the course of a discretization/approximation scheme, by simulation or by a limited sample information, they can result from a preliminary analysis of the problem with probabilities of their occurence that may reflect an ad hoc belief or a subjective opinion of an expert.
Naturally, one is interested in results for the true underlying problem; as we shall see, this is not always possible. The next item is robustness of the obtained approximate optimal solution and optimal value: The procedure should be robust in the sense that small perturbances of the input, i.e., of the chosen scenarios and of their probabilities, should impair the outcome only slightly so that the obtained results remain close to the unperturbed ones and that somewhat larger perturbations do not cause a catastroph. The importance of robust procedures increases with the complexity of the model and with its dimensionality.
We shall deal with stochastic programs (1), (2) that can be reformulated so that the expectations appear only in the objective function:

$$\text{Minimize} \qquad F(\boldsymbol{x}, P_0) := E_{P_0} f(\boldsymbol{x}, \omega) \quad \text{on the set} \quad \mathcal{X} \subset R^n \tag{3}$$

and we shall confine ourselves to approximations of the true distribution P_0 by a discrete distribution P carried by a finite number of scenarios.
Scenario based two-stage stochastic linear programs known also under the name *stochastic linear programs with recourse* belong to this class. Their generic form that corresponds to the formulation (3) reads
minimize

$$E_{P_0} \left\{ \boldsymbol{c}(\omega)^\top \boldsymbol{x} + Q(\boldsymbol{x}, \omega) \right\} \quad \text{on the set} \quad \mathcal{X} = \left\{ \boldsymbol{x} \in R_+^{n_1} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \right\} \tag{4}$$

with the recourse costs $Q(\boldsymbol{x}, \omega)$ defined for a given $\boldsymbol{x}$ and ω as the optimal value of the auxiliary *second-stage program*

$$\text{minimize} \quad \boldsymbol{q}(\omega)^\top \boldsymbol{y} \tag{5}$$

$$\text{subject to} \quad \boldsymbol{y} \in R_+^{n_2} \quad \text{that satisfy} \quad \boldsymbol{W}(\omega)\boldsymbol{y} + \boldsymbol{T}(\omega)\boldsymbol{x} = \boldsymbol{h}(\omega).$$

For a discrete distribution P carried by a finite number of scenarios, (4), (5) can be rewritten as a large linear program

$$\text{minimize} \quad \boldsymbol{c}^\top \boldsymbol{x} + \sum_{s=1}^{S} p_s \boldsymbol{q}_s^\top \boldsymbol{y}_s \tag{6}$$

$$\text{subject to} \quad \begin{array}{lllllll} \boldsymbol{A}\boldsymbol{x} & & & & & & = \boldsymbol{b} \\ \boldsymbol{T}_1\boldsymbol{x} & + & \boldsymbol{W}_1\boldsymbol{y}_1 & & & & = \boldsymbol{h}_1 \\ \boldsymbol{T}_2\boldsymbol{x} & + & & \boldsymbol{W}_2\boldsymbol{y}_2 & & & = \boldsymbol{h}_2 \\ \vdots & & & & \ddots & & \vdots \\ \boldsymbol{T}_S\boldsymbol{x} & + & & \dots & + & \boldsymbol{W}_S\boldsymbol{y}_S & = \boldsymbol{h}_S \end{array} \tag{7}$$

$$\boldsymbol{x} \geq 0, \ \boldsymbol{y}_s \geq 0, \ s = 1, \dots, S,$$

where $\omega_s = [\boldsymbol{q}_s, \boldsymbol{T}_s, \boldsymbol{W}_s, \boldsymbol{h}_s], s = 1, \dots, S$ are scenarios or atoms at which the probability distribution P is concentrated and $p_s \geq 0, s = 1, \dots, S$ are their probabilities, $\sum_s p_s = 1$. Given scenarios and their probabilities, (6), (7) is nothing else but a large linear program of a special structure. The *scenario independent* first-stage solutions $\boldsymbol{x} \in \mathcal{X}$ are compensated by scenario dependent second-stage solutions $\boldsymbol{y}_s$ such that the total expected cost of the two-stage decision procedure is minimal. The optimal value and the optimal solutions $\boldsymbol{x}^*, \boldsymbol{y}_s^* \quad \forall s$ depend on the choice of scenarios. Modifications of scenarios influence in general the whole large linear program (6), (7) so that classical postoptimality techniques known from linear programming are tractable only exceptionally. This illustrates the importance of designing proper scenario generation procedures and the necessity of a detailed analysis of the impact of the chosen scenarios on the sought output of the chosen approximation of the underlying problem.

The same applies to *scenario based multistage stochastic linear programs with recourse*:

Minimize

$$\boldsymbol{c}_0^\top \boldsymbol{x}_0 + \sum_{k_2=1}^{K_2} p_{k_2} \boldsymbol{c}_{k_2}^\top \boldsymbol{x}_{k_2} + \sum_{k_3=K_2+1}^{K_3} p_{k_3} \boldsymbol{c}_{k_3}^\top \boldsymbol{x}_{k_3} + \dots + \sum_{k_T=K_{T-1}+1}^{K_T} p_{k_T} \boldsymbol{c}_{k_T}^\top \boldsymbol{x}_{k_T} \tag{8}$$

subject to constraints

$$
\begin{array}{llllll}
\boldsymbol{A}_0\boldsymbol{x}_0 & & & & = & \boldsymbol{b}_0 \\
\boldsymbol{B}_{k_2}\boldsymbol{x}_0 & +\boldsymbol{A}_{k_2}\boldsymbol{x}_{k_2} & & & = & \boldsymbol{b}_{k_2}, \\
 & & & & & k_2 = 1,\ldots,K_2 \\
 & \boldsymbol{B}_{k_3}\boldsymbol{x}_{a(k_3)} & +\boldsymbol{A}_{k_3}\boldsymbol{x}_{k_3} & & = & \boldsymbol{b}_{k_3}, \\
 & & & & & k_3 = K_2+1,\ldots,K_3 \\
 & \ddots & \ddots & & \vdots & \\
 & & \boldsymbol{B}_{k_T}\boldsymbol{x}_{a(k_T)} & +\boldsymbol{A}_{k_T}\boldsymbol{x}_{k_T} & = & \boldsymbol{b}_{k_T}, \\
 & & & & & k_T = K_{T-1}+1,\ldots,K_T
\end{array}
\tag{9}
$$

$$\boldsymbol{l}_t \le \boldsymbol{x}_{k_t} \le \boldsymbol{u}_t, \qquad k_t = K_{t-1}+1,\ldots,K_t, \quad t = 1,\ldots,T$$

based on scenarios, i.e., on finitely many *sequences* of considered realizations $(\boldsymbol{c}_{k_t}, \boldsymbol{A}_{k_t}, \boldsymbol{B}_{k_t}, \boldsymbol{b}_{k_t})$ of the random coefficients in the objective and in the constraints for the stage t, $t = 2,\ldots,T$ and on the path probabilities p_{k_t} of subsequences of these realizations, $p_{k_t} > 0 \;\; \forall\; k_t$, $\sum_{k_t=K_{t-1}+1}^{K_t} p_{k_t} = 1$, $\; t = 2,\ldots,T$ that identify the discrete distribution P. The probabilities p_s of individual scenarios ω_s are equal to the path probabilities $p_{k_T}, k_T = K_{T-1}+1,\ldots,K_T$. In program (8), (9), $\boldsymbol{x}_{k_1} \equiv \boldsymbol{x}_0$ denotes the first stage decision variable, $K_1 = 0$ and $a(k_t)$ denotes the immediate ancestor of k_t, so that $a(k_2) = k_1 = 0, \quad k_2 = 1,\ldots,K_2$.

Clearly, the number of scenarios is a crucial factor from the point of view of the size of the linear program (8), (9) and its numerical tractability. The input information that leads to (8), (9) can be represented in the form of a *scenario tree* or *event tree*. Each path through the tree from its root to one of its leaves corresponds to one scenario, i.e., to a particular sequence of realizations of random coefficients. The nodes where branching occures correspond to stages and they are indexed by k_t. In our notation, the root is indexed by $k_1 = 0$, there are $K_t - K_{t-1}$ nodes indexed by $k_t = K_{t-1}+1,\ldots,K_t$ for the stage t (with $K_0 = K_1 = 0$); particularly, the $K_T - K_{T-1}$ leaves indexed by k_T correspond to scenarios. For each node of the scenario tree, an entire set of decision variables is introduced; for instance, the vector of the first-stage decision variables $\boldsymbol{x}_0$ corresponds to the root and subvectors $\boldsymbol{x}_{k_t}$ of the tth stage decision variables are assigned to the nodes k_t, respectively. In each of stages, the subvectors of decision variables exploit only the information that comes from the previous stages (preceding nodes of the tree) and the choice of decisions which are based on the available past information and which at the same time allow for

the continuation of the decision process in the subsequent stages is one of positive important features of multistage stochastic programs.
The purpose of this paper is to survey and classify various approaches to scenario generation (Section 2) and to give an idea about the methodological advances developed for stability and postoptimality studies for the scenario based approximate problems (Section 3).

2 Scenarios and their generation

We have introduced scenarios as the atoms of the discrete probability distribution P that is used to approximate the underlying probability distribution P_0. Correspondingly, the primary aim of scenario generation is to represent the distribution P_0 in a reasonable way. This is an ambitious task in which compromise is needed between precision of the approximation and the size of the approximate problem, which, moreover, often requires a specific form of the input (e.g., a scenario tree for multistage problems). The origin of scenarios can be very diverse. It is problem specific, their generation should reflect both the problem structure and the available information about the underlying probability distribution. To generate them it is natural to use historical data (if any) in conjunction with an assumed background model, to apply suitable estimation, simulation and sampling procedures and also to reflect the opinion the experts based on their experience, heuristics, etc. These are the reasons why generation of scenarios cannot be reduced solely to forecasting future development of the complex system under consideration.
Concerning the level of the available information, we can distinguish four types of problems:

2.1 Full knowledge of P_0. The distribution P_0 is fully specified, hence, the scenarios can be obtained by sampling from this distribution or by application of a discretization or simulation scheme. Related to the chosen approximation technique, there are various possibilities how to draw conclusions about the optimal solution of the original problem. In its pure form this situation appears mostly in the context of testing the designed models and/or the performance of newly developed solvers; this is the case of the first papers [1, 16] as well as papers surveyed in [62]. The assumed fully known distribution (mostly normal [10, 95], uniform [76] or discrete one) may stem both from a theoretical model and, similarly as in 2.4, from an experience of an expert.

2.2 Known parametric family. The given parametric form of the distribution P_0 should be preferably based on a theoretical model and the parameters of the distribution are to be estimated from the available data. The choice of the parametric form of the probability distribution or of the stochastic process corresponds to the choice of model, the estimation of parameters to calibration of the model and a subsequent simulation, sampling or discretization procedure follows similarly as in 2.1. An example is sampling from the estimated model for loads and water quality in [92].

This type of information appears frequently in stochastic programming problems in finance and water resources management and planning, partly due to the fact that the relevant stochastic models of interest rates and assets prices or those of water inflows came to the attention relatively early and have been well developed and supported by historical data. At present, it seems to be an increasing interest in building appropriate models for distributions of demand, e. g. [10, 45].

One of the first applications was described and exploited already in [8, 9]: An estimated distribution of changes of one-year interest rates on good grade municipal bonds is used to forecast (conditional) changes in the 20 and 30 years rates and to forecast the future yield curves. Scenarios correspond to a discrete approximation of the obtained normal distribution concentrated at three points (suitably chosen quantiles). Yield curves are fitted for each branch of the obtained scenario tree and used to get the input coefficients of the designed model of the type (8), (9).

Similarly, historical data on long term and short term interest rates for government securities together with the random walk model of their changes are used to simulate logarithmic yield curves and, consequently, the corresponding empirical distribution of the terminal debt (cf. [22]).

Moreover, in financial models there is often only one prevailing one-dimensional random factor; for instance, the short term forward interest rate with its evolution in time serves as *the underlying random phenomena* which determines the values of prices, of holding period returns and of other coefficients in bond portfolio management models. Various stochastic models of interest rates, based on random walk hypotheses, were developed; the examples are the diffusion process [14] or the discrete binomial process [7]. Parameters of these models (variances or volatilities, base rates, etc.) are fitted to the observed data. The subsequent non-trivial sampling and simulation procedures have to reflect several contradicting requirements: The number of scenarios is limited because of the numerical tractability of the problem, the precision of the approximate optimal solutions is important and at the same time, the tree structure of the information cannot be forgotten. Applications of this approach are reported in [33, 34, 47, 48, 51, 77, 99, 100, 102]. It is necessary to say that the sampling procedure may destroy the postulated non-arbitrage assumption which is behind the interest rate model [24, 65] and that the historical data are not always satisfactory for various reasons; the case of non-equilibrium market or existence of change-points are two examples. Even in such situation, an additional expert knowledge can help to build sensible scenarios.

The above models can be extended to the case of several intercorrelated random factors whose behavior is modeled by diffusion processes: cf. interest rates and exchange rates considered in [50], interest rates and inflation in [70].

Monte Carlo simulation is often used in the last step of the scenario generation procedure. For instance, nonlinear constrained regression model is used in [15] to estimate the parameters of a chosen parametric discount function and, at the same time, to estimate the corresponding term structure of the forward rates in such a way that the difference between actual market prices of the bonds and prices implied by the model is minimal. The short term interest rate scenarios result then from Monte Carlo simulation based on a lognormal model with a mean reversion term that forces the simulated rates to stay within historically acceptable limits and they are used again to evaluate the coefficients of the stochastic programming model. The mean reversion term substitutes the non-arbitrage assumptions which are not present in this type of approach.

Also models of hydrological time series have been well elaborated and a technology of generation of artificial time series of water inflows has been frequently used in practice. The scenarios are simulated according to an autoregression model whose parameters have been estimated from historical data and their significance tested. Such approach is sufficient for two-stage multiperiod models or for the tracking models [20, 21]. In the context of multistage stochastic programs, however, the artificial time series do not provide the scenario tree. This is one of stumbling blocks for model building and for applications. A general idea how to use such type of replicas of time series for construction of scenario tree was suggested in [63], for a specific approach to generation of inflow scenarios see [54].

2.3 Sample information. The available sample information about the true probability distribution is mostly based on observed past data, e.g. [42, 43, 66, 87]. If the data are homogeneous enough, independent, identically distributed, use of the empirical distribution is straightforward. Otherwise, one could think of a preprocessing procedure (e.g. in [42] for electricity demand) or of an adjustment for to get specific values of sample moments

[11, 49, 73]. This applies also to the case of multistage problems provided that the required structure of the input information mentioned in Section 1 is preserved; cf. [13, 49].
Two distribution free procedures for generating forecasts (multidimensional scenarios) of the joint asset returns are suggested in [71, 72]: For a model over T periods, scenarios can be generated by assessing equal probabilities to observed T-tuples of successive past joint asset returns. Another suggestion is to apply principal components analysis to reduce the dimensionality of the random vector and, at the same time, to preserve the correlation structure given by the historical data. Principal components are uncorrelated and their simulated values together with estimated scores are used for generating scenarios, i.e., for generating the future joint assets returns that enter the considered two-stage multiperiod model designed for investment planning. Instead of principal components and scores one can use factors and factor loadings; this has been suggested in [11, 18].
Based on observed data or on sample moments, simple extremal scenarios can be constructed; branching to "upper" and "lower" cases and the expected value scenario are typical examples, e.g. [10]. These scenarios are useful also for obtaining bounds for the optimal value; the theoretical background goes back to [56, 69]. See [31, 57, 59, 82] for further references and [39, 78, 79] for applications.

2.4 Low information level. The above mentioned procedures fail if there are no reliable data. Under such circumstances, scenarios and their probabilities are mostly based on experts forecasts (see e.g. [3, 85] for predictions of future economic development, [38] for demand outlooks in production planning or [68] in power generation planning, [81] for optimization of a technological process, [91] for financial models) or even on governmental regulations (e.g., New York State regulation requires 7 specific scenarios to be analyzed when an insurance company proposes changes). Postoptimality and stability analysis of the obtained solutions is crucial even though it is impossible to draw conclusions about the optimal solution of the true underlying problem.

The four indicated groups of the available input information are not strictly distinguishable; just the opposite is true. In most of the mentioned applications one can trace interactions of the four information levels and to use all available information is the best thing to do; cf. [2, 11, 70]. Past information is often combined with experts opinion ([81]) and this is probably one possible origin of the postulated form of the true probability distribution [66]. Different information levels can apply to distinct parameters of the model separately. In portfolio management, for instance, different classes of securities require different treatment, cf. [11, 70], deposits and liabilities can be driven by external factors such as mortality rates, see e.g. [66] and [32], respectively. Similarly, the inflow scenarios in [20, 21] are based on a model fitted to historical data whereas the scenarios of electricity demand are roughly forecasted.

Also the structure of the problem influences essentially the possibilities of an adequate scenario generation. The easy case is the simple recourse problem with random right hand sides for which scenario generation reduces to a discretization of one-dimensional marginal distributions of the random right-hand sides. The most complicated problems are multistage stochastic programs with interstage dependent coefficients. They have been treated mainly for scenario trees designed by experts; for the first ideas how to deal with Markov type of interstage dependences consult [13, 18, 22, 53]. The multistage models with interstage *independent* coefficients are uncomparably easier but they pay for the simplified generation of scenarios not only by a loss of adequacy but also by an exponentially increasing size.

The general strategy of scenario generation is concerned with the number of scenarios to be employed because of the evident trade-off between the dimensionality of the resulting problem and the information preserved; for instance, the number of scenarios reported in a model for expansion planning of a multi-area power system of a seemingly realistic size ([19]) was 5.10^6, a modest discretization of demand for a communication network containing 100 of pairs leads to 5^{100} scenarios [87], portfolio management with mortgage backed securities based on the full universe of the binomial lattice scenarios would mean to use 2^{360} scenarios (see e.g. [51, 100, 102]). Careful non-redundant formulation [40], well elaborated sampling strategies [75, 103], aggregation of scenarios [11], clustering [2], EVPI reduction [23], development of parallel computation techniques [17, 46, 51, 100, 102] and algorithmic procedures that use sampling within the optimization process help to maintain numerical tractability of scenario based stochastic programs; let us mention here the use of importance sampling within the Benders decomposition algorithm [17, 18, 19, 52, 53] and the stochastic decomposition method [44, 87].

3 How to draw inference about the true problem?

Scenario generation and the adjoined problems of stability and postoptimality belong definitely to the areas of stochastic programming that deserve a focused effort both from the theoretical point of view and implementation. To ascertain the effect of the chosen probability distribution and/or of the selected scenarios, results of repeated runs of a perturbed basic model are reported; cf. [41, 47, 66].

Possibilities of drawing conclusions about the optimal solutions and the optimal value of the true stochastic program (3) using the results of the approximate scenario based program depend essentially on the structure of the solved problem as well as on the origin of scenarios. Generally speaking, the output can hardly be more precise than the input and it is easier to answer questions concerning precision of the obtained optimal values than those concerning the sets of optimal solutions. We will not deal with suitable simulation or sampling techniques and their precision neither with possible stopping rules needed for some of scenario based algorithms. We shall discuss briefly the general methodological devices which can be applied to analysis of the scenario based problems that arise as an approximation of the true problem. Again, it is expedient to deal separately with the four basic levels of distributional information as specified in Section 2. As we shall see, there are various approaches ready for possible exploitation. Their application, however, is problem specific and nontrivial and evaluation of their efficiency requires extensive numerical tests and experiments.

3.1 Full knowledge of P_0. If this assumption is true one can exploit efficient simulation or sampling techniques to generate a discrete distribution P_ν which approximates the original one with a prescribed precision, say, $d(P_0, P_\nu) < \epsilon$ for a suitable distance d of probability measures. The question is how the choosen precision ϵ of the approximation influences the differences of the optimal values $\varphi(P_0) - \varphi(P_\nu)$ and the distances of the sets of optimal solutions $\psi(P_0), \psi(P_\nu)$ for the true problem and for the approximate one, respectively. To this purpose, one has to choose the distance d and to apply relevant quantitative stability results, see e.g. [83, 84, 89]. More stringent assumptions concerning the structure of the problem (3) are needed for stability results for optimal solutions; namely, the case of multiple true optimal solutions requires a special treatment. For to be able to translate the assumed closedness of the true and approximate probability distribution to a similar statement for optimal solutions, one mostly assumes a *growth condition*, e. g.,

$$F(\boldsymbol{x}, P_0) \geq \inf_{\boldsymbol{x} \in \mathcal{X}} F(\boldsymbol{x}, P_0) + \alpha \, [d(\boldsymbol{x}, \psi(P_0))]^2$$

for all $\boldsymbol{x}$ belonging to a neighborhood of $\psi(P_0)$ and for some $\alpha > 0$; for details consult [83, 84, 89].

In a similar way, one can try to answer the question how far is the optimal output from the true one if the true probability measure P_0 has been misspecified. The errors due to misspecification compose with those due to approximation of the misspecified probability measure. Another possibility is to use a contaminated probability distribution to study the influence of deviations from the assumed probability distribution P_0: For another fixed probability distribution $\overline{P}$ one considers probability distributions of the form $P_\lambda = (1-\lambda)P_0 + \lambda\overline{P}$ with $0 \leq \lambda \leq 1$ so that the considered perturbation is represented only by a scalar parameter λ which enters the objective function in a linear way and this opens various possibilities for exploitation of parametric programming results. For development of the theory under various model assumptions see [26, 27, 29, 30], for an application to scenario based financial planning models see [34].

3.2 Known parametric family means that the true probability distribution belongs to $\mathcal{P} = \{P_\theta, \theta \in \Theta\}$ and is identified by an unknown parametr value, say, $\theta_0 \in \Theta$. The straightforward approach is to estimate the parameter θ_0 say, by $\hat{\theta}$, to generate scenarios using the estimated probability distribution $P_{\hat{\theta}}$ and to exploit the quantitative stability results. Again, two types of errors have to be considered - those arising by estimation of parameters (cf. asymptotic results [25, 27]) and errors due to simulation from the estimated probability distribution. An adaptive Bayesian approach which exploits a sample information about the true probability distribution P_{θ_0} was suggested in [55] and elaborated for special cases of the two-stage stochastic linear programs with random right-hand sides.
An alternative idea has appeared recently: To use simultaneously estimation, simulation and optimization; cf. [86, 90]. It means in our context to start simulation with a rough estimate of θ_0 and to solve the approximate problem without high precision requirements. Then to update the parameter value, generate new scenarios by sampling from the updated probability distribution and to solve the new approximate problem using the former approximate optimal solution as the starting point for the algorithm; etc.

3.3 Sample information about the true probability distribution often leads directly to scenarios. It is used to construct an empirical probability distribution P_ν based on a sample of size ν, the correponding empirical objective function $F(\boldsymbol{x}, P_\nu)$, the set of empirical optimal solutions $\psi(P_\nu)$ and the optimal empirical value $\varphi(P_\nu)$. The *asymptotic properties* of these estimates, such as consistency, rates of convergence, asymptotic distributions and large deviations have been studied in a host of papers; see e. g. [35, 60], the surveys [27, 98] or Chapter 6 of [86] and references ibid. Again, multiple optimal solutions have to be treated carefully and it is useful to use epi-consistency instead of the usual consistency [64]. For to get *non-asymptotic properties* one can apply directly inequalities for differences of the true and empirical expectations (e.g., Hoeffding inequality in [61] or more general results by [93] used in [80]) or to rely on the quantative stability results combined with bounds on the distance $d(P_0, P_\nu)$ of the true and the empirical probability distribution; see [83] for an outline of the general methodology. An example are inequalities for the Kolmogorov -Smirnov distance which is used, for instance, in [84, 89].

3.4 Low information level corresponds to *scenarios designed by experts* without any obvious relation to the true probability distribution or to the case when the true probability distribution is known to belong into a *non-parametric family* of probability distributions identified, e.g., by fixed values of some moments, by a given support or by qualitative attributes such as unimodality.
In the first case conclusions about the true problem cannot be expected; still, it is both possible and important to study the resistance of the output to changes of scenarios and their probabilities, including the impact of inclusion of out-of-sample scenarios; see recent papers [29, 30, 32, 33, 34] which apply results of parametric programming and the contamination technique to these postoptimality studies.
In the second case, one can construct at least minimax and maximax bounds on the optimal value of the true program:

$$\min_{\boldsymbol{x}\in\mathcal{X}} \inf_{P\in\mathcal{P}} F(\boldsymbol{x}, P) \leq \varphi(P_0) \leq \min_{\boldsymbol{x}\in\mathcal{X}} \sup_{P\in\mathcal{P}} F(\boldsymbol{x}, P) \tag{10}$$

Under special assumptions about the family $\mathcal{P}$ and about the structure of the underlying stochastic program (3) such as convexity and/or separability of the random objective $f(\boldsymbol{x}, \omega)$ with respect to ω , this approach provides the best case and the worst case discrete distributions carried by fully specified scenarios [39]. For further discussions in this direction see [28, 31, 36, 57, 59, 82] and references ibid.

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