Spinors, Spectral Geometry, and Riemannian Submersions

PETER B. GILKEY† Department of Mathematics, University of Oregon, Eugene Or 97403 USA. email: gilkey@math.uoregon.edu.

JOHN V. LEAHY⋆ Department of Mathematics, University of Oregon, Eugene Or 97403 USA. email:leahy@darkwing.uoregon.edu.

JEONGHYEONG PARK‡ Department of Mathematics, Honam University, 506-090 Kwangju South Korea email:jhpark@honam.honam.ac.kr.

†Research partially supported by the NSF (USA)
⋆ Research partially supported by the GARC
‡Research partially supported by KOSEF 971-0104-016-2 and BSRI 98-1425, the Korean Ministry of Education

Preface

These lecture notes appeared as Lecture Notes Series Number 40 in the Research Institute of Mathematics - Global Analysis Research Center (Seoul National University, Seoul 151-742 Korea). Professor Sang Moon Kim, who is Director of GARC, has kindly given his permission for us to post them on the EMIS web server.

We note that with the permission of the GARC, these notes have been completely rewritten and expanded to form Chapter Four of the book **Spectral Geometry, Riemannian Submersions, and the Gromov-Lawson Conjecture** by Gilkey, Leahy and Park. This book is expected to appear in summer 1999 from CRC press.

We are are grateful to the EMIS for making these Lecture Notes available on the world wide web.

Contents

Abstract

Abstract. We study the spectral geometry of Riemannian submersions. We study when the pull back of eigenforms of the Laplacian on the base are eigenforms of the Laplacian on the total space. This is a comparatively rare phenomena. We study the corresponding question for the Dolbeault Laplacian for Hermitian submersions and for the spin Laplacian in the context of principal bundles.

1991 *Mathematics Subject Classification.* Primary 58G25 Secondary 53C21, 53A50.

Key words and phrases. Riemannian submersion, spectral geometry, Bochner Laplacian, Spin Laplacian, Laplace Beltrami operator.

Introduction

This book studys the spectral geometry of Riemannian submersions. We work for the most part with the form valued Laplacian in the class of smooth compact manifolds without boundary. Let Δ^p be the p form valued Laplacian and let $\pi : Z \to Y$ be a Riemannian submersion. We will study the relationship (if any) between the spectrum of Δ^p on Y and Δ^p on Z. Generically, the pull back of an eigenform on Y is not an eigenform on Z. We say that an eigenform is preserved if the pullback of an eigenform on the base Y is an eigenform on the total space Z; this is a relatively rare phenomena. Suppose an eigenform is preserved. We say that an eigenvalue is preserved if the corresponding eigenvalue is the same and that an eigenvalue changes if the corresponding eigenvalue changes. We will study these phenomena not only for the form valued Laplacian but also for the Bochner Laplacian and the spinor Laplacian in suitable settings.

In Chapter One, we introduce the basic differential geometry with which we shall be working. We define the mean curvature and integrability tensors which will be crucial to our study. We will normalize the local coordinate systems and discuss connections and curvature. We will introduce the de Rham complex and recall the de Rham and the Hodge theorems. We derive the fundamental formula which describes the extent to which π^* fails to intertwine the coderivatives by computing $\delta_Z\pi^* - \pi^*\delta_Y$. We discuss the first Chern class and the basic differential geometry of circle bundles. We apply this discussion to construct certain circle bundles over tori with harmonic curvatures. We introduce the Hopf fibration and the Hopf manifold; these are fundamental examples in the subject. We discuss the geometry of sphere bundles and principal bundles; these provide very natural families of examples.

In Chapter Two, we discuss operators of Laplace type. This chapter is a basic primer to the subject; we refer to Gilkey [64] for further details. We discuss the symbol of an operator and discuss the properties of the discrete spectral resolution of a self-adjoint operator of Laplace type. We introduce spherical harmonics as an example; these give the discrete spectral resolution of the Laplacian on a sphere. We discuss the intertwining formulas for the coderivative in the complex category and the corresponding formulas in for the spin Dirac operator. We also discuss the category of manifolds with boundary.

Chapter Three is devoted to rigidity theorems. We give necessary and sufficient conditions to ensure that the pull back of every eigenform on Y is an eigenform on Z; the eigenvalues do not change if all eigenforms are preserved. We show that if a single eigensection is preserved, then eigenvalues do not change for the scalar Laplacian or the Bochner Laplacian; for the form valued Laplacian, we show that if an eigenform is preserved, then the corresponding eigenvalue can only increase. There are similar results in the complex setting. We show that if π is a principal bundle with structure group G so that $H^1(G;\mathbb{R}) = 0$, then eigenvalues do not change; there are similar theorems for the sphere bundle of a vector bundle of rank at least 3. The spinor setting is quite different and is treated separately. If the manifold has non-trivial boundary, and if we impose Neumann boundary conditions, we show that eigenvalues can decrease and that the form valued Neumann Laplacian can have negative eigenvalues; these results are somewhat surprising. We conclude Chapter Three with a brief treatment of the heat content asymptotics.

Chapter Four is devoted to giving examples where eigenvalues change; this is a relatively rare phenomena for manifolds without boundary. We give examples where the submersion is either a circle bundle or more generally a principal bundle with structure group G where the first cohomology group $H^1(G;\mathbb{R})$ is non-trivial. There are similar results in the complex setting. We show that eigenvalues can decrease in the spinor setting.

In Chapter Five, we take up related topics involving questions of positive curvature. We discuss briefly the relationship between the Dirac operator and the existence of metrics of positive scalar curvature in the spinor context that relates to the Gromov-Lawson conjecture. We also discuss the construction of bundle metrics of positive Ricci curvature on principal bundles. Both of these areas involve Riemannian submersions and some spectral geometry although the main focus of this investigation is not on the question of when eigenforms are preserved but rather on the existence of metrics with positive scalar curvature or with positive Ricci curvature. We present a brief list of unsolved problems relating to the area of spectral geometry and Riemannian submersions.

The appendix contains additional material. We give a brief introduction to spin geometry and the Dirac operator and discuss spinors on the torus and on the sphere. We also provide a brief introduction to complex and Hodge geometry. We have provided a fairly extensive bibliography on Riemannian submersions, the Hopf fibration, manifolds of positive scalar curvature, and related subjects to assist the reader.

The book is dedicated to Emily, to George, and to Jun Min. The first and third authors acknowledge with gratitude the support of MPIM (Germany) where they collaborated mathematically. All three authors acknowledge with gratitude the facilities provided by the Institute of Theoretical Science at the University of Oregon which enabled them to further their collaboration. The first author and the second author acknowledge with gratitude the efforts of L. Davis, J. Rice, and A. Tedards on behalf of the University of Oregon.

Chapter One: Riemannian Submersions

1.1 Introduction

In this chapter, we review the geometry of Riemannian submersions. In §1.2, we introduce the basic notational conventions and define the fundamental integrability tensors θ and ω which are associated with a Riemannian submersion $\pi : Z \to Y$. The fibers of π are minimal if and only if $\theta = 0$. The horizontal distribution $\mathcal{H}$ of π is integrable if and only if $\omega = 0$. In §1.3, we discuss normalized local coordinates for a Riemannian submersion. We define the horizontal lift and show it is bracket preserving. In §1.4, we recall the facts concerning the exterior algebra, the de Rham theorem, and the Hodge decomposition theorem that we shall need. The exterior derivative is natural with respect to pull back, the coderivative δ is not. In §1.5, we express the difference $\delta_Z\pi^* - \pi^*\delta_Y$ in terms of the tensors defined in §1.2. in terms of the tensors θ and ω. In §1.6, we express the exterior derivative of the fiber volume element in terms of these tensors.

Examples are crucial. If $\omega = 0$, the distribution has integrable horizontal distribution. In §1.7, we study the geometry of such distributions and show locally they have the form $Z = X \times Y$ where $\pi(x, y) = y$ and where

$$ds_Z^2 = g_{ij}(x, y)dx^i \circ dx^j + h_{ab}(y)dy^a \circ dy^b.$$

Thus this sort of twisted product is a Riemannian submersion. Note that we have adopted the Einstein convention and sum over repeated indices.

Fiber products discussed are a way of constructing new Riemannian submersions from given ones. Let $\pi_i : Z_i \to Y$ be Riemannian submersions. In §1.8, we define the fiber product $W(Z_1, Z_2)$ and express the integrability tensors θ_W and

ω_W for the fiber product in terms of the corresponding tensors θ_i and ω_i for the submersions π_i.

In §1.9, we discuss connections on real and complex vector bundles and define the curvature tensor. In §1.10, we discuss the geometry of principal circle bundles. These are Riemannian submersions where the fibers are totally geodesic circles; the tensor θ vanishes and the tensor ω is the curvature of the associated complex line bundle. In §1.11, we discuss the Chern class of a circle bundle. In §1.12, we construct specific line bundles over tori that we shall need later; these are examples of Riemannian submersions with harmonic curvature tensors and totally geodesic fibers.

Let $\pi : S^{2n+1} \to \mathbb{CP}^n$ be the *Hopf fibration.* We give $\mathbb{CP}^n$ the *Fubini-Study* metric and suitably normalize the round metric on the sphere S^{2n+1}. Then π is a Riemannian submersion. In §1.13 we discuss the geometry of the fibration $S^3 \to S^2$ in some detail; this is a Riemannian submersion with totally geodesic fiber circles. In §1.14, we discuss a complex analogue defined by the Hopf manifold $S^1 \times S^3$.

In §1.15, we discuss the geometry of the sphere bundle of a vector bundle of higher rank to construct more examples of Riemannian submersions. Principal bundles, see §1.16, are also Riemannian submersions. In these all these examples, the fibers are totally geodesic so the tensor θ vanishes while the tensor ω gives the curvature.

In §1.17, we discuss integration over the fiber to define the push forward π_* of differential forms. Let $V(y)$ be the volume of the fiber $\pi^{-1}(y)$ of a Riemannian submersion. We show that $\pi_*\theta = -d_Y \log(V)$. Thus if $\theta = 0$, then the fibers have constant volume.

1.2 Mean curvature and integrability tensors

Unless otherwise noted, all manifolds are assumed to be compact, connected, smooth, without boundary, and Riemannian. Let T_PM denote the tangent space to a manifold M at a point P of M and let T_P^*M denote cotangent space of M.

1.2.1 Definition.

(1) If $\pi : Z \to Y$ is a smooth map, we have push forward $\pi_* : T_zZ \to T_{\pi z}Y$ and pull back $\pi^* : T^*_{\pi z}Y \to T^*_zZ$. If π is surjective and if π_* is surjective, we say that π is a *submersion.*

(2) Let $\pi : Z \to Y$ be a submersion. The *fiber* X of π is the inverse image of the base point; $X := \pi^{-1}(y_0)$. Up to diffeomorphism, the fiber is independent of the particular point y_0 of Y chosen. If y is any point of Y, there exists a neighborhood $\mathcal{O}_y$ of y and a diffeomorphism from $\pi^{-1}(\mathcal{O}_y)$ to $X \times \mathcal{O}_y$ so that π is projection on the second factor. This means that π defines a *fiber bundle.*

(3) The *vertical distribution* $\mathcal{V} := \ker \pi_*$ is a sub-bundle of the tangent bundle which is determined by the projection π; it does not depend on the metric chosen.

(4) The *horizontal distribution* $\mathcal{H} := \mathcal{V}^\perp$ is a sub-bundle of the tangent bundle of Z which depends both on the projection π and on the metric ds^2_Z. The distributions $\mathcal{V}$ and $\mathcal{H}$ are smooth complementary distributions of TZ.

(5) If z is any point of Z, then the push forward π_* is a linear isomorphism from the horizontal distribution $\mathcal{H}_z$ at z to the tangent space $T_{\pi z}Y$ at the corresponding point of Y. If the push forward map is an isometry, then π is called a *Riemannian submersion*.

1.2.2 Notational conventions. Let π be a Riemannian submersion with base Y and total space Z. We shall use capital letters for tensors on Y and lower case letters for tensors on Z. Let $\rho_{\mathcal{V}}$ and $\rho_{\mathcal{H}}$ be orthogonal projection on the vertical and horizontal distributions $\mathcal{V}$ and $\mathcal{H}$. Let $\mathcal{V}^*$ and $\mathcal{H}^*$ be the dual co-distributions of the cotangent bundle T^*Z. We shall use indices a, b, and c to index local orthonormal frames $\{f_a\}$, $\{f^a\}$, $\{F_a\}$, and $\{F^a\}$ for $\mathcal{H}$, $\mathcal{H}^*$, TY, and T^*Y. We shall use indices i, j, and k to index local orthonormal frames $\{e_i\}$ and $\{e^i\}$ for $\mathcal{V}$ and $\mathcal{V}^*$. We shall adopt the *Einstein convention* and sum over repeated indices. Let Γ be the *Christoffel symbols* of the Levi-Civita connection ∇.

There are two *integrability tensors* which are naturally associated with a Riemannian submersion $\pi : Z \to Y$.

1.2.3 Definition.

(1) Let $\theta := -g_Z([e_i, f_a], e_i)f^a = {}^Z\Gamma_{iia}f^a \in C^\infty(\mathcal{H})$. The tensor θ is the unnormalized *mean curvature covector* of the fibers of π; we have omitted the usual normalizing constant of $\dim(X)^{-1}$ to simplify later formulas.

(2) Let $\omega := \omega_{abi} = \frac{1}{2}g_Z(e_i, [f_a, f_b]) = \frac{1}{2}({}^Z\Gamma_{abi} - {}^Z\Gamma_{bai})$. The tensor ω is the *integrability curvature tensor.*

The following observation is immediate from the definitions we have given:

1.2.4 Lemma. *Let $\pi : Z \to Y$ be a Riemannian submersion.*

(1) *The following assertions are equivalent:*
   1-a) *The fibers of π are minimal.*
   1-b) *π is a harmonic map.*
   1-c) *$\theta = 0$.*

(2) *The following assertions are equivalent:*
   2-a) *The distribution $\mathcal{H}$ is integrable.*
   2-b) *$\omega = 0$.*

1.3 Normalization of local coordinates

1.3.1 Definition. Let $\mathfrak{H}$ be the *horizontal lift* operator. If F is a tangent vector field on Y, then $\mathfrak{H}F$ is the vector field on Z which is characterized by the following two properties:

(1) If z is any point in Z, then we have the intertwining relation $\pi_*\mathfrak{H}F_z = F_{\pi z}$.
(2) We have that $\mathfrak{H}F$ is a horizontal vector field i.e. $\mathfrak{H}F \in C^\infty\mathcal{H}$.

The horizontal lift is bracket preserving.

1.3.2 Lemma. *Let $\pi : Z \to Y$ be a Riemannian submersion.*

(1) *If $f_i := \mathfrak{H}F_i$, then $\pi_*[f_1, f_2] = [F_1, F_2]$.*
(2) *We have ${}^Z\Gamma_{abc} = \pi^*({}^Y\Gamma_{abc})$.*

Proof. Let $\psi_\mu(t)$ and $\Psi_\mu(t)$ be the flows of the vector fields f_μ and F_μ on Z and on Y respectively for $\mu = 1, 2$. For fixed values of the parameter t, $\psi_\mu(t)$ is a diffeomorphism of Z and Ψ_μ is a diffeomorphism of Y. The curve $t \mapsto \psi_\mu(t)(z_0)$ is an integral curve for the vector field f_μ starting at the point $z_0 \in Z$; similarly the curve $t \mapsto \Psi_\mu(t)(y_0)$ is an integral curve for the vector field F_μ starting at the point $y_0 \in Y$. Since $\pi_* f_\mu = F_\mu$, π intertwines these flows; i.e. we have the relationship:

$$\pi\psi_\mu(t) = \Psi_\mu(t)\pi.$$

We can use the flows to compute the bracket. Let

$$\begin{aligned}
h(t) &:= \psi_1(-\sqrt{t})\psi_2(-\sqrt{t})\psi_1(\sqrt{t})\psi_2(\sqrt{t})(z_0)\\
H(t) &:= \pi h(t) = \Psi_1(-\sqrt{t})\Psi_2(-\sqrt{t})\Psi_1(\sqrt{t})\Psi_2(\sqrt{t})(\pi z_0)
\end{aligned}$$

for $z_0 \in Z$. The first assertion of the Lemma now follows from the observation that:

$$\dot{h}(0) = [f_1, f_2](z_0) \text{ and } \pi_*\dot{h}(0) = \dot{H}(0) = [F_1, F_2](\pi z_0).$$

Let $\{F_a\}$ be a local orthonormal frame for the tangent bundle of Y and let $\{f_a\}$ be the horizontal lift to an orthonormal set; $f_a := \mathfrak{H}F_a$. We can compute the Christoffel symbols in terms of the bracket and use assertion (1) to prove assertion (2) by computing:

$$\begin{aligned}
{}^Z\Gamma_{abc} &= \tfrac{1}{2}\{g_Z([f_a, f_b], f_c) - g_Z([f_b, f_c], f_a) + g_Z([f_c, f_a], f_b)\}\\
&= \tfrac{1}{2}\{g_Y(\pi_*[f_a, f_b], \pi_* f_c) - g_Y(\pi_*[f_b, f_c], \pi_* f_a) + g_Y(\pi_*[f_c, f_a], \pi_* f_b)\}\\
&= \tfrac{1}{2}\{g_Y([F_a, F_b], F_c) - g_Y([F_b, F_c], F_a) + g_Y([F_c, F_a], F_b)\}\\
&= {}^Y\Gamma_{abc}. \quad \square
\end{aligned}$$

We can normalize the local coordinates as follows. Let $\mathcal{O}$ be a neighborhood of a point $y_0 \in Y$. We use coordinates on $\mathcal{O}$ to identify $\mathcal{O}$ with Euclidean space $\mathbb{R}^m$ and to identify y_0 with the origin 0 of $\mathbb{R}^m$. We let $\{\partial_\alpha^y\}$ and $\{dy^\alpha\}$ be the coordinate frames for the tangent and cotangent bundles.

1.3.3 Lemma. *Let $\pi : Z \to Y$ be a Riemannian submersion with fiber X. Choose a point y_0 of Y. There exists a neighborhood $\mathcal{O}$ of y_0 and a local diffeomorphism T from $X \times \mathcal{O}$ to Z so that*

(1) *We have $T(x, 0) = x$.*

(2) *We have $\pi(T(x, y)) = y$.*

(3) *We have $T_*(\partial_a^y)(x, 0) = \mathfrak{H}(\partial_a^y)$*

Proof. If $y \in \mathbb{R}^m$, then y determines a vector field on $\mathcal{O}$ and the horizontal lift $\mathfrak{H}(y)$ determines a vector field on $\pi^{-1}\mathcal{O}$. Let $\phi(t, y, x)$ be the corresponding flow from a point $x \in X$. There is a constant C so that ϕ is a smooth map defined for $|ty| \le C$. Note that we have

$$\phi(ts, y, x) = \phi(t, sy, x).$$

We define the coordinate transformation we need by defining $T(x, y) := \phi(1, y, x)$ and restrict the domain of T to those values of y so that $|y| \le C^{-1}$. $\square$

Note that the two tensors θ and ω of Definition 1.2.3 give obstructions to constructing coordinate systems with the normalizations of Lemma 1.3.3 being valid not just at the base point y_0 but in a full neighborhood of y_0. The following observation is an immediate consequence of the Lemma.

1.3.4 Corollary. *The tensors θ and ω are linear in the 1 jets of the structures involved and are the only tensorial expressions linear in the 1 jets of the structures.*

1.4 The exterior algebra and de Rham cohomology groups

1.4.1 Definition. If M is a Riemannian manifold, let $\Lambda^p(M)$ be the bundle of *exterior differential p forms* on M and let $C^\infty\Lambda^p(M)$ be the space of smooth p forms on M. If $u = (u^1, ..., u^m)$ is a system of local coordinates on M and if $A = \{a_1 < ... < a_p\}$, we define

$$du^A := du^{a_1} \wedge ... \wedge du^{a_p} \text{ and } |A| = p.$$

The $\{du^A\}_{|A|=p}$ form a local frame for $\Lambda^p(M)$.

1.4.2 Definition. *Exterior differentiation* d from the space of smooth p forms to the space of smooth $p+1$ forms is given by:

$$d(\sum_A f_A du^A) = \sum_{A,i} \partial_i^u(f) du^i \wedge du^A.$$

Let δ denote the dual map *interior differentiation*; δ lowers the degree by 1 and d raises the degree by 1. These two operators are related by the following duality identity in L^2:

$$(d\Phi, \Psi)_{L^2} = (\Phi, \delta\Psi)_{L^2}.$$

If $\{\sigma_a\}$ is a local orthonormal frame for TM, let $\{\sigma^a\}$ be the corresponding dual orthonormal frame for T^*M. The bundle of p forms inherits a natural inner product and $\{\sigma^A := \sigma^{a_1} \wedge ... \wedge \sigma^{a_p}\}_{|A|=p}$ is a local orthonormal frame.

1.4.3 Definition. Let σ be a covector. We use left *exterior multiplication* by σ to define the linear operator $\mathrm{ext}(\sigma)$ on the bundle of exterior differential forms $\Lambda^* M$:

$$\mathrm{ext}(\sigma)\Phi := \sigma \wedge \Phi.$$

Let $\mathrm{int}(\sigma)$ be the dual operation of *interior multiplication*; these two operations are related by the duality equation:

$$(\mathrm{ext}(\sigma)\Phi_1, \Phi_2) = (\Phi_1, \mathrm{int}(\sigma)\Phi_2).$$

These operators are 0^{th} order operators; in contrast to the operators d and δ, they depend only on the value of an exterior differential form Φ at the point in question.

Suppose that σ is a unit covector. Then we can choose an orthonormal basis $\{\sigma_i\}$ for TM so that σ is the first element of the dual orthonormal basis $\{\sigma^i\}$ for T^*M, i.e. $\sigma = \sigma^1$. Relative to such an adapted orthonormal basis we have:

$$\mathrm{ext}(\sigma)(\sigma^A) := \begin{cases} 0 & \text{if } a_1 = 1, \\ \sigma^1 \wedge \sigma^{a_1} \wedge ... \wedge \sigma^{a_p} & \text{if } a_1 > 1, \end{cases}$$

and

$$\mathrm{int}(\sigma)(\sigma^A) := \begin{cases} \sigma^{a_2} \wedge ... \wedge \sigma^{a_p} & \text{if } a_1 = 1, \\ 0 & \text{if } a_1 > 1. \end{cases}$$

In other words, in this adapted orthonormal basis, exterior multiplication by σ adds the index '1' to A while interior multiplication by σ removes the index '1' from A.

Let ∇ be the Levi-Civita connection and define the Christoffel symbols relative to this local orthonormal frame by the identity:

$$\nabla_{\sigma_a} \sigma_b = \Gamma_{abc}\sigma_c.$$

We extend the Levi-Civita connection in a natural fashion to $C^\infty \Lambda M$.

1.4.4 Lemma. *Let* $\phi=\phi_A\sigma^A\in C^\infty\Lambda M$. *Then*

(1) *We have* $\nabla\phi=\sigma^a\otimes(\partial_a\phi_A)\sigma^A+\Gamma_{abc}\sigma^a\otimes\operatorname{ext}(\sigma^c)\operatorname{int}(\sigma^b)\phi)$.

(2) *We have* $d\phi=\operatorname{ext}(\sigma^a)\{(\partial_a\phi_A)\sigma^A+\Gamma_{abc}\operatorname{ext}(\sigma^c)\operatorname{int}(\sigma^b)\phi\}$.

(3) *We have* $\delta\phi=-\operatorname{int}(\sigma^a)\{(\partial_a\phi_A)\sigma^A+\Gamma_{abc}\operatorname{ext}(\sigma^c)\operatorname{int}(\sigma^b)\phi\}$.

Proof. If ϕ is a function or a 1 form, then assertion (1) is immediate. Since ∇ acts as a graded derivation on the exterior algebra, assertion (1) now follows in general.

The following operators are invariantly defined first order partial differential operators on the exterior algebra with the same leading symbols as d and δ:

$$\hat d:=(\operatorname{ext}\otimes I_\Lambda)\circ\nabla\text{ and }\hat\delta:=-(\operatorname{int}\otimes I_\Lambda)\circ\nabla.$$

Thus $\mathcal{A}_1:=d-\hat d$ and $\mathcal{A}_2:=\delta-\hat\delta$ are invariantly defined 0^{th} order operators. This shows that the $\mathcal{A}_i$ are invariantly defined endomorphisms which are linear in the 1 jets of the metric with coefficients which are smooth functions of the metric tensor. Given a point $P\in M$, we can always choose coordinates so the 1 jets of the metric vanish at P. This implies that $\mathcal{A}_i(P)=0$. Since the $\mathcal{A}_i$ are invariantly defined, we see they vanish identically. This shows that

$$d=(\operatorname{ext}\otimes I_\Lambda)\circ\nabla\text{ and }\delta=-(\operatorname{int}\otimes I_\Lambda)\circ\nabla.$$

The remaining assertions now follow. □

1.4.5 Definition. The *de Rham cohomology* groups are defined by

$$H^p(M;\mathbb{R}):=\frac{\ker(d_p:C^\infty\Lambda^pM\to C^\infty\Lambda^{p+1}M)}{\operatorname{image}(d_{p-1}:C^\infty\Lambda^{p-1}M\to C^\infty\Lambda^pM)}.$$

The *de Rham theorem* establishes an isomorphism between these groups and the topological cohomology groups of M; the *Hodge decomposition* theorem expresses the de Rham cohomology groups of a Riemannian manifold in terms of the kernel of the Laplace operators. Let

$$\Delta_M^p:=\delta_pd_p+d_{p-1}\delta_{p-1}\text{ on }C^\infty\Lambda^pM.$$

If ϕ is a smooth p form, then we have

$$\phi\in\ker(\Delta_M^p)\iff d\phi=0\text{ and }\delta\phi=0.$$

Thus if $\phi\in\ker(\Delta_M^p)$ is harmonic, $[\phi]$ defines a class in de Rham cohomology. We refer to Gilkey [64, Theorem 1.5.2] for the proof of the following result.

1.4.6 Theorem (de Rham-Hodge). *Let M be a compact Riemannian manifold without boundary.*

(1) *There is a natural isomorphism between the de Rham cohomology groups $H^p(M;\mathbb{R})$ defined above and the topological cohomology groups of M which are defined either using sheaf cohomology or using singular cohomology.*

(2) *The map $\phi \to [\phi] \in H^p(M;\mathbb{R})$ defines an isomorphism from $\ker(\Delta_M^p)$ to $H^p(M;\mathbb{R})$.*

There is a generalization of this result to local coefficient systems discussed in §3.9. There is also a corresponding analogue in the complex setting for Kaehler manifolds that identifies

$$\ker(\Delta_M^{(p,q)}) = H^q(M;\mathcal{O}\Lambda^{(p,0)})$$

where $\mathcal{O}\Lambda^{(p,0)}$ is the sheaf of holomorphic sections to the holomorphic vector bundle $\Lambda^{(p,0)}$; the case $p=0$ corresponds to the sheaf of holomorphic functions on M. We omit details as we shall not need this fact.

1.4.7 Definition. Let M be an orientable manifold and let ν_M be the oriented volume form. The Hodge $\star$ operator is characterized by the duality identity linking the metric with wedge product:

$$g(\phi_p,\psi_p)\nu_M = \phi_p \wedge \star\psi_p.$$

If $\{\sigma_a\}$ is an oriented orthonormal local frame for TM and if ϵ is a permutation of the indices from 1 through m, then

$$\star(\sigma^{a_{i_1}} \wedge \ldots \wedge \sigma^{a_{i_p}}) = \operatorname{sign}(\epsilon)\sigma^{a_{i_{p+1}}} \wedge \ldots \wedge \sigma^{a_{i_m}}.$$

The Hodge $\star$ operator $\star : \Lambda^p M \to \Lambda^{m-p}M$ has the properties:

$$\begin{aligned}\star_{m-k}\star_k &= (-1)^{k(m-k)}\\ \delta_k &= (-1)^{mk+1}\star_{m-k} d_{m-k-1}\star_{k+1}\,.\end{aligned}$$

Consequently we have $\star_k : \ker(\Delta_M^k) \to \ker(\Delta_M^{m-k})$ and hence by the Hodge decomposition theorem given above, we have the *Poincare duality* isomorphism:

$$H^p(M;\mathbb{R}) = H^{m-p}(M;\mathbb{R}).$$

1.5 Intertwining the coderivatives

Let $\pi : Z \to Y$ be a Riemannian submersion. If $\Phi := \sum_I \Phi_I dy^I$ is a differential form on the base, then the *pull back operator*

$$\pi^* : C^\infty \Lambda Y \to C^\infty \Lambda Z$$

is defined by:

$$\pi^*\Phi := \sum_I (\Phi_I \circ \pi) d(y^{i_1} \circ \pi) \wedge ... \wedge d(y^{i_p} \circ \pi).$$

We have π^* is a graded ring homomorphism. Furthermore, pull back commutes with exterior differentiation, i.e.

$$d_Z \pi^* = \pi^* d_Y \text{ on } C^\infty \Lambda^p(Y).$$

The adjoint, interior differentiation, does not in general commute with pullback. Recall we defined in Definition 1.2.3:

$$\begin{aligned}
\theta &:= -g_Z([e_i, f_a], e_i) f^a = {}^Z\Gamma_{iia} f^a \in C^\infty(\mathcal{H}),\\
\omega &:= \omega_{abi} = \tfrac{1}{2} g_Z(e_i, [f_a, f_b]) = \tfrac{1}{2}({}^Z\Gamma_{abi} - {}^Z\Gamma_{bai}).
\end{aligned}$$

We now define:

$$\begin{aligned}
\mathcal{E} &:= \omega_{abi} \operatorname{ext}_Z(e^i) \operatorname{int}_Z(f^a) \operatorname{int}_Z(f^b),\\
\Xi &:= \operatorname{int}_Z(\theta) + \mathcal{E}.
\end{aligned} \tag{1.5.a}$$

The following formula will be crucial to our study.

1.5.1 Lemma. *Let $\pi : Z \to Y$ be a Riemannian submersion.*

(1) *We have $\delta_Z \pi^* - \pi^* \delta_Y = \Xi \pi^*$.*

(2) *We have $\Delta_Z \pi^* - \pi^* \Delta_Y = \{\Xi d_Z + d_Z \Xi\} \pi^*$.*

Proof. Let $\Phi \in C^\infty \Lambda^p(Y)$. We expand $\Phi = \Phi_A F^A$ and use Lemma 1.4.4 to compute:

$$\begin{aligned}
\delta_Z \pi^* \Phi &= \delta_Z \pi^*(\Phi_A) f^A\\
&= -\operatorname{int}_Z(e^i) e_i(\pi^*\Phi_A) f^A && (1.5.b)\\
&\quad - \operatorname{int}_Z(e^i) {}^Z\Gamma_{iab} \operatorname{ext}_Z(f^b) \operatorname{int}_Z(f^a) \pi^*\Phi && (1.5.c)\\
&\quad - \operatorname{int}_Z(e^i) {}^Z\Gamma_{iaj} \operatorname{ext}_Z(e^j) \operatorname{int}_Z(f^a) \pi^*\Phi && (1.5.d)\\
&\quad - \operatorname{int}_Z(f^a) f_a(\pi^*\Phi_A) f^A && (1.5.e)\\
&\quad - \operatorname{int}_Z(f^a) {}^Z\Gamma_{abc} \operatorname{ext}_Z(f^c) \operatorname{int}_Z(f^b) \pi^*\Phi && (1.5.f)\\
&\quad - \operatorname{int}_Z(f^a) {}^Z\Gamma_{abi} \operatorname{ext}_Z(e^i) \operatorname{int}_Z(f^b) \pi^*\Phi. && (1.5.g)
\end{aligned}$$

Since horizontal covector fields are annihilated by $\operatorname{int}_Z(e^i)$, the terms in (1.5.b) and in (1.5.c) vanish. Furthermore in (1.5.d) we must have $i = j$. By definition, $\theta = -{}^Z\Gamma_{iai} f^a$ so the terms in (1.5.d) yield $\operatorname{int}_Z(\theta)\pi^*\Phi$. Since ${}^Y\Gamma_{abc} = {}^Z\Gamma_{abc}$, the terms in (1.5.e) and (1.5.f) yield $\pi^*\delta_Y\Phi$. Note that

$$\begin{aligned}-\operatorname{int}_Z(f^a)\operatorname{ext}_Z(e^i)\operatorname{int}_Z(f^b) =& \operatorname{ext}_Z(e^i)\operatorname{int}_Z(f^a)\operatorname{int}_Z(f^b)\\ =& -\operatorname{ext}_Z(e^i)\operatorname{int}_Z(f^b)\operatorname{int}_Z(f^a).\end{aligned}$$

Thus we may anti-symmetrize to see the terms in (1.5.g) yield

$$\tfrac{1}{2}\operatorname{ext}_Z(e^i)({}^Z\Gamma_{abi} - {}^Z\Gamma_{bai})\operatorname{int}_Z(f^a)\operatorname{int}_Z(f^b)\pi^*\Phi = \mathcal{E}\pi^*\Phi.$$

The first assertion now follows. Since $d_Z\pi^* = \pi^* d_Y$, we prove the second assertion by computing:

$$\begin{aligned}\Delta_Z\pi^* - \pi^*\Delta_Y =& d_Z\delta_Z\pi^* + \delta_Z d_Z\pi^* - \pi^* d_Y\delta_Y - \pi^*\delta_Y d_Y\\ =& d_Z(\delta_Z\pi^* - \pi^*\delta_Y) + (\delta_Z\pi^* - \pi^*\delta_Y)d_Y\\ =& d_Z\Xi\pi^* + \Xi\pi^* d_Y = (d_Z\Xi + \Xi d_Z)\pi^*. \quad\square\end{aligned}$$

1.6 The derivative of the fiber volume element

Let $\pi : Z \to Y$ be a Riemannian submersion. Recall our notational convention that $\{e_i\}$ and $\{e^i\}$ are local orthonormal frames for the vertical distributions and codistributions $\mathcal{V}$ and $\mathcal{V}^*$ and that $\{f_a\}$ and $\{f^a\}$ are local orthonormal frames for the horizontal distributions and codistributions $\mathcal{H}$ and $\mathcal{H}^*$.

Suppose the fiber X of π is orientable. Let

$$\nu_X := e^1 \wedge \ldots \wedge e^{\dim X}$$

be the volume form of the fiber. We can express $d_Z(\nu_X)$ in terms of the tensors θ and ω.

1.6.1 Lemma. *Let $\pi : Z \to Y$ be a Riemannian submersion with orientable fiber X. Then $d_Z(\nu_X) = -\theta \wedge \nu_X - \omega_{abi}\operatorname{ext}_Z(f^a)\operatorname{ext}_Z(f^b)\operatorname{int}_Z(e^i)\nu_X$.*

Proof. We apply the formulas of Lemma 1.4.4 to the manifold Z and separate the indices into horizontal and vertical parts to compute:

$$\begin{aligned}&(1.6.a)\qquad & d_Z(\nu_X) =& \Gamma_{ijk}\operatorname{ext}_Z(e^i)\operatorname{ext}_Z(e^k)\operatorname{int}_Z(e^j)\nu_X\\ &(1.6.b)\qquad & &+ (\Gamma_{ija} - \Gamma_{aji})\operatorname{ext}_Z(e^i)\operatorname{ext}_Z(f^a)\operatorname{int}_Z(e^j)\nu_X\\ &(1.6.c)\qquad & &+ \Gamma_{ajb}\operatorname{ext}_Z(f^a)\operatorname{ext}_Z(f^b)\operatorname{int}_Z(e^j)\nu_X.\end{aligned}$$

The form ν_X has maximal vertical degree. Thus the terms in equation (1.6.a) yield 0 for dimensional reasons. In equation (1.6.b), the terms with $i \neq j$ vanish and we therefore set $i = j$. Since we are dealing with an orthonormal frame field, $\Gamma_{aii} = 0$. Thus $\Gamma_{iia} - \Gamma_{aii} = \Gamma_{iia}$ and the terms in equation (1.6.b) yield $-\theta \wedge \nu_X$. Finally, the terms from equation (1.6.c) yield

$$-\tfrac{1}{2}(\Gamma_{abj} - \Gamma_{baj}) = -\omega_{abj}. \quad \square$$

1.7 Integrable horizontal distributions

1.7.1 Definition. We say that a Riemannian submersion $\pi : Z \to Y$ is *flat* if the horizontal distribution is integrable.

We can improve Lemma 1.3.3 if the horizontal distribution is flat.

1.7.2 Lemma. *Let $\pi : Z \to Y$ be a Riemannian submersion with fiber X and integrable horizontal distribution $\mathcal{H}$.*

(1) *We can find local coordinates $z = (x, y)$ on Z so $\pi(x, y) = y$.*

(2) *In these coordinates, $ds^2_Z = g_{ij}(x, y)dx^i \circ dx^j + h_{ab}(y)dy^a \circ dy^b$.*

(3) *If we set $g_X := \det(g_{ij})^{1/2}$, then $\theta = -d_Y \ln(g_X)$.*

Proof. Choose local coordinates $y = (y^a)$ defined on a neighborhood $\mathcal{O}$ of some point y_0 in Y. Let $\tilde f_a$ be the horizontal lift of the coordinate vector fields ∂^y_a from Y to Z over $\pi^{-1}(\mathcal{O})$; these vector fields do *not* form an orthonormal frame. We use Lemma 1.3.2 to see that we have

$$\pi_*[\tilde f_a, \tilde f_b] = [\partial^y_a, \partial^y_b] = 0.$$

Since we have assumed that the horizontal distribution $\mathcal{H}$ is integrable, we have $[\tilde f_a, \tilde f_b] \in \mathcal{H}$. Consequently $[\tilde f_a, \tilde f_b] = 0$. Choose local coordinates $x = (x^i)$ for the fiber $X = \pi^{-1}(y_0)$ near z_0. We apply the Frobenius theorem to extend x to a system of coordinates $z = (x, w)$ on a neighborhood of z_0 so that $\tilde f_a = \partial^w_a$. The projection of the integral curves of the vector fields ∂^w_a are the integral curves of the vector fields ∂^y_a. Therefore $y = \pi(x, w) = w$ and π is projection on the second factor. At a point $z = (x, w)$, we have

$$\mathcal{V}_z = T_x X = \operatorname{span}\{\partial^x_i\} \perp \mathcal{H}_z = T_w Y = \operatorname{span}\{\partial^w_a\}.$$

Since π_* is an isometry from the horizontal distribution $\mathcal{H}$ to the tangent bundle of TY of Y, the metric locally has the form given in the second assertion.

Since $\omega = 0$, we may apply Lemma 1.6.1 to prove the final assertion by computing

$$\begin{aligned}\nu_X &= g_X dx^1 \wedge \ldots \wedge dx^{\dim(X)}\\ d_Z(\nu_X) &= d_Y(g_X) \wedge dx^1 \wedge \ldots \wedge dx^{\dim(X)}\\ &= d_Y(g_X) g_X^{-1} \wedge \nu_X = -\theta \wedge \nu_X. \quad \square\end{aligned}$$

Let $\pi : Z \to Y$ be a flat Riemannian submersion. We cover Y by open sets $\mathcal{O}_\alpha$ and use Lemma 1.7.2 to construct local diffeomorphisms T_α from $X \times \mathcal{O}_\alpha$ to $\pi^{-1}\mathcal{O}_\alpha$ so that $\pi T_\alpha(x, y) = y$ is projection on the second factor and so the vector fields $(T_\alpha)_* \partial_a^y$ span the horizontal distribution. Then on the overlap $\mathcal{O}_\alpha \cap \mathcal{O}_\beta$ we have must have

$$T_\alpha^{-1} \circ T_\beta(x, y) = (\Phi_{\alpha\beta}(x), y).$$

In this equation, the diffeomorphism $\Phi_{\alpha\beta}$ is independent of y. Thus the fiber bundle $\pi : Z \to Y$ is a flat fiber bundle; the glueing transition functions are locally constant. Thus this fiber bundle is defined by a representation of the fundamental group $\pi_1(Y)$ of Y into the group of diffeomorphisms of the fiber. In particular, if Y is simply connected, then the bundle admits a global trivialization and the Riemannian submersion is globally product with the metric having the form of Lemma 1.7.2.

1.7.3 Definition. We say that a flat Riemannian submersion has *SL structure group* if the transition functions $\Phi_{\alpha\beta}$ described above can be chosen to preserve some volume element on the fibers or equivalently if there exists a measure ν on the fibers so that the Lie derivative $\mathcal{L}_H \nu = 0$ vanishes for every horizontal lift H; instead of the tangent map taking values in the full general linear group of automorphisms of the tangent bundle of the fiber X, the structure group reduces to the special linear group of the automorphisms of the tangent bundle of the fiber relative to a suitably chosen fiber metric. This is always the case if the fundamental group of Y is finite.

1.8 Fiber products

The following construction is a very useful one for building new Riemannian submersions.

1.8.1 Definition. We suppose given two Riemannian submersions π_α mapping U_α to the same base manifold Y. We denote the corresponding vertical and horizontal distributions of these submersions by $\mathcal{H}_\alpha$ and $\mathcal{V}_\alpha$. Let

$$W := \{w = (u_1, u_2) \in U_1 \times U_2 : \pi_1(u_1) = \pi_2(u_2)\}.$$

be the *fiber product*. Define smooth submersions π_W from W to Y and σ_α from W to U_α by

$$\pi_W(w) := \pi_1(u_1) = \pi_2(u_2) \text{ and } \sigma_\alpha(w) = u_\alpha.$$

The vertical space of π_W is given by

$$\mathcal{V}_W(w) = \mathcal{V}_1(u_1) \oplus \mathcal{V}_2(u_2)$$

with respect to the usual embedding of TW in $TU_1 \oplus TU_2$. It is now clear that π_W and $(\pi_W)_*$ are surjective so π_W is a submersion. Let

$$\mathcal{H}_W(w) := \{(\xi_1, \xi_2) \in \mathcal{H}_1(u_1) \oplus \mathcal{H}_2(u_2) : (\pi_1)_*\xi_1 = (\pi_2)_*\xi_2\}$$

define a complementary splitting. We define a metric on W by requiring that:

(1) We have the relations: $\mathcal{H}_W \perp \mathcal{V}_1$, $\mathcal{H}_W \perp \mathcal{V}_2$, and $\mathcal{V}_1 \perp \mathcal{V}_2$.
(2) The metrics on $\mathcal{V}_1$ and $\mathcal{V}_2$ are induced from the metrics on U_1 and U_2.
(3) We have that $\pi_W(w)_* : \mathcal{H}_W(w) \to TY(\pi(w))$ is an isometry.

Note that the metric on $\mathcal{H}_W$ differs from the subspace metric by a factor of $\frac{1}{\sqrt{2}}$; the diagonal in a right equilateral triangle has length $\sqrt{2}$. With these definitions, we have:

(1) $\pi_W : W \to Y$ is a Riemannian submersion.
(2) $\sigma_\alpha : W \to U_\alpha$ is a Riemannian submersion.

1.8.2 Examples.

(1) If the U_i are vector bundles over Y, then the fiber product of U_1 with U_2 is the *Whitney sum* vector bundle $U_1 \oplus U_2$.
(2) If $U_i = X_i \times Y$ are trivial submersions where the π_i are projection on the second factors, then the fiber product is given by $W = (X_1 \times X_2) \times Y$ and the projection is again projection on the second factor.

Let θ_α and θ_Z be the mean curvature vectors defined by the submersions π_α and π_Z; let ω_α and ω_Z be the corresponding curvature tensors.

1.8.3 Lemma. *Let $\pi_\alpha : U_\alpha \to Y$ be Riemannian submersions. Let W be the fiber product of two Riemannian submersions. Then*

(1) *We have $\theta_W = \sigma_1^*\theta_1 + \sigma_2^*\theta_2$ and $\omega_W = \sigma_1^*\omega_1 + \sigma_2^*\omega_2$.*
(2) *If $\Phi \in E(\lambda, \Delta_Y^p)$, and if for $\alpha = 1, 2$ we have $\pi_\alpha \in E(\lambda + \epsilon_\alpha, \Delta_{U_\alpha}^p)$, then we have $\pi_W^*\Phi \in E(\lambda + \epsilon_1 + \epsilon_2, \Delta_W^p)$.*

Proof. Let $\{F_a\}$ be a local orthonormal frame for TY. Let f_a^α and f_a^W be the horizontal lifts with respect to the submersions π_α and π_W. Note that f_a^W is also the horizontal lift of f_a^α with respect to the submersion σ_α. Let $\{e_i\}$ and $\{\hat{e}_j\}$ be local orthonormal frames for the vertical distributions of π_1 and π_2 and let $\{e_i^W\}$ and $\{\hat{e}_j^W\}$ be horizontal lifts to W with respect to the submersions σ_1

and σ_2. Then $\{e_i^W, \hat{e}_j^W\}$ is a local orthonormal frame for $\mathcal{V}(\pi_W)$, $\{e_i^W\}$ is a local orthonormal frame for $\mathcal{V}(\sigma_2)$, and $\{\hat{e}_j^W\}$ is a local orthonormal frame for $\mathcal{V}(\sigma_1)$. We use Lemma 1.3.2 to compute:

$$\begin{aligned}\theta_W &= -\{g_W(e_i^W, [e_i^W, f_a^W]) + g_W(\hat{e}_j^W, [\hat{e}_j^W, f_a^W])\}\pi_W^*(F^a)\\ &= -\sigma_1^*\{g_1(e_i, [e_i, f_a^1])\pi_1^*(F^a)\} - \sigma_2^*\{g_2(\hat{e}_j, [\hat{e}_j, f_a^2])\pi_2^*(F^a)\}\\ &= \sigma_1^*\theta_1 + \sigma_2^*\theta_2.\end{aligned}$$

Since f_a^W and e_i^W are the horizontal lifts of f_a^1 and e_i with respect to σ_1 and f_a^W and $\hat{e}_j^W$ are the horizontal lifts of f_a^2 and $\hat{e}_j$ with respect to σ_2, we use Lemma 1.3.2 to complete the proof by computing

$$\begin{aligned}\omega_{abi}^W &= g_W(e_i^W, [f_a^W, f_b^W])/2 = \sigma_1^*\{g_1(e_i, [f_a^1, f_b^1])\}/2 = \sigma_1^*\omega_{abi},\\ \hat{\omega}_{abj}^W &= g_W(\hat{e}_j^W, [f_a^W, f_b^W])/2 = \sigma_2^*\{g_2(\hat{e}_j, [f_a^2, f_b^2])\}/2 = \sigma_2^*\hat{\omega}_{abj}.\end{aligned}$$

This establishes the first assertion. The second assertion is an immediate consequence of the first assertion and of the intertwining formulas of Lemma 1.5.1. □

1.9 Connections and curvature

1.9.1 Definition. A connection ∇ on a real or complex vector bundle V over a manifold M is a generalized directional derivative. It is a first order partial differential operator

$$\nabla : C^\infty(V) \to C^\infty(T^*M \otimes V)$$

which satisfies the Leibnitz rule. If $s \in C^\infty V$ is a smooth section to V and if $f \in C^\infty M$ is a smooth function on V, then we have:

$$\nabla(fs) = df \otimes s + f\nabla s.$$

There is a natural extension of the connection to an operator

$$\nabla : C^\infty(\Lambda^p(M) \otimes V) \to C^\infty(\Lambda^{p+1}(M) \otimes V)$$

which is defined by setting

$$\nabla(\Phi_p \otimes s) = d\Phi_p \otimes s + (-1)^p\Phi_p \wedge \nabla s.$$

In contrast to ordinary exterior differentiation, the square of the covariant derivative ∇^2 need not vanish. However,

$$\nabla^2(fs) = ddf \otimes s - df \wedge \nabla s + df \wedge \nabla s + f\nabla^2 s = f\nabla^2 s.$$

1.9.2 Definition. The operator ∇^2 is a 0^{th} order partial differential operator called the *curvature* $\mathcal{F}$. If s_i is a local frame for V, we define the connection 1 form by expanding

$$\nabla s_i = A_i^j \otimes s_j.$$

In the next section, we will add an extra factor of $\sqrt{-1}$ to define the normalized connection one form. For the moment, however, we omit this normalizing constant. We compute that

$$\nabla^2 s_i = (dA_i^j - A_i^k \wedge A_k^j) \otimes s_j$$

and thus the curvature is given by

$$\mathcal{F}_i^j = (dA_i^j - A_i^k \wedge A_k^j).$$

If $\tilde{s} = g_i^j s_j$ is another local frame, we show that the curvature is an invariantly defined endomorphism by checking that $\tilde{\mathcal{F}} = g\mathcal{F}g^{-1}$.

1.9.3 Definition. A connection ∇ is a *Riemannian connection* if we have

$$(\nabla s_1, s_2) + (s_1, \nabla s_2) = d(s_1, s_2).$$

We restrict to such a connection henceforth. Relative to a local orthonormal frame, the curvature $\mathcal{F}$ is skew-symmetric. Note that the Levi-Civita connection is a Riemannian connection on the tangent bundle.

1.9.4 Remark. If ∇ is a Riemannian connection on a complex vector bundle over Y, there a natural metric on V. Let $S(V)$ be the unit sphere bundle. Then $\pi : S(V) \to Y$ is a Riemannian submersion. The curvature of the connection ∇ and the tensor ω defined in Definition 1.2.3 encode essentially the same information; we refer to §1.15 for further details. In the next section, we consider the special case of circle bundles.

1.10 The geometry of circle bundles

Let L be a complex line bundle over Y. We suppose that Y is equipped with a smooth fiber metric and a Riemannian connection ${}^L\nabla$. Let

$$S(L) := \{\xi \in L : |\xi| = 1\}$$

be the associated circle bundle and let $\pi : S(L) \to Y$ be the natural associated projection. Let $S^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ be the unit circle. Then S^1 acts transitively on the fibers of $S(L)$ by complex multiplication so $S(L)$ is a *principal circle bundle*; conversely, of course, every principal circle bundle arises in this fashion. We note that there are circle bundles which are not principal circle bundles. We refer to §1.16 for further details concerning principal circle bundles.

Let s be a local orthonormal section to L and let

$$\nabla s = \sqrt{-1} A_s s$$

define the normalized connection 1 form A_s. We have adopted a slightly different normalization convention from that given in Definition 1.9.2; the factor of $\sqrt{-1}$ will make later computations easier.

We introduce local coordinates (t, y) on $S(L)$ by

$$(t, y) \mapsto e^{\sqrt{-1}t} s(y).$$

We recall the definition of the tensor $\mathcal{E}$ given in equation (1.5.a):

$$\mathcal{E} := \omega_{abi} \operatorname{ext}_Z(e^i) \operatorname{int}_Z(f^a) \operatorname{int}_Z(f^b).$$

Since the fibers of π are totally geodesic, θ vanishes so Lemma 1.5.1 shows that

$$\delta_S \pi^* - \pi^* \delta_Y = \mathcal{E}\pi^*.$$

Understanding the tensor $\mathcal{E}$ will be crucial in subsequent sections. The following Lemma summarizes the results we shall need concerning the geometry of circle bundles:

1.10.1 Lemma. *Let $\pi : S(L) \to Y$ be a principal circle bundle.*

(1) *The fibers of π are totally geodesic.*

(2) *We have ∂_t is an invariantly defined unit tangent vector spanning $\mathcal{V}$.*

(3) *If $\tilde{s} = e^{\sqrt{-1}\Phi} s$, then $\partial_t = \partial_{\tilde{t}}$, $\partial_a^y = \partial_a^{\tilde{y}} - \partial_a^y \Phi \partial_t$, and $\tilde{\mathcal{A}}_s = \mathcal{A}_s + d_Y \Phi$.*

(4) *The horizontal lift of a vector field Ψ on Y is given by $\mathfrak{H}\Psi := \Psi - \mathcal{A}_s(\Psi)\partial_t$.*

(5) *We have $e^1 := dt + \pi^* \mathcal{A}_s$ is dual to ∂_t and spans $\mathcal{V}^*$.*

(6) *The normalized curvature $\mathcal{F} := d_Y \mathcal{A}_s$ is invariantly defined.*

(7) *We have $de^1 = \pi^* \mathcal{F}$ and $\mathcal{E} = -\operatorname{ext}_S(e^1)\pi^* \operatorname{int}_Y(\mathcal{F})$.*

Proof. The flow $v \to e^{\sqrt{-1}t} v$ for $v \in S(L)$ and $t \in \mathbb{R}$ is invariantly defined; ∂_t is the associated unit vertical Killing vector field. Assertions (1) and (2) now follow. Since ${}^L\nabla$ is unitary, $\mathcal{A}_s$ is a real 1-form. Let $\tilde{s} = e^{\sqrt{-1}\Phi(y)} s$. Assertion (3) follows from the observation:

$$(\tilde{y}, \tilde{t}) = (y, t - \Phi).$$

We show that the lifting operator $\mathfrak{H}$ is invariantly defined by computing:

$$\partial_a^y - \mathcal{A}_s(\partial_a^y)\partial_t = \partial_a^{\tilde{y}} - \partial_a^y \Phi \partial_t - \mathcal{A}_s(\partial_a^y)\partial_t = \partial_a^{\tilde{y}} - \mathcal{A}_{\tilde{s}}(\partial_a^y)\partial_t.$$

Fix $y_0 \in Y$ and choose Φ so $(\mathcal{A}_s + d_Y\Phi)(y_0) = 0$. Since $\mathcal{A}_{\tilde{s}}(y_0) = 0$, the $\partial_a^{\tilde{y}}$ are horizontal. Thus at y_0, $\mathfrak{H}\partial_a^y = \partial_a^{\tilde{y}}$ is horizontal. Since $\mathfrak{H}\Psi$ is invariantly defined, $\mathfrak{H}\Psi$ is the horizontal lift. Since $e^1(\mathfrak{H}\Psi) = 0$ for all Ψ and since $e^1(dt) = 1$, e^1 is the vertical projection of dt and is invariantly defined. By assertion (3), we have

$$d_Y\mathcal{A}_{\tilde{s}} = d_Y\mathcal{A}_s.$$

This shows that the curvature $\mathcal{F}$ is invariantly defined. Clearly $de^1 = \pi^*\mathcal{F}$. We compute:

$$\begin{aligned}\mathcal{E} :=& \tfrac{1}{2}\operatorname{ext}_S(e^1)g_S(\partial_t, [\mathfrak{H}\partial_a^y, \mathfrak{H}\partial_b^y])\pi^*\operatorname{int}_Y(dy^a)\operatorname{int}_Y(dy^b)\\ =& \tfrac{1}{2}\operatorname{ext}_S(e^1)\pi^*\{-\partial_a^y\mathcal{A}_b + \partial_b^y\mathcal{A}_a\}\operatorname{int}_Y(dy^a)\operatorname{int}_Y(dy^b)\\ =& -\operatorname{ext}_S(e^1)\pi^*\operatorname{int}_Y(\mathcal{F}). \quad \square\end{aligned}$$

In §5.3, we will examine the curvature tensor of the manifold S in a slightly more general setting where take a metric where the fibers are not totally geodesic and have different lengths.

1.11 The first Chern class

1.11.1 Definition. Let ${}^L\nabla$ be a connection on a complex line bundle L over a manifold Y. If we choose a local section s to L, let

$${}^L\nabla s = \sqrt{-1}\mathcal{A}_s s \text{ and } \mathcal{F} = d_Y A_s$$

define the associated normalized connection 1 form and the curvature tensor. The first *Chern form* is an invariantly defined real closed 2 form on Y given by:

$$c_1({}^L\nabla) := -\tfrac{1}{2\pi}\mathcal{F}({}^L\nabla)$$

If ${}^L\tilde{\nabla}$ is another connection on the line bundle L, let

$$({}^L\tilde{\nabla} - {}^L\nabla)s = \sqrt{-1}\Psi s;$$

the 1 form Ψ is invariantly defined on Y. Since

$$\mathcal{F}({}^L\nabla) = \mathcal{F}({}^L\tilde{\nabla}) + d\Psi,$$

The de Rham cohomology class is independent of the connection chosen. We define

$$c_1(L) := [c_1({}^L\nabla)] \in H^2(M;\mathbb{R}).$$

We now present a brief digression into algebraic topology to establish a theorem giving the existence of line bundles with non-trivial harmonic curvatures if $H^2(Y;\mathbb{R}) \neq 0$. We use both classifying space theory and the Hodge-de Rham theorem. We shall shall only deal with the subject briefly as it is tangential to the main thrust of our exposition at this point.

Let $L_{\mathbb{CP}}$ be the *classifying line bundle* over the infinite dimensional complex projective space $\mathbb{CP}^\infty$. Let $\mathrm{Vect}^1_{\mathbb{C}}(Y)$ be the set of isomorphism classes of complex line bundles over Y and let $[Y,\mathbb{CP}^\infty]$ denote the set of homotopy classes of maps from Y to $\mathbb{CP}^\infty$. Let $f: Y \to \mathbb{CP}^\infty$. The map $f \to f^*L_{\mathbb{CP}}$, which associates to f the pull back of the classifying bundle, defines a natural isomorphism of functors:

$$[Y,\mathbb{CP}^\infty] = \mathrm{Vect}^1_{\mathbb{C}}(Y). \tag{1.11.2}$$

Note that $\mathbb{CP}^\infty$ can be replaced by $\mathbb{CP}^k$ for any k so that $2k > \dim(Y)$ and that $L_{\mathbb{CP}}$ can be replaced by the Hopf bundle in this construction.

We use tensor product to give $\mathrm{Vect}^1_{\mathbb{C}}(Y)$ the structure of an Abelian group. The natural H-group structure on $\mathbb{CP}^\infty$ gives the set of homotopy classes $[X,\mathbb{CP}^\infty]$ a group structure and the isomorphism of equation (1.11.2) is a group isomorphism.

1.11.3 Definition. Let $H^2(Y;\mathbb{Z})$ denote the integer cohomology groups of Y; these can either be defined using sheaf cohomology or singular cohomology. Let x generate $H^2(\mathbb{CP}^\infty;\mathbb{Z}) = \mathbb{Z}$. The topological first Chern class is defined to be:

$$c_1^{\mathbb{Z}}(L) := f^*(x) \in H^2(Y;\mathbb{Z}).$$

Then c_1 is a natural transformation of functors which is an isomorphism from $\mathrm{Vect}^1_{\mathbb{C}}(Y)$ to $H^2(Y;\mathbb{Z})$. We use the *universal coefficient theorem* to see that

$$H^2(Y;\mathbb{R}) = H^2(Y;\mathbb{R}) \otimes_{\mathbb{Z}} \mathbb{R};$$

see, for example, Spanier [178]. Let

$$\kappa([\theta]) := [\theta] \otimes_{\mathbb{Z}} 1_{\mathbb{R}}$$

define a natural transformation of functors map κ from $H^2(Y;\mathbb{Z})$ to $H^2(Y;\mathbb{R})$. The link between algebraic topology and differential geometry is then given by the identity:

$$\kappa \circ c_1^{\mathbb{Z}} = c_1.$$

1.11.4 Lemma. *Assume $H^2(Y;\mathbb{R}) \neq 0$. There exists a unitary connection on a complex line bundle L over Y so that the curvature $\mathcal{F}$ is harmonic and non-trivial.*

Proof. Suppose that $H^2(Y;\mathbb{R}) \neq 0$. Choose $L \in \operatorname{Vect}^1_{\mathbb{C}}(Y)$ so that $0 \neq \kappa(c_1(L))$ in $H^2(Y;\mathbb{R})$. We use the Hodge decomposition theorem (see Theorem 1.4.6) to find $0 \neq \Phi \in E(0, \Delta^2_Y)$ so that

$$[-\tfrac{1}{2\pi}\Phi] = \kappa(c_1(L)) \text{ in } H^2(Y;\mathbb{R}).$$

Let ${}^L\tilde{\nabla}$ be any unitary connection on L. Then

$$[\mathcal{F}({}^L\tilde{\nabla})] = [\Phi] \text{ in } H^2(Y;\mathbb{R}).$$

Thus we can find a smooth 1-form Ψ on Y so that $\Phi = \mathcal{F}({}^L\tilde{\nabla}) + d\Psi$. Let

$${}^L\nabla := {}^L\tilde{\nabla} + \sqrt{-1}\Psi.$$

This connection is a unitary connection on L and $\mathcal{F} = \mathcal{F}({}^L\nabla) = \Phi$ is harmonic. $\square$

1.12 Line bundles over the torus

Give $\mathbb{T}^2 := S^1 \times S^1$ the flat product metric. If x^i for $0 \leq x^i \leq 2\pi$ are the usual periodic parameters on $\mathbb{T}^2$, then

$$ds^2 = (dx^1)^2 + (dx^2)^2.$$

Lemma 1.11.4 shows there exists a line bundle over $\mathbb{T}^2$ with non-trivial *harmonic curvature*. We can exhibit this line bundle quite explicitly.

1.12.1 Lemma. *There exists a unitary connection ∇ on a complex line bundle L over $\mathbb{T}^2$ so that*

$$\mathcal{F}(L) = \tfrac{1}{2}(dx^1 \wedge dx^2).$$

Proof. We decompose

$$\mathbb{T}^2 = [0, 2\pi] \times S^1/\cong \text{ where } (0, w) \cong (2\pi, w).$$

Let the line bundle L be defined by:

$$L := [0, 2\pi] \times [0, 2\pi] \times \mathbb{C}/\cong \text{ where } (0, w, z) \cong (2\pi, w, e^{-\sqrt{-1}x^2} z).$$

The complex line bundle L has a natural fiber metric since the *clutching function* $e^{-\sqrt{-1}x^2}$ is unitary. Let

$$A(x) := \tfrac{1}{2\pi}(x^1 dx^2)$$

be the connection 1-form. The clutching or transition function, which describes how fiber at $x_1 = 0$ is glued to the fiber at $x_1 = 2\pi$, is defined by $\Phi = -x^2$. Since

$$\mathcal{A}(0, w) = \mathcal{A}(2\pi, w) + d\Phi,$$

relation (3) in Lemma 1.10.1 is satisfied so $\mathcal{A}$ defines a Riemannian connection ∇ on L with associated normalized curvature

$$\tfrac{1}{2\pi}(dx^1 \wedge dx^2). \quad \square$$

1.13 The Hopf fibration

We can use Lemma 1.11.4 to see there exists a line bundle over S^2 with non-trivial harmonic curvature. Again, this line bundle can be given very explicitly. Let S^3 be the unit sphere in the quaternions $\mathbb{H}$; S^3 inherits a natural group structure. If $\vec{x} \in S^3$, let

$$\vec{x} = (x^0, x^1, x^2, x^3) = (z^0, z^1) = x^0 + x^1 i + x^2 j + x^3 k$$

where $z^0 = x^0 + ix^1$ and $z^1 = x^2 + ix^3$. Let $\zeta_1(\vec{x}) = i \cdot \vec{x}$, $\zeta_2(\vec{x}) = j \cdot \vec{x}$, and $\zeta_3(\vec{x}) = k \cdot \vec{x}$. The vectors

$$\{\vec{x}, \zeta_1(\vec{x}), \zeta_2(\vec{x}), \zeta_3(\vec{x})\}$$

form an orthonormal basis for $\mathbb{R}^4$ so $\{\zeta_1, \zeta_2, \zeta_3\}$ is an orthonormal frame for the tangent bundle of the Lie group S^3. Let $\{\zeta^1, \zeta^2, \zeta^3\}$ be the dual coframe for the cotangent bundle of S^3. The ζ^i and ζ_i are invariant under *right* multiplication and are a basis for the associated Lie algebra $\mathfrak{so}(3)$. We compute:

$$\begin{aligned}
\zeta_1 &= -x^1\partial_0 + x^0\partial_1 - x^3\partial_2 + x^2\partial_3, \ [\zeta_2, \zeta_3] = -2\zeta_1,\\
\zeta_2 &= -x^2\partial_0 + x^3\partial_1 + x^0\partial_2 - x^1\partial_3, \ [\zeta_3, \zeta_1] = -2\zeta_2,\\
\zeta_3 &= -x^3\partial_0 - x^2\partial_1 + x^1\partial_2 + x^0\partial_3, \ [\zeta_1, \zeta_2] = -2\zeta_3,\\
\zeta^1 &= -x^1dx^0 + x^0dx^1 - x^3dx^2 + x^2dx^3, \ d\zeta^1 = 2\zeta^2 \wedge \zeta^3,\\
\zeta^2 &= -x^2dx^0 + x^3dx^1 + x^0dx^2 - x^1dx^3, \ d\zeta^2 = 2\zeta^3 \wedge \zeta^1,\\
\zeta^3 &= -x^3dx^0 - x^2dx^1 + x^1dx^2 + x^0dx^3, \ d\zeta^3 = 2\zeta^1 \wedge \zeta^2.
\end{aligned}$$

We let the circle $S^1 \subset \mathbb{C}$ act on S^3 by complex multiplication from the left; let π be the natural projection from S^3 to the quotient manifold $S^1 \backslash S^3 = \mathbb{CP}^1 = S^2$; this is the *Hopf fibration.* In terms of coordinates, $\pi : S^3 \to S^2$ is defined by:

$$\begin{aligned}\pi(\vec{x}) =&(2Re(z^0\bar{z}^1), 2Im(z^0\bar{z}^1), |z^0|^2 - |z^1|^2)\\ =&(2(x^0x^2 + x^1x^3), 2(x^1x^2 - x^0x^3), x^0x^0 + x^1x^1 - x^2x^2 - x^3x^3).\end{aligned}$$

Since $\vec{x} \to e^{it} \cdot \vec{x}$ defines the 1-parameter flow for the vector field ζ_1, $\pi_*\zeta_1 = 0$. If $y = (y^0, y^1, y^2)$ are the standard coordinates on $\mathbb{R}^3$, then

$$\pi^*(dy^0 \circ dy^0 + dy^1 \circ dy^1 + dy^2 \circ dy^2) = 4\zeta^2 \circ \zeta^2 + 4\zeta^3 \circ \zeta^3.$$

We let g_n be the standard metric on S^n. Then π_* is a Riemannian submersion from (S^3, g_3) to $(S^2, \frac{1}{4}g_2)$ with vertical distribution spanned by ζ_1 and horizontal distribution spanned by $\{\zeta_2, \zeta_3\}$. We can check this normalization by computing

$$2\pi^2 = \mathrm{vol}(S^3, g_3) = \pi \cdot (2\pi) = \mathrm{vol}(S^2, \tfrac{1}{4}g_2) \cdot \mathrm{vol}(S^1, g_1).$$

If ν_2 is the volume element on $(S^2, \frac{1}{4}g_2)$, then

$$\pi^*\nu_2 = \zeta^2 \wedge \zeta^3.$$

Note that although the vector fields $\{\zeta_2, \zeta_3\}$ and covector fields $\{\zeta^2, \zeta^3\}$ are horizontal, they are *not* horizontal lifts of vector and covector fields on S^2. In §1.14, we will define a complex analogue of the Hopf fibration.

The vertical distribution is spanned by ζ_1; thus $e^1 = \zeta^1$ and $de^1 = 2\pi^*\nu_2$. We use Lemma 1.10.1 to see that the curvature tensor takes the form:

(1.13.a) $$\mathcal{F} = 2\nu_2.$$

We will need this result in the proof of Theorem 4.2.1. We also note that

$$\frac{1}{2\pi}\int_{S^2} \mathcal{F} = \frac{1}{\pi}\mathrm{vol}(S^2) = 1.$$

More generally, we can let S^1 act by complex multiplication on the unit sphere S^{2k+1} in $\mathbb{C}^{k+1}$ and let $\mathbb{CP}^k$ be the resulting quotient manifold. The natural projection

$$\pi : S^{2k+1} \to \mathbb{CP}^k$$

is a Riemannian submersion from (S^{2k+1}, g_{2k+1}) to $(\mathbb{CP}^k, g_{FS})$. The metric g_{FS} is called the *Fubini-Study metric* on $\mathbb{CP}^k$.

We can also let S^3 act by quaternion multiplication on the unit sphere S^{4k+3} in $\mathbb{H}^{k+1}$ and let $\mathbb{HP}^k$ be the resulting quotient manifold. The projection

$$\pi_{\mathbb{H}} : (S^{4k+3}, g_{4k+3}) \to (\mathbb{HP}^k, g_{\mathbb{HP}})$$

is a Riemannian submersion where $g_{\mathbb{HP}}$ is a suitably chosen metric. The metric $g_{\mathbb{HP}}$ is a homogeneous metric with isometry group $PSP(3)$; this metric has positive scalar curvature. Riemannian submersions with fibers $(\mathbb{HP}^2, g_{\mathbb{HP}})$ play an important role in the Gromov-Lawson conjecture. We refer to §5.2 for further details.

Note that the fibration $S^{2n+1} \to \mathbb{CP}^n$ defines a principal circle bundle. The associated complex line bundle L is, modulo a possible sign convention, the Hopf line bundle; taking the limit as $n \to \infty$ defines the classifying line bundle $L_{\mathbb{CP}}$ over $\mathbb{CP}^\infty$ discussed in §1.11. Thus the Hopf fibration plays an important role not only in differential geometry but also in algebraic topology. The quaternion Hopf fibration $S^{4n+1} \to \mathbb{HP}^n$ in a similar way defines the classifying $SU(2)$ principal bundle over $\mathbb{HP}^\infty$. If 'B' denotes the 'classifying functor', then

$$\mathbb{CP}^\infty = BU(1) \text{ and } \mathbb{HP}^\infty = BSU(2).$$

1.14 The Hopf manifold

The Hopf fibration discussed in §1.13 has a natural generalization to the complex setting. We refer to the material in §A.5 of the appendix for further information about complex geometry. Let $Z := S^1 \times S^3$ be the *Hopf manifold*. Let ζ_0 and ζ^0 be the usual orthonormal basis for the tangent and cotangent bundles of the circle and let ζ_i and ζ^i be the right invariant vector fields and covector fields defined in §1.13. We give Z the product metric

$$ds^2 = \zeta^0 \circ \zeta^0 + \zeta^1 \circ \zeta^1 + \zeta^2 \circ \zeta^2 + \zeta^3 \circ \zeta^3. \tag{1.14.a}$$

We define an almost complex structure on Z by defining:

$$J(\zeta_0) = \zeta_1,\ J(\zeta_1) = -\zeta_0,\ J(\zeta_2) = \zeta_3,\ \text{and } J(\zeta_3) = -\zeta_2.$$

Since J is unitary with respect to the metric defined in equation (1.14.a), ds^2 is a Hermitian metric. The holomorphic tangent bundle is spanned by

$$\xi_0 := \frac{1}{2}(\zeta_0 - \sqrt{-1}\zeta_1) \text{ and } \xi_1 := \frac{1}{2}(\zeta_2 - \sqrt{-1}\zeta_3).$$

Note that we have

$$\begin{aligned} 4[\xi_0, \xi_1] =& -\sqrt{-1}\{[\zeta_1, \zeta_2] - \sqrt{-1}[\zeta_1, \zeta_3]\} \\ =& 2\sqrt{-1}\zeta_3 - 2\zeta_2 = -4\xi_1. \end{aligned}$$

Thus the complex tangent bundle $T_{\mathbb{C}}Z := \operatorname{span}_{\mathbb{C}}\{\xi_0, \xi_1\}$ is integrable; if Ξ_i are smooth sections to $T_{\mathbb{C}}Z$, then $[\Xi_1, \Xi_2]$ is a smooth sections to $T_{\mathbb{C}}Z$. Thus the complex Frobenius integrability condition is satisfied so the Nirenberg-Neulander integrability theorem (see §A.5 Theorem A.5.1) implies that (Z, J) is a complex manifold.

We can also show that (Z, J) is a complex manifold directly. This can also be seen directly. Fix a real number $r > 1$. Let $\mathbb{Z}$ act on $\mathbb{C}^2 - \{0\}$ by

$$n : (z_1, z_2) \to r^n(z_1, z_2).$$

This defines a fixed point free holomorphic representation of $\mathbb{Z}$ on $\mathbb{C}^2 - \{0\}$ and we let Z be the resulting quotient manifold. We introduce coordinates (u, θ) on $\mathbb{C}^2 - \{0\}$ by

$$(u, \theta) \mapsto e^u\theta \text{ for } u \in \mathbb{R}, \theta \in S^3.$$

The action described above then takes the form $n : (u, \theta) \mapsto (u + \ln(r), \theta)$ so with a suitable choice of $r = e^{2\pi}$ we see that Z is isometric to $S^1 \times S^3$. If we identify ∂_r with ξ_0, then the complex structure is the one given above:

$$J\xi_0 = i\xi_0 = \xi_1 \text{ and } J\xi_2 = i\xi_2 = \xi_3$$

We give S^2 the standard complex structure. Let $\pi : S^3 \to S^2$ be the Hopf fibration defined in §1.13. Let

$$\tilde{\pi}(\lambda, \vec{x}) = \pi(\vec{x}) : Z \to S^2$$

define a fibration from the Hopf manifold to S^2. Since π is a Riemannian submersion, $\tilde{\pi}$ is a Riemannian submersion as well. We have $\tilde{\pi}_*\xi_0 = 0$. It is easily checked that $\tilde{\pi}_*\xi_1$ is a holomorphic tangent vector on S^2. This shows that $\tilde{\pi}_*$ is a holomorphic Riemannian submersion and the metrics involved are Hermitian. Note that the metric on S^2 is Kaehler but that the metric on Z is not Kaehler.

1.15 The geometry of sphere bundles

In §1.10 we studied the geometry of circle bundles; these were defined by complex line bundles or equivalently by orientable real vector bundles of rank 2. We can also study the higher rank case.

1.15.1 Definition. Let V be a real vector bundle of rank $r \geq 3$ over a Riemannian manifold Y. We assume that V is equipped with a fiber metric and let $S(V)$ be the unit sphere bundle. We use a Riemannian connection ∇ on the bundle V to split

$$T(V) = \mathcal{V} \oplus \mathcal{H};$$

see Lemma 1.15.2 below for further details. We use this splitting to define a Riemannian metric g_V so $\pi : V \to Y$ is a Riemannian submersion. The restriction of π to $S(V)$ defines a projection $\pi_S : S(V) \to Y$ which is a Riemannian submersion.

We now develop some of the Riemannian geometry of this situation. Introduce local coordinates $y = (y^a)$ on Y. Let $s = (s_i)$ be a local orthonormal frame for V. The map $(x, y) \to s_i(y)x^i$ introduces local coordinates on V. Let

$$\nabla s_i = A_{aij}(y)dy^a s_j$$

define the components of the connection 1-form A of ∇ relative to the given local frame; we omit the $\sqrt{-1}$ normalizing constant of §1.10. Let ∂_i^y and ∂_a^x be the coordinate frames for the tangent bundle of the total space V. The curvature of the bundle is given by the tensor ω.

1.15.2 Lemma.

(1) *We have* $g_V(\partial_a^y, \partial_b^y) = g_Y(\partial_a, \partial_b) + x^i x^j A_{aik}A_{bjk}$.

(2) *We have* $g_V(\partial_a^y, \partial_i^x) = A_{aji}x^j$.

(3) *We have* $g_V(\partial_i^x, \partial_j^x) = \delta_{ij}$.

(4) *We have* $\pi_S : S(V) \to Y$ *is a Riemannian submersion.*

(5) *The fibers of* π_S *are totally geodesic.*

(6) *If* $\tilde{f}_a := \partial_a^y - x^j A_{aji}\partial_i^x$, *then* $\tilde{f}_a$ *is the horizontal lift of* ∂_a^y.

(7) *Let* R_{abij} *be the curvature of the connection on* V. *Then* $2\omega_{abi} = -R_{abji}x^j$.

Proof. Let $f : \mathcal{O} \to O(r)$ define a local gauge transformation $\tilde{s} = f(y)s$. Fix $y_0 \in Y$ and choose f so that

$$f(y_0) = I \text{ and } df(y_0) = -A(y_0).$$

Then $\tilde{s}(y_0) = s(y_0)$ and $\nabla\tilde{s}(y_0) = 0$. Let $(\tilde{x}, \tilde{y})$ be the new system of local coordinates; the identity $x^i s^i = \tilde{x}^i\tilde{s}^i$ implies $y^a = \tilde{y}^a$ and $x^j = \tilde{x}^i f_{ij}$. Thus at a point $z_0 \in \pi^{-1}(y_0)$ we have:

$$\partial_i^{\tilde{x}} = \partial_i^x \text{ and } \partial_a^{\tilde{y}} = \partial_a^y - A_{aij}x^i\partial_j^x. \tag{1.15.a}$$

Since $\nabla\tilde{s}(y_0) = 0$, $\mathcal{H}(z_0) = \operatorname{span}\{\partial_a^{\tilde{y}}\}$ and $\mathcal{V}(z_0) = \operatorname{span}\{\partial_i^{\tilde{x}}\}$. Therefore

$$\begin{aligned} &g_V(\partial_a^{\tilde{y}}, \partial_b^{\tilde{y}})(z_0) = g_Y(\partial_a^y, \partial_b^y)(y_0), \\ &g_V(\partial_a^{\tilde{y}}, \partial_i^x)(z_0) = 0, \text{ and } g_V(\partial_i^x, \partial_j^x)(z_0) = \delta_{ij}. \end{aligned} \tag{1.15.b}$$

Assertions (1-3) now follow from equations (1.15.a) and (1.15.b). Assertion (4) is immediate. To prove assertion (5), normalize the choice of coordinates on Y so the 1-jets of the metric vanish at y_0 and choose the frame so $\nabla s(y_0) = 0$. Then the 1-jets of metric on V vanish at (x, y_0). Thus the curve $\gamma(t) = (x + t\bar{x}, y_0)$ is a geodesic in V and the fibers of V are totally geodesic. Let $\xi \in S_0$. Since orthogonal projection of $T_\xi V_0$ on $T_\xi S$ is contained in S_0, S_0 is a totally geodesic submanifold of S so (5) follows. Since $g_V(\tilde f_a, \partial_k^x) = 0$ and since $\{\partial_k^x\}$ span the vertical space, $\tilde f_a$ is the horizontal lift of ∂_a^y and (6) follows. We choose s so $A(y_0) = 0$ and compute

$$\begin{aligned}2\omega_{abi}(x, y_0) =& g_V([\tilde f_a, \tilde f_b], e_i)(x, y_0) = (\partial_b^y A_{aji} - \partial_a^y A_{bji})(y_0)x^j \\ =& R_{baji}(y_0)x^j.\end{aligned}$$

Since ω and R are tensorial, assertion (7) follows. $\square$

1.16 Principal bundles

We refer to Eguchi et al [47] for further details concerning the material of this section.

1.16.1 Definition. Let g_G be a bi-invariant Riemannian metric on a compact Lie group G. We say that $\pi : P \to Y$ is a *principal G bundle* if

(1) The manifold P has a right action by G that preserves π, i.e. $\pi(z \cdot g) = \pi(z)$.

(2) The group G acts transitively and without fixed points on the fibers of π.

Let $\mathfrak{g}_i$ be a basis for the Lie algebra of left invariant vector fields on G and let $\mathfrak{g}^i$ be the corresponding dual basis for the left invariant 1 forms on G. Let $g_i(t)$ be the 1 parameter subgroup of G corresponding to the vector fields $\mathfrak{g}_i$. Then right multiplication by $g_i(t)$ defines a flow for the vector fields $\mathfrak{g}_i$ on G. Right multiplication by the $g_i(t)$ also defines a flow on the principal bundle P. By an abuse of notation, we will also denote the corresponding vector fields on P by $\mathfrak{g}_i$; these vector fields extend the given vector fields $\mathfrak{g}_i$ from G to P. The vertical distribution $\mathcal{V}$ of π is spanned by the the vector fields $\mathfrak{g}_i$.

1.16.2 Definition. A metric g_P on P is called a *bundle* metric if

(1) The metric g_P is invariant under the action of G.

(2) The restriction of g_P to the fibers of π is the given bi-invariant metric g_G.

(3) The map π is a Riemannian submersion.

Note that the fibers of π are necessarily totally geodesic if g_P is a bundle metric.

1.16.3 Examples.

(1) The circle bundles discussed in §1.10 are principle bundles with fiber S^1.

(2) The bundle $S^{4k+3} \to \mathbb{HP}^k$ discussed in §1.13 is a principal bundle with structure group $S^3 = SU(2)$.

(3) If Y is a Riemannian manifold, then the bundle of all orthonormal frames for the tangent bundle TY is a principal bundle with structure group the orthogonal group $O(\dim(Y))$.
(4) If Y is an orientable Riemannian manifold, then the bundle of oriented orthonormal frames for the tangent bundle TY is a principal bundle with structure group the special orthogonal $SO(\dim(Y))$.
(5) If Y is a Hermitian manifold, then the bundle of all unitary frames for the complex tangent bundle $T_{\mathbb{C}}Y$ is a principal bundle with structure group the unitary group $U(\dim_{\mathbb{C}}(Y))$.

Let $y = (y^1, ..., y^{\dim(Y)})$ be a system of local coordinates on the base Y and let s be a local section to the principal bundle P. The map $(g, y) \to s(y) \cdot g$ gives local coordinates to P. The horizontal distribution $\mathcal{H}$ of π is invariant under the G action and is spanned by vector fields of the form

$$f_a := \partial_a^y + \Gamma_a{}^i(y)\mathfrak{g}_i.$$

1.16.4 Definition. The Lie-algebra valued 1-form

$$\mathcal{A} := \Gamma_a{}^i(y) dy^a \otimes \mathfrak{g}_i$$

defines a principal connection. Suppose that G is a matrix group. Let V be the associated vector bundle. Then the principal connection on P defines a linear connection ∇ on V and the curvature of ∇ is given by $\mathcal{A}$; the connection is Riemannian if $G = O(k)$, $G = SO(k)$, or $G = U(k)$.

1.16.5 Definition. The curvature $\mathcal{F}$ of the principal bundle P is the Lie-algebra valued 2-form given by

$$\begin{aligned}\mathcal{F} :=& dy^a \wedge dy^b \otimes (\partial_a^y \Gamma_b{}^i \mathfrak{g}_i - \partial_b^y \Gamma_a{}^i \mathfrak{g}_i + \Gamma_a{}^i \Gamma_b{}^j [\mathfrak{g}_i, \mathfrak{g}_j])\\ =& g^{ij}\omega_{abi} dy^a \wedge dy^b \otimes \mathfrak{g}_j.\end{aligned}$$

Thus in this setting, the tensor ω is the principle bundle connection. Again, if G is a matrix group, $\mathcal{F}$ gives the curvature of the associated connection on the associated vector bundle.

Let $\mathfrak{H}F$ be the horizontal lift to P of a vector field F on Y. If ${}^Y\vec{s} = (F_1, ...)$ is a local orthonormal frame field for TY, let

(1.16.a) $$^P\vec{s} := (f_1, ..., \mathfrak{g}_1, ...) \text{ where } f_a := \mathfrak{H}F_a$$

be the corresponding local orthonormal frame field for TP. This shows that TP and TY are defined by the same transition functions; this will play an important role in §2.7 when we discuss projectable spinors.

Let Γ^P and Γ^Y be the Christoffel symbols of P and Y with respect to these local frames.

1.16.6 Lemma.

(1) *The fibers of* P *are totally geodesic, i.e.* $\Gamma^P_{ija} = 0$.

(2) *We have* $[\mathfrak{g}_i, f_a] = 0$.

(3) *We have* $\Gamma^P_{abc} = \Gamma^Y_{abc}$, $\Gamma^P_{abi} = -\Gamma^P_{aib} = -\Gamma^P_{iab} = \omega_{abi}$, *and* $\Gamma^P_{iaj} = \Gamma^P_{aij} = 0$.

Proof. If ${}^G e \in \mathfrak{g}$, let e be the corresponding vertical vector field on P. The 1-parameter group generated by e acts by isometries since the metric on P is invariant under right multiplication. Furthermore, the length of e is constant on P. It then follows that the integral curves of e, which lie in the fibers of π, are geodesics; this implies the fibers of π are totally geodesic so the second fundamental form $\Gamma^P_{ija} = 0$; this proves the first assertion. Let F be a local tangent vector field on Y. The horizontal distribution is invariant under the action of G; thus the vector field $f := \mathfrak{H}(F)$ is invariant under the flow defined by e so $[e, f] = 0$; the second assertion now follows. As $\pi_*(\mathfrak{H}F) = F$, $\pi_*[f_a, f_b] = [F_a, F_b]$. Thus, since π is a Riemannian submersion,

$$g_P([f_a, f_b], f_c) = g_Y([F_a, F_b], F_c).$$

We express Γ in terms of the Lie bracket to see $\Gamma^P_{abc} = \Gamma^Y_{abc}$. Since $[\mathfrak{g}_i, f_a] = 0$,

$$\Gamma^P_{aij} = \Gamma^P_{iaj} = -\Gamma^P_{ija} = 0 \text{ and } \Gamma^P_{abi} = -\Gamma^P_{aib} = -\Gamma^P_{iab}.$$

This shows Γ^P_{abi} is skew symmetric in a and b. Thus

$$\Gamma^P_{abi} = \tfrac{1}{2}(\Gamma^P_{abi} - \Gamma^P_{bai}) = \omega_{abi}. \quad \Box$$

1.17 Integration over the fibers

Let $\pi : Z \to Y$ be a Riemannian submersion with fiber $X(y) := \pi^{-1}y$ over a point $y \in Y$. Let $X = X(y_0)$ be the fiber over the basepoint. Let ν_X be the volume form on fiber. If dvol_Z and dvol_Y are the volume forms on Z and Y respectively, then we have $\mathrm{dvol}_Z := \nu_X \wedge \pi^* \mathrm{dvol}_Y$.

Let $\pi : Z \to Y$ be a Riemannian submersion with fiber X. Pullback defines a natural morphism $\pi^* : C^\infty\Lambda^p(Y) \to C^\infty\Lambda^p(Z)$.

1.17.1 Definition. We average over the fibers to define the *push forward.* This is a map π_* from $C^\infty\Lambda^p(Z)$ to $C^\infty\Lambda^p(Y)$. Let $\phi \in C^\infty\Lambda^p(Z)$. Let F_1, ..., F_p be tangent vectors at $y \in Y$. Let $V(y) := \int_{x\in X(y)} \nu_X(x, y)$ be the *volume of the fiber.* We define:

$$\pi_*(\phi)(F_1, ..., F_p) := V(y)^{-1} \int_{x\in X(y)} \phi(\mathfrak{H}F_1, ..., \mathfrak{H}F_p)\nu_X(x, y).$$

There is an alternate description which is also useful. Let $\rho^{\mathcal{H}}$ be the orthogonal projections of $\Lambda^p(Z)$ on $\Lambda^p(\mathcal{H}^*) = \pi^*\Lambda^p(Y)$. We decompose $\rho^{\mathcal{H}}$ and then compute:

$$\rho^{\mathcal{H}}(\phi) = \sum_{|A|=p} c_A(x,y)\pi^*(dy^A) \text{ and}$$
$$\pi_*\phi = V(y)^{-1}\sum_A \int_{x\in X(y)} c_A(x,y)\nu_X(x,y)dy^A.$$

It is immediate from the definition that $\pi_*\pi^*$ is the identity on $C^\infty\Lambda^p(Y)$.

We can relate the volume of the fiber and the push forward of θ as follows:

1.17.2 Lemma. *Let $\pi : Z \to Y$ be a Riemannian submersion with fiber X.*

(1) *We have $\pi_*\theta = -d_Y \log(V)$.*

(2) *If $\theta = 0$, then V is constant.*

Proof. We introduce a natural bigrading and decompose $\Lambda^*(Z) = \oplus_{p,q}\Lambda^p(\mathcal{H}^*)\otimes\Lambda^q(\mathcal{V}^*)$. Let $\rho^{(p,q)}$ be the corresponding orthogonal projections. Fix $y_0 \in Y$. We use assertion Lemma 1.3.3 to choose a local trivialization of π. This permits us to assume that locally we have $Z = X \times \mathcal{O}$ and $\mathcal{H}(x, y_0) = \operatorname{span}\{\partial_a^y\}$. We choose local coordinates $x_\alpha = (x_\alpha^s)$ on the fiber X and let ϕ_α be a partition of unity subordinate to this cover. Let

$$X_\alpha^s := \rho^{(0,1)}dx_\alpha^s = dx_\alpha^s - C_{\alpha,a}^s dy^a, \text{ and}$$
$$\nu_X = g_\alpha(x,y)X^1 \wedge ... \wedge X^{n-m}.$$

We use Lemma 1.6.1 to see that $\rho^{(1,n-m)}d_Z(\nu_X) = -\theta\wedge\nu_X$. We evaluate at a point $(x,0)$ and use the fact $C_a^i(x,0) = 0$ to see $\rho^{(1,n-m)}d_Z(\nu_X) = (g_\alpha^{-1}\partial_a^y g_\alpha)dy^a \wedge \nu_X$. Consequently $-\theta(\partial_a^y)(x,0) = g_\alpha^{-1}\partial_a^y g_\alpha$. We complete the proof of the first assertion by computing:

$$\begin{aligned}\partial_a^y V(y) =& \sum_\alpha \int_X \phi_\alpha g_\alpha^{-1}(\partial_a^y g_\alpha)\nu_X = -\sum_\alpha \int_X \phi_\alpha \theta(\partial_a^y)\nu_X \\ =& - V(y)\pi_*\theta(\partial_a^y).\end{aligned}$$

The second assertion follows immediately from the first. $\square$

We will use the first assertion of this Lemma in §3.5 when we discuss the complex Laplacian. We will use the second assertion of the Lemma in §3.10 when we discuss the heat content asymptotics.

Chapter Two: Operators of Laplace Type

2.1 Introduction

In this chapter, we study partial differential operators which are of Laplace type. In §2.2, we define the symbol of an operator. In §2.3, we discuss the discrete spectral resolution of an operator of Laplace type. In §2.4, we use spherical harmonics to construct the discrete spectral resolution of the scalar Laplacian on the sphere. In §2.5, we use spherical harmonics to construct the discrete spectral resolution of the scalar Laplacian for the Hopf manifold. In §2.6, we give a normal form for any operator of Laplace type and discuss the Bochner Laplacian. We state the Bochner and Lichnerowicz vanishing theorems. We also present a brief introduction to the heat equation asymptotics; if π is a Riemannian submersion, there is no simple relationship between the invariants of the base, the invariants of the fiber, and the invariants of the total space.

In §2.7, we generalize the intertwining formulas of §1.5 from the real to the complex category. In the spinor setting, the situation is quite different and we assume $\pi : Z \to Y$ is a principal bundle. We choose a spinor on the Lie group in question to define the notion of pull back; this is the construction of *projectable* spinors of Moroianu. Once this is done, we can determine analogous intertwining formulas for the Dirac operators on Y and on Z. We refer to the material of the appendix §A.2 and §A.3 for further details concerning spinors.

In §2.8, we discuss elliptic operators on manifolds with boundary. We discuss Green's formula and introduce absolute, relative, Dirichlet, and Neumann boundary conditions. In §2.9, we discuss Riemannian submersions in the category of manifolds with boundary. We show that pullback preserves Dirichlet and absolute boundary conditions. We give necessary and sufficient conditions that relative and Neumann boundary conditions are preserved. We show that the Neumann

Laplacian on forms need not be positive definite; this is a somewhat surprising result.

2.2 The symbol of an operator

Let $u = (u^1, ..., u^m)$ be a system of local coordinates on a Riemannian manifold M. Let $\partial_i^u = \partial/\partial u^i$. If $\alpha = (\alpha_1, ..., \alpha_m)$ is a multi-index, let

$$\partial_\alpha^x := (\partial_1^x)^{\alpha_1}...(\partial_m^x)^{\alpha_m}, \ \xi^\alpha := \xi_1^{\alpha_1}...\xi_m^{\alpha_m}, \text{ and } |\alpha| := \alpha_1 + ... + \alpha_m.$$

Let V and W be smooth complex vector bundles over M and let P mapping $C^\infty(V)$ to $C^\infty(W)$ be a partial differential operator of order n. Choose local frames for V and W to decompose

$$P = \textstyle\sum_{|\alpha|\leq n} a_\alpha(x)\partial_\alpha^x$$

where for any point $x \in M$, the coefficients $a_\alpha(x)$ are linear maps from the fiber of V over x to the fiber of W over x. We define the *leading symbol* of P by replacing differentiation with multiplication:

$$\sigma_L(P)(x,\xi) := \textstyle\sum_{|\alpha|=n} a_\alpha(x)\xi^\alpha.$$

If we identify ξ with the cotangent vector $\xi := \sum_i \xi_i dx^i$, then $\sigma_L(P)(x,\xi)$ is an invariantly defined map which is homogeneous of degree n from the cotangent bundle T^*M to the bundle of endomorphisms from V to W. We have

$$\sigma_L(P^*) = (-1)^n \sigma_L(P)^* \text{ and}$$
$$\sigma_L(P_1 \circ P_2) = \sigma_L(P_1) \circ \sigma_L(P_2).$$

The leading symbol is sometimes defined with factors of $\sqrt{-1}$ to make formulas involving the Fourier transform and the adjoint more elegant; we delete these factors in the interests of simplicity.

2.2.1 Definition. Let $D : C^\infty(V) \to C^\infty(V)$ be a second order operator. We say that D is of *Laplace type* if the leading symbol of D is scalar and is given by the metric tensor. This means that locally D has the form:

$$D = -g^{ij}\partial_i^u \partial_j^u \cdot I_V + a^k \partial_k^u + b$$

where the a^k and b are endomorphisms of V.

2.2.2 Definition. We say that a first order operator $A : C^\infty(V) \to C^\infty(V)$ is of *Dirac type* if A^2 is of Laplace type. This means that locally A has the form:

$$A = c^i \partial_i^u + c_0$$

where $c^i c^j + c^j c^i = -2g^{ij} I_V$.

Let $\mathrm{Clif}(T^*M)$ be the universal unital algebra bundle generated by the cotangent bundle of M subject to the *Clifford commutation rules*

$$\xi * \eta + \eta * \xi = -2(\xi, \eta)1.$$

We refer to §A.2 for a further discussion of the Clifford algebra. A first order operator A is of Dirac type if and only if the leading symbol of A gives V a $\mathrm{Clif}(T^*M)$ module structure.

2.2.3 Example. We can use Theorem 1.4.4 to see that the operator $d + \delta$ has leading symbol $\mathrm{ext} - \mathrm{int}$ and therefore the leading symbol of the operator Δ is $-|\xi|^2$. Thus $d + \delta$ is an operator of Dirac type and Δ is an operator of Laplace type. Furthermore $c(\xi) := \mathrm{ext}(\xi) - \mathrm{int}(\xi)$ gives the exterior algebra a Clifford module structure.

2.3 Spectral resolution

Let D be a second order operator of Laplace type on a smooth vector bundle V over a Riemannian manifold M. We assume V is equipped with a smooth inner product we use to define $L^2(V)$. We say that D is *self-adjoint* if we have

$$(D\phi, \psi)_{L^2(V)} = (\phi, D\psi)_{L^2(V)}$$

for every ϕ and ψ in $C^\infty V$. We refer to Gilkey [64] for the proof of the following result. It shows that self-adjoint operators of Laplace type have a *discrete spectral resolution.*

2.3.1 Theorem. *Let D be a self-adjoint operator of Laplace type on the space of smooth sections to a vector bundle V over a compact Riemannian manifold M of dimension m without boundary.*

(1) *There exists a complete orthonormal basis $\{\phi_n\}$ for $L^2(V)$ where the ϕ_n belong to $C^\infty(V)$ so that $D\phi_n = \lambda_n \phi_n$.*

(2) *We have $\lim_{n\mapsto\infty} \lambda_n = \infty$. Order the eigenvalues so that $\lambda_1 \le \lambda_2 \ldots$. For any $\epsilon > 0$, there exists an integer $n(\epsilon)$ so that*

$$n^{\frac{2}{m}-\epsilon} \le \lambda_n \le n^{\frac{2}{m}+\epsilon} \text{ for } n \ge n(\epsilon).$$

(3) *If* $\phi \in L^2(V)$, *let* $c_n = (\phi, \phi_n)_{L^2(V)}$ *be the Fourier coefficients. Then* ϕ *belongs to* $C^\infty(V)$ *if and only if the* c_n *are rapidly decreasing, i.e.* $\lim_{n\mapsto\infty} n^k c_n = 0$ *for any natural number* k.

(4) *Let* ∇ *be a Riemannian connection on* V *and let* $|\phi|_k$ *be the sup norm of the* k^{th} *covariant derivative of* ϕ. *Then there exists* $j(k)$ *so that* $|\phi_n|_k \le n^{j(k)}$ *if* n *is sufficiently large.*

If ϕ is smooth, it follows from this Theorem that the series

$$\phi = \sum_n c_n \phi_n$$

converges absolutely to ϕ in the C^∞ topology. We will use this observation in the proof of Lemma 3.5.7 and in the proof of Theorem 3.8.1. Let

$$E(\lambda, D) = \{\phi \in C^\infty(V) : D\phi = \lambda\phi\}$$

be the associated eigenspaces; these are finite dimensional for all λ and non trivial only for a discrete set of eigenvalues λ. Then we have an orthogonal direct sum decomposition

$$L^2(V) = \oplus_\lambda E(\lambda, D).$$

Note that the linear span of the eigenfunctions is dense in $C^\infty(V)$ in the C^∞ topology.

2.4 Spherical harmonics

Let $M = S^1$ be the unit circle with the usual periodic parameter θ. The Laplacian on functions is given by $-\partial_\theta^2$ and the discrete spectral resolution takes the form $\{e^{\sqrt{-1}n\theta}, n^2\}$ for $n \in \mathbb{Z}$. This means that the functions $e^{\sqrt{-1}n\theta}$ form a complete orthonormal basis for $L^2(S^1)$ and we have

$$-\partial_\theta^2(e^{\sqrt{-1}n\theta}) = n^2 e^{\sqrt{-1}n\theta}.$$

The usual decomposition of a smooth function as a Fourier series is the eigen function decomposition described in §2.3.

The spectral resolution of the Laplacian on the unit sphere S^m is given in terms of *spherical harmonics*. Let $x = (x^0, ..., x^m) \in S^m$. Let $S(m, j)$ be the set of all polynomials $p(x)$ which are homogeneous of order j. Let $\Delta_e := -\partial_0^2 - ... - \partial_m^2$ be the Euclidean Laplacian. Let $H(m, j) \subset S(m, j)$ be the subset of homogeneous harmonic polynomials. Let $\tilde{H}(m, j)$ be the restriction of these functions to S^m.

2.4.1 Lemma. *Let Δ_S be the scalar Laplacian on $C^\infty(S^m)$.*

(1) *We have* $\dim \tilde{H}(m,j) = \binom{m+j}{m} - \binom{m+j-2}{m}$.

(2) *The discrete spectral resolution of* Δ_S *is* $\{\tilde{H}(m,j), j(j+m-1)\}_{j\geq 0}$.

2.4.2 Remark. In Example 3.9.8 we will use spherical harmonics to discuss the spectral resolution of the Laplacian on a lens space $S^{2k-1}/\mathbb{Z}_\ell$.

Proof. We use an argument which is is due to Calderon. It is clear that

$$S(m,j) = x_m \cdot S(m,j-1) \oplus S(m-1,j).$$

This relation implies that

$$\dim S(m,j) = \dim S(m,j-1) + \dim S(m-1,j).$$

Note that $\dim S(m,0) = 1$ and $\dim S(0,j) = 1$. We now use induction to see that

$$\dim S(m,j) = \binom{m+j}{m}.$$

If $p(x)$ is a polynomial symbol, we expand

$$p(x) = \sum_\alpha p_\alpha x^\alpha; \text{ let } P(p) := \sum_\alpha p_\alpha \partial_\alpha^x$$

be the associated partial differential operator with symbol p. Define a positive definite symmetric inner product on $S(m,j)$ by:

$$(p,q) = \sum_\alpha \alpha! p_\alpha q_\alpha = P(p)q.$$

If $p \in S(m,j-2)$ and if $q \in S(m,j)$, we have

$$(-p, \Delta_e q) = (r^2 p, q)$$

since Δ_e corresponds to $-r^2 \in S(m,2)$. Since multiplication by r^2 is injective, the cokernel of r^2 is the kernel of Δ_e. It now follows that

$$S(m,j) = r^2 \cdot S(m,j-2) \oplus H(m,j). \tag{2.4.a}$$

The restriction of a homogeneous polynomial to the sphere defines an injective map from $H(m,j)$ to $\tilde{H}(m,j)$. The first assertion of the Lemma now follows from the observation that:

$$\dim H(m,j) = \dim S(m,j) - \dim S(m,j-2).$$

Let

$$\mathcal{A} := \sum_j H(m,j) \subset C^\infty(S^m)$$

be the subspace generated by the harmonic polynomials. Since $r^2 = 1$ on S^m, we can use equation (2.4.a) to see that $\mathcal{A} = \sum_j S(m,j)$ as well. Thus $\mathcal{A}$ forms a unital algebra separating points. We use the *Stone-Weierstrauss* theorem to see that $\mathcal{A}$ is dense in $L^2(S^m)$.

Introduce polar coordinates (r,θ) for $r \in \mathbb{R}^+$ and $\theta \in S^m$ on $\mathbb{R}^{m+1}$. We can then decompose the Euclidean Laplacian in the form:

$$\Delta_e = -\partial_r^2 - mr^{-1}\partial_r + r^{-2}\Delta_S. \tag{2.4.b}$$

If $p \in H(m,j)$, then $\Delta_e p = 0$. Since p is homogeneous of degree j, we use equation (2.4.b) to see

$$\Delta_S p = j(j+m-1)p.$$

Thus

$$\tilde{H}(m,j) \subset \ker(\Delta_S - j(j+m-1)) = E(j(j+m-1), \Delta_S).$$

Since Δ_S is self-adjoint, the eigenspaces are orthogonal and therefore

$$\tilde{H}(m,j) \perp \tilde{H}(m,k) \text{ in } L^2(S^m) \text{ if } j \neq k.$$

Since $\mathcal{A}$ is dense in $C^\infty(S^m)$, it is dense in $L^2(S^m)$ and thus

$$L^2(S^m) = \oplus_{j\geq 0}\tilde{H}(m,j).$$

The second assertion now follows; we have

$$\tilde{H}(m,j) = E(j(j+m-1), \Delta_S). \quad \square$$

2.5 Spectral resolution of the Hopf manifold

Let $Z = S^1 \times S^3$ be the *Hopf manifold* discussed in §1.14. We refer to §A.5 for a definition of the complex Laplacian $\Delta^{(p,q)}$.

2.5.1 Lemma. *Let Z be the Hopf manifold.*

(1) *We have* $\Delta_Z^0 = -(\zeta_0^2 + \zeta_1^2 + \zeta_2^2 + \zeta_3^2)$ *and* $2\Delta_Z^{(0,0)} = \Delta_Z^0 + 2\sqrt{-1}\zeta_1$.

(2) *We have* $\Delta_Z^0 \Delta_Z^{(0,0)} = \Delta_Z^{(0,0)}\Delta_Z^0$.

Proof. Let $\star$ be the Hodge operator defined in Definition 1.4.7. Note that the adjoint of the operator $\bar\partial$ is given by $\bar\partial^* = -\star d\star$ on $C^\infty\Lambda^{(0,1)}$. We compute:

$$\begin{aligned}
&\xi_0 = \tfrac12(\zeta_0 - \sqrt{-1}\zeta_1), && \xi_1 = \tfrac12(\zeta_2 - \sqrt{-1}\zeta_3),\\
&\xi^0 = \zeta^0 + \sqrt{-1}\zeta^1, && \xi^1 = (\zeta^2 + \sqrt{-1}\zeta^3),\\
&\partial f = \xi_0 f\cdot\xi^0 + \xi_1 f\cdot\xi^1, && \bar\partial f = \bar\xi_0 f\cdot\bar\xi^0 + \bar\xi_1 f\cdot\bar\xi^1,\\
&\star(\bar\xi^0) = -\tfrac12\bar\xi^0\wedge\xi^1\wedge\bar\xi^1, && \star(\bar\xi^1) = \tfrac12\bar\xi^1\wedge\xi^0\wedge\bar\xi^0,\\
&d(\star\bar\xi^0) = d(\star(\bar\xi^1) = 0.
\end{aligned}$$

We can now determine the associated complex Laplacian $\Delta^{(0,0)}$ on functions:

$$\begin{aligned}
\Delta_Z^{(0,0)} f &= -\star d\star(\bar\xi_0 f\cdot\bar\xi^0 + \bar\xi_1 f\cdot\xi^1)\\
&= \tfrac12\star d(\bar\xi_0 f\cdot\bar\xi^0\wedge\xi^1\wedge\bar\xi^1 + \bar\xi_1 f\cdot\bar\xi^1\wedge\xi^0\wedge\bar\xi^0)\\
&= \tfrac12\star(\xi_0\bar\xi_0 f + \xi_1\bar\xi_1 f)\xi^0\wedge\bar\xi^0\wedge\xi^1\wedge\bar\xi^1\\
&= -2(\xi_0\bar\xi_0 + \xi_1\bar\xi_1)f\\
&= -\tfrac12(\zeta_0^2 + \zeta_1^2 + \zeta_2^2 + \zeta_3^2 + \sqrt{-1}[\zeta_0,\zeta_1] + \sqrt{-1}[\zeta_2,\zeta_3])\\
&= -\tfrac12(\zeta_0^2 + \zeta_1^2 + \zeta_2^2 + \zeta_3^2 - 2\sqrt{-1}\zeta_1),\\
\Delta_Z^{(0,0)} &+ \bar\Delta_Z^{(0,0)} = \Delta_Z^0 = -(\zeta_0^2 + \zeta_1^2 + \zeta_2^2 + \zeta_3^2)\\
\Delta_Z^{(0,0)} &- \bar\Delta_Z^{(0,0)} = 2\sqrt{-1}\zeta_1, \text{ and}\\
2\Delta_Z^{(0,0)} &= \Delta_Z^0 + 2\sqrt{-1}\zeta_1.
\end{aligned}$$

We complete the proof by checking:

$$\begin{aligned}
-[\zeta_1,\Delta_Z^0] &= \{[\zeta_1,\zeta_2^2] + [\zeta_1,\zeta_3^2]\}.\\
&= [\zeta_1,\zeta_2]\zeta_2 + \zeta_2[\zeta_1,\zeta_2] + [\zeta_1,\zeta_3]\zeta_3 + \zeta_3[\zeta_1,\zeta_3]\\
&= -2\zeta_3\zeta_2 - 2\zeta_2\zeta_3 + 2\zeta_2\zeta_3 + 2\zeta_3\zeta_2 = 0. \quad \square
\end{aligned}$$

We can now give the discrete spectral resolution of $\Delta^{(0,0)}$ on the Hopf manifold; these results also follow from work of Bedford and Suwa [11] who used a different approach. The map

$$e^{it}:(\lambda,\vec x)\to(e^{it}\lambda,\vec x)$$

gives a circle action on Z by isometries. We decompose

$$L^2(Z) = \oplus_j\lambda^j\cdot L^2(S^3)$$

under this action. The real and complex Laplacians on Z respect this decomposition so it suffices to study $\Delta_{S^3}^0$ and $\Delta_{S^3}^0 + 2\sqrt{-1}\zeta_1$ to determine the spectral resolution. Let $\vec x$ be the standard coordinates on S^3. Let $H(3,k)$ be the space of harmonic polynomials on S^3 which are homogeneous of degree k as discussed in §2.4. If $p\in H(3,k)$, then $\Delta_Z^0 p = k(k+2)p$. This proves

2.5.2 Theorem. *The decomposition $L^2(Z) = \oplus_{j,k}\lambda^j \cdot H(3,k)$ is the spectral resolution of the real Laplacian Δ_Z^0 on the Hopf manifold $S^1 \times S^3$; the associated eigenvalue is $j^2 + k(k+2)$.*

The vector field ζ_1 is left quaternion multiplication by $\sqrt{-1}$ and generates an isometric circle action

$$\phi(t) : z \to (\cos(t) + \sqrt{-1}\sin(t))z$$

which commutes with the real Laplacian. We decompose the eigenspace of the Laplacian on S^3 into eigenspaces of this action

$$H(3,k) := \oplus_\ell H_\ell(3,k) \text{ where } \phi(t)^*p = e^{\sqrt{-1}\ell t}p.$$

The following theorem now follows from Lemma 2.5.1 and from Theorem 2.5.2:

2.5.3 Theorem. *The decomposition $L^2(Z) = \oplus_{j,k,\ell}\lambda^j \cdot H_\ell(3,k)$ is an eigenvalue decomposition of both the real and complex Laplacians on the Hopf manifold. The associated eigenvalue of the real Laplacian Δ_Z^0 is $j^2 + k(k+2)$. The associated eigenvalue of the complex Laplacian $\Delta_Z^{0,0}$ is $\frac{1}{2}j^2 + k(k+2) - 2\ell$.*

Let $z^0 = x^0 + \sqrt{-1}x^1$ and $z^1 = x^2 + \sqrt{-1}x^3$. These are not, of course, holomorphic functions on Z. We note $\zeta_1 z^i = \sqrt{-1}z^i$ and $\zeta_1 \bar{z}^0 = -\sqrt{-1}\bar{z}^i$. The functions $\{z^0, z^1, \bar{z}^0, \bar{z}^1\}$ generate the algebra of all polynomials. Thus we see that we need $-k \le \ell \le k$ when studying $H_\ell(3,k)$ so

$$L^2(Z) = \oplus_{j,|\ell|\le k}\lambda^j \cdot H_\ell(3,k).$$

The space $j = 0$ and $\ell = 0$ corresponds to functions which are invariant under both circle actions; such functions are the pull back of eigenfunctions on the 2-sphere. Since S^2 is Kaehler, $\Delta_{S^2}^0 = 2\Delta_{S^2}^{(0,0)}$.

2.6 The Bochner Laplacian

There is a normal form for any operator of Laplace type.

2.6.1 Definition. If ∇ is a Riemannian connection on V and if E is an auxiliary self-adjoint endomorphism of V, we can define

$$D(\nabla, E) := -(g^{\nu\mu}\nabla_\nu\nabla_\mu + E)$$

where $g^{\nu\mu}$ denotes the metric tensor on the cotangent bundle. The associated operator $-g^{\nu\mu}\nabla_\nu\nabla_\mu$ is called the *Bochner Laplacian*.

We refer to [64] for the proof of the following result

2.6.2 Lemma. *Let D be a self-adjoint operator of Laplace type. There exists a unique Riemannian connection ∇ on V and a unique self-adjoint endomorphism E of V so that $D = D(\nabla, E)$. If $D = -(g^{\nu\mu} I_V \partial_\nu \partial_\mu + a^\sigma \partial_\sigma + b)$, then the connection 1 form $\tilde{A}$ of ∇ and endomorphism E are given by*

$$\begin{aligned}
\tilde{A}_\delta &= \tfrac{1}{2} g_{\nu\delta}(a^\nu + g^{\mu\sigma}\Gamma_{\mu\sigma}{}^\nu I_V) \\
E &= b - g^{\nu\mu}(\partial_\nu \tilde{A}_\mu + \tilde{A}_\nu \tilde{A}_\mu - \tilde{A}_\sigma \Gamma_{\nu\mu}{}^\sigma)
\end{aligned}$$

We will use this decomposition in §3.10 when we discuss the heat content asymptotics.

2.6.3 Example. Let d be the exterior derivative and δ the coderivative. We form the Laplacian $\Delta = d\delta + \delta d$. The Weitzenböch formulas express Δ in the form given by Lemma 2.6.2. The associated connection is the Levi-Civita connection and the endomorphism E is given by

$$E = -\tfrac{1}{8} R_{ijkn} c^l(e^i) c^l(e^j) c^r(e^k) c^r(e^n) - \tfrac{1}{4} R_{ijji} I$$

where $c^l = \text{ext}^l - \text{int}^l$ is defined by left exterior and $c^r = \text{ext}^r - \text{int}^r$ is defined by right exterior and interior multiplication. If $p = 0$, the endomorphism E vanishes and we have

$$\Delta^0 = -g^{\nu\mu}\nabla_\nu \nabla_\mu.$$

If $p = 1$, the endomorphism E is given by the Ricci tensor. Let $\rho_{ij} := R_{ikkj}$ be the Ricci tensor. If ϕ is a 1-form, then the action of the Ricci tensor on ϕ is defined by:

$$\rho(\phi) := \rho_{ij}\,\text{ext}(e^i)\,\text{int}(e^j)\phi.$$

The Weitzenböch formulas then become:

$$\Delta^1 = -g^{\nu\mu}\nabla_\nu \nabla_\mu + \rho.$$

2.6.4 Example. Let $\mathcal{R} := R_{ijji}$ be the scalar curvature. Let $D^{\mathcal{C}}$ be the spin Laplacian (see §A.2). The associated connection ∇ is the spin connection and the Lichnerowicz formula [123] shows

$$D^{\mathcal{C}}\Phi = \text{Tr}(\nabla^2 \Phi) + \tfrac{1}{4}\mathcal{R}\Phi, \text{ i.e.} E^{\mathcal{C}} = -\tfrac{1}{4}\mathcal{R}.$$

We can use the Weitzenböch and Lichnerowicz formulas to prove the following theorem which will be used in §5.2.

2.6.5 Theorem. *Let M be a compact Riemannian manifold*

(1) *Suppose the Ricci tensor ρ of M is positive definite. Then $H^1(M;\mathbb{R})=0$.*
(2) *Suppose the scalar curvature $\mathcal{R}$ of M is positive and M is spin. Then there are no harmonic spinors on M.*

Proof. Suppose the Ricci tensor ρ is positive. We can find $\epsilon>0$ so that we have the estimate $g(\rho\phi,\phi)\ge\epsilon g(\phi,\phi)$. Suppose that ϕ is a harmonic 1 form. We use the Weitzenböch formula and integrate by parts to compute:

$$\begin{aligned}0=&(\Delta^1\phi,\phi)_{L^2}=(\nabla\phi,\nabla\phi)_{L^2}+(\rho\phi,\phi)_{L^2}\\ \ge&\epsilon(\phi,\phi)_{L^2}\ge 0.\end{aligned}$$

This implies $|\phi|_{L^2}=0$ and hence $\phi=0$ as ϕ is smooth. Consequently there are no non-trivial harmonic 1 forms. The Hodge decomposition theorem (Theorem 1.4.6) identifies $H^1(M;\mathbb{R})$ with $\ker(\Delta^1)$; the first assertion now follows. The proof of the second assertion is similar. Suppose that the scalar curvature $\mathcal{R}$ is positive and that M is spin. Choose $\epsilon>0$ so that $\mathcal{R}\ge\epsilon$. We integrate by parts to compute:

$$\begin{aligned}0=&(\Delta^{\mathcal{C}}\phi,\phi)_{L^2}=(\nabla\phi,\nabla\phi)_{L^2}+\tfrac14(\mathcal{R}\phi,\phi)_{L^2}\\ \ge&\tfrac14\epsilon(\phi,\phi)_{L^2}\ge 0.\end{aligned}$$

The second assertion now follows. □

Remark. In §2.8, we will generalize Theorem 2.6.5 (2) to the case of manifolds with boundary. We will impose spectral boundary conditions. This will create an additional boundary integrand which fortunately also is positive semi-definite.

At this stage we digress briefly to illustrate the use of Lemma 2.6.2. Let M be a smooth compact Riemannian manifold without boundary of dimension m. Let $\{\phi_n,\lambda_n\}$ be the discrete spectral resolution of a self-adjoint operator D of Laplace type acting on the space of smooth sections to a smooth vector bundle V. The trace of the fundamental solution of the heat equation e^{-tD} has an asymptotic trace as $t\downarrow 0^+$ of the form:

$$\operatorname{Tr}_{L^2}(e^{-tD})=\sum_n e^{-t\lambda_n}\sim\sum_{k\ge0}(4\pi)^{-m/2}t^{(k-m)/2}a_k(D).$$

The coefficients $a_k(D)$ are spectral invariants of the operator D. They are locally computable and vanish for k odd if the boundary of M is empty. We refer to Gilkey [64] for the proof of the following result:

2.6.6 Theorem. *Let M be a compact Riemannian manifold without boundary. Let D be an operator of Laplace type on the space of smooth sections to a smooth vector bundle over M. Decompose $D = -\operatorname{Tr}(\nabla^2) - E$. Let ';' denote multiple covariant differentiation with respect to the connection ∇ and the Levi-Civita connection. Let Ω be the curvature of the connection ∇ and let R be the curvature of the Levi-Civita connection. We then have:*

(1) $a_0(D) = \int_M \dim(V)$.

(2) $a_2(D) = \frac{1}{6}\int_M \operatorname{Tr}\{6E + \mathcal{R}I_V\}$.

(3) $a_4(D) = \frac{1}{360}\int_M \operatorname{Tr}\{60E_{;kk} + 60\mathcal{R}E + 180E^2 + (12\mathcal{R}_{;kk} + 5\mathcal{R}^2 - 2|\rho|^2$
$+2|R|^2)I_V + 30\Omega_{ij}\Omega_{ij}\}$.

(4) $a_6(D) = \frac{1}{360}\int_M\{(\frac{18}{7!}\mathcal{R}_{;iijj} + \frac{17}{7!}\mathcal{R}_{;k}\mathcal{R}_{;k} - \frac{2}{7!}\rho_{ij;k}\rho_{ij;k} - \frac{4}{7!}\rho_{jk;n}\rho_{jn;k}$
$+\frac{9}{7!}R_{ijkl;n}R_{ijkl;n} + \frac{28}{7!}\mathcal{R}\mathcal{R}_{;nn} - \frac{8}{7!}\rho_{jk}\rho_{jk;nn} + \frac{24}{7!}\rho_{jk}\rho_{jn;kn}$
$+\frac{12}{7!}R_{ijk\ell}R_{ijk\ell;nn} + \frac{35}{9\cdot 7!}\mathcal{R}^3 - \frac{14}{3\cdot 7!}\mathcal{R}\rho^2 + \frac{14}{3\cdot 7!}\mathcal{R}R^2 - \frac{208}{9\cdot 7!}\rho_{jk}\rho_{jn}\rho_{kn}$
$-\frac{64}{3\cdot 7!}\rho_{ij}\rho_{kl}R_{ikjl} - \frac{16}{3\cdot 7!}\rho_{jk}R_{jn\ell i}R_{kn\ell i} - \frac{44}{9\cdot 7!}R_{ijkn}R_{ij\ell p}R_{kn\ell p}$
$-\frac{80}{9\cdot 7!}R_{ijkn}R_{i\ell kp}R_{j\ell np})I_V + \frac{1}{45}\Omega_{ij;k}\Omega_{ij;k} + \frac{1}{180}\Omega_{ij;j}\Omega_{ik;k}$
$+\frac{1}{60}\Omega_{ij;kk}\Omega_{ij} + \frac{1}{60}\Omega_{ij}\Omega_{ij;kk} - \frac{1}{30}\Omega_{ij}\Omega_{jk}\Omega_{ki} - \frac{1}{60}R_{ijkn}\Omega_{ij}\Omega_{kn}$
$-\frac{1}{90}\rho_{jk}\Omega_{jn}\Omega_{kn} + \frac{1}{72}\mathcal{R}\Omega_{kn}\Omega_{kn} + \frac{1}{60}E_{;iijj} + \frac{1}{12}EE_{;ii} + \frac{1}{12}E_{;ii}E$
$+\frac{1}{12}E_{;i}E_{;i} + \frac{1}{6}E^3 + \frac{1}{30}E\Omega_{ij}\Omega_{ij} + \frac{1}{60}\Omega_{ij}E\Omega_j + \frac{1}{30}\Omega_{ij}\Omega_{ij}E + \frac{1}{36}\mathcal{R}E_{;kk}$
$+\frac{1}{90}\rho_{jk}E_{;jk} + \frac{1}{30}\mathcal{R}_{;k}E_{;k} - \frac{1}{60}E_{;j}\Omega_{ij;i} + \frac{1}{60}\Omega_{ij;i}E_{;j} + \frac{1}{12}EE\mathcal{R} + \frac{1}{30}E\mathcal{R}_{;kk}$
$+\frac{1}{72}E\mathcal{R}^2 - \frac{1}{180}E\rho_{jk}\rho_{jk} + \frac{1}{180}E|R|^2\}$.

Let $\pi : Z \to Y$ be a Riemannian submersion. The following two examples show that there is no simple relationship between the heat equation asymptotics of the fiber, of the base, and of the total space.

2.6.7 Example. Let $Y = S^2$, let $F = S^1$, and let $Z = S^1 \times Y$. We then have

$$\Delta_Z^0 = \Delta_Y^0 \otimes 1 + 1 \otimes \Delta_F^0.$$

It then follows that the spectral resolution of the Laplacian on Z is just the product of the spectral resolution of the Laplacian on Y and of the Laplacian on S^1. Since the circle is flat, $a_n(S^1) = 0$ for $n \geq 1$. Once the normalizing constants are taken into account, we have the relation:

$$a_n(S^1 \times S^2) = \text{volume}(S^1)a_n(S^2).$$

2.6.8 Example. Let $Y = S^2$, let $F = S^1$, and let $Z = S^3$ define the Hopf fibration . A simple calculation using Theorem 2.6.6 shows there is no constant c so that shows we do not have $a_n(S^3) = a_n(S^2)$ for all n. We can use Theorem 3.2.1 to see that the eigenfunctions on S^2 pull back to eigenfunctions with the same eigenvalue on S^3; this in essence takes care of the zero mode spectrum of S^1. However the remaining eigenvalues of S^1 are not so easily accounted for and there is no simple relationship between the two spectra.

We remark that McKean and Singer have determined the heat equation asymptotics a_n for S^2 and S^3 for all n and there is no obvious relationship between these invariants. Let

$$Z(M,t) := (4\pi t)^{m/2}\mathrm{Vol}(M)^{-1}\,\mathrm{Tr}_{L^2}\, e^{-t\Delta_M^0}.$$

Their results (see [129] page 63) show that

$$\begin{aligned}
&Z(S^1,t) = 1 + O(t^k) \text{ for any k}\\
&Z(S^2,t) = \frac{e^{t/4}}{\sqrt{\pi t}}\int_0^1 \frac{e^{-x/t}}{\sin\sqrt{x}}dx = 1 + \frac{t}{3} + \frac{t^2}{15} + ...\\
&Z(S^1\times S^2,t) = Z(S^2,t)Z(S^1,t)\\
&Z(S^3),t) = e^t = 1 + t + \frac{1}{2}t^2 + ...
\end{aligned}$$

In §3.9, we will generalize the results of §3.2 and §3.4 to the Laplacian with coefficients in a flat bundle defined by a unitary representation of the fundamental group.

In §3.10, we will discuss the heat content asymptotics. The heat equation asymptotics discussed here and in sections and §2.8 do not behave well under Riemannian submersion as noted above. The heat content asymptotics, by contrast, do behave well. If $\pi : Z \to Y$ is a Riemannian submersion of smooth compact manifolds with boundary, we will show that the heat content asymptotics β_n for Z and Y are related by the simple relationship

$$\beta_n(Z) = \mathrm{volume}(F)\beta_n(Y)$$

if the mean curvature vector vanishes. Thus these invariants do behave well in the context of Riemannian submersions.

2.7 Pullback

In this section, we study the complex and spinor analogues of Lemma 1.5.1. We refer to §A.2 for further information concerning spin geometry and to §A.5 for further information concerning holomorphic geometry.

2.7.1 Definition. We say that a Riemannian submersion π from Z to Y is a *Hermitian submersion* if

(1) The manifolds Z and Y are complex manifolds.

(2) The map π is a complex analytic mapping.

(3) The metrics on Z and on Y are Hermitian.

We extend pull back π^* to be complex linear on the complexified exterior algebra. We then have
$$\pi^* : C^\infty\Lambda_Y^{(p,q)} \to C^\infty\Lambda_Z^{(p,q)}.$$
The pull-back π^* is a graded ring homomorphism. We decompose $d_M = \partial_M + \bar\partial_M$ where we have
$$\bar\partial_M : C^\infty\Lambda^{(p,q)}M \to C^\infty\Lambda^{(p,q+1)}M \text{ and}$$
$$\partial_M : C^\infty\Lambda^{(p,q)}M \to C^\infty\Lambda^{(p+1,q)}M.$$
If π is a Hermitian submersion, then pull back commutes with the complex exterior derivative, i.e.
$$\bar\partial_Z\pi^* = \pi^*\bar\partial_Y \text{ on } C^\infty\Lambda^{(p,q)}(Y).$$
We refer to §A.5 for further details concerning complex geometry.

Let $\pi^{(p,q)}$ be orthogonal projection of Λ^* on $\Lambda^{(p,q)}$ in the complex setting. Recall that we defined
$$\begin{aligned}&\theta := -g_Z([e_i, f_a], e_i)f^a = {}^Z\Gamma_{iia}f^a \in C^\infty(\mathcal{H}),\\ &\omega := \omega_{abi} = \tfrac{1}{2}g_Z(e_i, [f_a, f_b]),\\ &\mathcal{E} := \omega_{abi}\operatorname{ext}_Z(e^i)\operatorname{int}_Z(f^a)\operatorname{int}_Z(f^b), \text{ and } \Xi := \operatorname{int}_Z(\theta) + \mathcal{E}.\end{aligned}$$
In Lemma 1.5.1, we showed:
$$\delta_Z\pi^* - \pi^*\delta_Y = \Xi\pi^* \text{ and } \Delta_Z\pi^* - \pi^*\Delta_Y = \{\Xi d_Z + d_Z\Xi\}\pi^*.$$
This result extends to the complex setting to become:

2.7.2 Lemma. *Let $\pi : Z \to Y$ be a Hermitian submersion.*

(1) *We have $\bar\partial_Z^*\pi^* - \pi^*\bar\partial_Y^* = \pi_Z^{(p,q-1)}\Xi\pi^*$ on $C^\infty\Lambda^{(p,q)}(Y)$.*

(2) *We have $\Delta_Z^{(p,q)}\pi^* - \pi^*\Delta_Y^{(p,q)} = \pi_Z^{(p,q)}(\bar\partial_Z\Xi + \Xi\bar\partial_Z)\pi^*$.*

Proof. Let $i_M^{p,q}$ denote the natural inclusion of $\Lambda^{(p,q)}(M)$ in $\Lambda^*(M)$. Dually then we have $\pi_M^{(p,q)} := (i_M^{(p,q)})^*$. We may express:
$$\begin{aligned}\bar\partial_M :=&\pi_M^{(p,q)} \circ d_M \circ i_M^{(p,q-1)} \text{ on } C^\infty\Lambda^{(p,q-1)}(M)\\ \bar\partial_M^* =&\pi_M^{(p,q-1)} \circ \delta_M \circ i_M^{(p,q)} \text{ on } C^\infty\Lambda^{(p,q)}(M).\end{aligned}$$

Since pullback commutes with both $i^{(p,q)}$ and $\pi^{(p,q)}$, we compute that

$$\begin{aligned}\bar\partial_Z^*\pi^* - \pi^*\bar\partial_Y^* &= \pi_Z^{(p,q-1)}\delta_Z i_Z^{(p,q)}\pi^* - \pi^*\pi_Y^{(p,q-1)}\delta_Y i_Y^{p,q}\\ &=\pi_Z^{(p,q-1)}(\delta_Z\pi^* - \pi^*\delta_Y)i_Y^{(p,q)} = \pi_Z^{(p,q-1)}\Xi\pi^* i_Y^{(p,q)}\end{aligned}\tag{2.7.a}$$

on $C^\infty\Lambda^{(p,q)}(M)$. We suppress the role of $i^{(p,q)}$ to complete the proof of the first identity; we use the identities

$$\bar\partial_Z\pi^* = \pi^*\bar\partial_Y \text{ and } \pi_Z^{(p,q)}\bar\partial_Z = \bar\partial_Z\pi_Z^{(p,q-1)}$$

together with equation (2.7.a) to complete the proof Lemma 2.7.2 by computing:

$$\begin{aligned}\Delta_Z^{(p,q)}\pi^* - \pi^*\Delta_Y^{(p,q)} &= \bar\partial_Z(\bar\partial_Z^*\pi^* - \pi^*\bar\partial_Y^*) + (\bar\partial_Z^*\pi^* - \pi^*\bar\partial_Y^*)\bar\partial_Y\\ &=\bar\partial_Z\pi_Z^{(p,q-1)}\Xi\pi^* + \pi_Z^{(p,q)}\Xi\pi^*\bar\partial_Y = \pi_Z^{(p,q)}(\bar\partial_Z\Xi + \Xi\bar\partial_Z)\pi^*. \quad\square\end{aligned}$$

There is an analogue of this intertwining result in spin geometry. The major technical difficulty is to construct a suitable notion of pull back. Let $\mathcal{S}$ denote the spin representation of the spinor group and let $\mathcal{C}$ denote the Clifford representation of the spinor group by left Clifford multiplication; we refer to the material of §A.2 and §A.3 for further details. If $m = 2k$, then $\mathcal{C} = 2^k\mathcal{S}$; there is a similar decomposition of m is odd. The representation $\mathcal{C}$ has the advantage that right multiplication gives $\mathcal{C}$ a $\text{Cliff}(\mathbb{R}^m)$ module structure. Also the decomposition formulas simplify greatly with respect to products. If M is a spin manifold, let $\mathcal{C}M$ be the associated bundle. Let $A_M^{\mathcal{C}}$ be the associated Dirac operator. Let $D_M^{\mathcal{C}} := (A_M^{\mathcal{C}})^2$ be the associated *Spin Laplacian*. Let c_M be the leading symbol of $A^{\mathcal{C}}M$. The endomorphism c_M gives a Clifford module structure to $\mathcal{C}M$ since

$$c_M(\xi)^2 = -|\xi|^2\cdot Id \text{ on } \mathcal{C}M.$$

The bundle $\mathcal{C}M$ inherits a natural connection $\nabla^{\mathcal{C}}$. Relative to an orthonormal frame $\{F_\mu\}$ for TM and relative to the induced orthonormal frame for $\mathcal{C}M$, the connection 1 form $A^{\mathcal{C}}$ of $\nabla^{\mathcal{C}}$, and the operator $A_M^{\mathcal{C}}$ are defined by

$$\begin{aligned}A^{\mathcal{C}} &:=\tfrac14 F^\mu\otimes c_M(F^\nu)c_M(F^\sigma)^M\Gamma_{\mu\nu\sigma}\in T^*M\otimes\text{End}(\mathcal{C}M), \text{ and}\\ A_M^{\mathcal{C}} &:=c_M\circ\nabla^{\mathcal{C}} : C^\infty(\mathcal{C}M)\to C^\infty(\mathcal{C}M).\end{aligned}\tag{2.7.b}$$

Let Y be a spin manifold and let $\pi : P \to Y$ be a principal G bundle with a principal bundle metric as defined in §1.16. We use the correspondence ${}^Y\vec{s}\mapsto {}^P\vec{s}$ of local frames on Y to local frames on P described in Equation (1.16.a) to identify the transition functions of the tangent bundle of P with the pull back of the transition functions for the tangent bundle of Y. The spin structure on Y permits us to lift these transition functions from $\mathrm{SO}(\dim Y)$ to $\mathrm{Spin}(\dim Y) \subset \mathrm{Spin}(\dim P)$. This defines a natural spin structure on P. Let $\mathfrak{g}$ be the Lie algebra of the Lie group G. With respect to this natural spin structure on P, we have the following relationship between the spinor bundles:

$$\mathcal{C}P = \mathcal{C}Y \otimes_{\mathbb{R}} \mathcal{C}\mathfrak{g}. \tag{2.7.c}$$

Let $\alpha \in \mathcal{C}\mathfrak{g}$ with $|\alpha| = 1$. We define the pullback

$$\pi_\alpha^* : C^\infty(\mathcal{C}Y) \to C^\infty(\mathcal{C}P) \text{ by } \Phi \mapsto \Phi \otimes \alpha. \tag{2.7.d}$$

This gives rise to the notion of *projectable spinors* introduced by Moroianu [136, 137, 138, 139, 140]; our point of view is a bit different as we are working with the bundle $\mathcal{C}$ rather than the bundle $\mathcal{S}$, but the construction is equivalent.

If the structure group G is simply connected, then $\pi_1(P) = \pi_1(Y)$ and this correspondence of spin structures is a bijective correspondence between spin structures on P and spin structures on Y. If $\pi_1(G) \neq 0$, however, the situation can be a bit more complicated; if we twist the canonical spin structure on G by taking coefficients in a flat line bundle over the fiber, we no longer have the isomorphism given in Equation (2.7.c) and hence can not define the pull back given in Equation (2.7.d).

The $\mathbb{Z}_2$ grading of the Clifford algebra decomposes $\mathcal{C} = \mathcal{C}^e \oplus \mathcal{C}^o$. Let χ be the parity operator defined in §A.2; $\chi = +1$ on $\mathcal{C}^eY$ and $\chi := -1$ on $\mathcal{C}^oY$. We have the following intertwining results:

$$\chi c_Y(F^a) + c_Y(F^a)\chi = 0 \text{ and } A_Y^{\mathcal{C}}\chi + \chi A_Y^{\mathcal{C}} = 0.$$

Let $\omega_{abi} := \frac{1}{2}(\Gamma_{abi}^P - \Gamma_{bai}^P)$ be the curvature tensor defined in §1.2. We also adopt the notation of §1.16.5 giving local frames $\{F_a, \mathfrak{g}_i\}$ and $\{F^a, \mathfrak{g}^i\}$ for the tangent and cotangent bundles of P. Let

$$\mathcal{E}_{spin} := \tfrac{1}{4}\omega_{abi}\chi c_Y(F^a)c_Y(F^b) \otimes c_{\mathfrak{g}}(\mathfrak{g}^i). \tag{2.7.e}$$

The tensor θ does not enter; we are only considering principal bundles with principal bundle metrics so the fibers are totally geodesic. With the notation established above, Lemma 1.5.1 has the following analogue:

2.7.3 Lemma. *Let $\pi : P \to Y$ be a principal bundle with structure group G. Let $A_P^{\mathcal{C}}$, $A_Y^{\mathcal{C}}$, and $A_G^{\mathcal{C}}$ be the Dirac operators on the manifolds P, Y, and G respectively. Then*

$$A_P^{\mathcal{C}}\pi_\alpha^* = \pi_\alpha^* A_Y^{\mathcal{C}} + \chi\pi_{A_G^{\mathcal{C}}\alpha}^* + \mathcal{E}_{spin}\pi_\alpha^*.$$

Proof. Let $\Phi \in C^\infty(\mathcal{C}Y)$. We must show that:

$$A_P^{\mathcal{C}}(\Phi \otimes \alpha) = A_Y^{\mathcal{C}}\Phi \otimes \alpha + \chi\Phi \otimes A_G^{\mathcal{C}}\alpha + \mathcal{E}_{spin}(\Phi \otimes \alpha).$$

Let (ξ_H, ξ_V) be horizontal and vertical tangent vector fields. Under the isomorphism given in Equation (2.7.c), we have

$$c_P(\xi_H, \xi_V) = c_Y(\xi_H) \otimes 1 + \chi \otimes c_{\mathfrak{g}}(\xi_V).$$

We use this identification of the symbol and definition given in equation $(2.7.b)$ to see that:

$$\begin{aligned} 4A_P^{\mathcal{C}}(\Phi \otimes \alpha) - & 4(A_Y^{\mathcal{C}}\Phi) \otimes \alpha \\ = & (2\Gamma^P_{abi} + \Gamma^P_{iab})\chi c_Y(F^a) c_Y(F^b)\Phi \otimes c_{\mathfrak{g}}(\mathfrak{g}^i)\alpha \\ & + (\Gamma^P_{aij} + 2\Gamma^P_{ija}) c_Y(F^a)\Phi \otimes c_{\mathfrak{g}}(\mathfrak{g}^i) c_{\mathfrak{g}}(\mathfrak{g}^j)\alpha \\ & + \Gamma^P_{ijk}\chi\Phi \otimes c_{\mathfrak{g}}(\mathfrak{g}^i) c_{\mathfrak{g}}(\mathfrak{g}^j) c_{\mathfrak{g}}(\mathfrak{g}^k)\alpha. \end{aligned}$$

We use Lemma 1.16.6 to see that $2\Gamma^P_{abi} + \Gamma^p_{iab} = \omega_{abi}$ and that $0 = \Gamma^P_{aij} = -\Gamma^P_{ija}$. The final terms yield $4\chi\Phi \otimes A_G^{\mathcal{C}}\alpha$. $\square$

2.8 Manifolds with boundary

Previously, we have assumed that M was without boundary. We now, for the moment, drop this assumption. We suppose, instead, that M is a compact Riemannian manifold with smooth boundary ∂M. We must impose suitable *boundary conditions* to ensure that our operators are self-adjoint. Let N_M and N_M^* be the inward unit normal vector and covector fields on the boundary ∂M. If $\xi \in T^*M$, let $\text{ext}(\xi)$ denote exterior multiplication and let $\text{int}(\xi)$ denote the dual, interior multiplication. The following assertions are well known; see for example Gilkey [64].

Lemma 2.8.1 (Green's Formula).

(1) $(d_M\Phi, \Psi)_{L^2(M)} = (\Phi, \delta_M\Psi)_{L^2(M)} - (\text{ext}(N_M^*)\Phi, \Psi)_{L^2(\partial M)}$.

(2) $(\Delta_M^p\Phi, \Psi)_{L^2(M)} = (d_M\Phi, d_M\Psi)_{L^2(M)} + (\delta_M\Phi, \delta_M\Psi)_{L^2(M)}$
$+(\text{int}(N_M^*)d_M\Phi, \Psi)_{L^2(\partial M)} - (\text{ext}(N_M^*)\delta_M\Phi, \Psi)_{L^2(\partial M)}$.

Let $\Phi \in C^\infty\Lambda^p(M)$. We say Φ satisfies *Dirichlet boundary conditions* if the restriction of Φ to the boundary of M vanishes, i.e. the Dirichlet boundary operator $\mathcal{B}_D\Phi := \Phi|_{\partial M}$ vanishes on Φ. Let ∇ be the Levi-Civita connection. We say Φ satisfies *Neumann boundary conditions* if the restriction of the normal derivative of Φ to the boundary of M vanishes, i.e the Neumann boundary operator $\mathcal{B}_N\Phi := \nabla_{N_M}\Phi|_{\partial M}$ vanishes on Φ.

There are other natural boundary conditions and operators. Let $i_M : \partial M \to M$ be the inclusion of the boundary ∂M into M. Let i_M^* be the pull back from $\Lambda^p(M)$ to $\Lambda^p(\partial M)$. We say that Φ satisfies *absolute boundary conditions* $\mathcal{B}_A$ if

$$i^* \operatorname{int}(N_M^*)\Phi = 0 \text{ and } i^* \operatorname{int}(N_M^*)d_M\Phi = 0.$$

Equivalently, let $y = (y^1, ..., y^{m-1})$ be a system of local coordinates on the boundary of M and let $x = (y, r)$ where r is the geodesic distance to the boundary. Let $dy^I := dy^{i_1} \wedge ... \wedge dy^{i_q}$ where $1 \le i_1 < ... < i_q \le m-1$. Expand $\Phi = \Sigma\Phi_I dy^I + \tilde{\Phi}_J dr \wedge dy^J$. Then Φ satisfies absolute boundary conditions if

$$\tilde{\Phi}_J|_{\partial M} = 0 \text{ and } \partial_r\Phi_I|_{\partial M} = 0.$$

Let $\star$ be the Hodge operator defined in §1.4. We say that Φ satisfies *relative boundary conditions* $\mathcal{B}_R$ if $\star\Phi$ satisfies absolute boundary conditions or equivalently as we shall show in Lemma 2.9.2 that:

$$i_M^*\Phi = 0 \text{ and } i_M^*\delta_M\Phi = 0.$$

Note that if $p = 0$, then absolute boundary conditions correspond to Neumann boundary conditions and relative boundary conditions correspond to Dirichlet boundary conditions. If $p = m$, the situation is reversed; absolute boundary conditions correspond to Dirichlet boundary conditions and relative boundary conditions correspond to Neumann boundary conditions. Let

$$E(\lambda, \Delta^p_{M,\mathcal{B}}) := \{\Phi \in C^\infty\Lambda^p(M) : \mathcal{B}\Phi = 0 \text{ and } \Delta_M^p\Phi = \lambda\Phi\}$$

be the associated eigenspaces given by the boundary condition defined by the Dirichlet, Neumann, relative, or absolute boundary operator. Theorem 1.4.6 generalizes to this setting. Let $H^p(M;\mathbb{R})$ be the absolute cohomology groups and let $H^p(M, \partial M;\mathbb{R})$ be the relative cohomology groups.

2.8.2 Theorem (Hodge-de Rham). *Let M be a compact Riemannian manifold with smooth boundary.*

(1) *We have* $E(0, \Delta^p_{M,\mathcal{B}_A}) = H^p(M;\mathbb{R})$.

(2) *We have* $E(0, \Delta^p_{M,\mathcal{B}_R}) = H^p(M, \partial M;\mathbb{R})$.

If M is oriented, the Hodge $\star$ operator intertwines $\Delta^p_{M,\mathcal{B}_A}$ and $\Delta^{m-p}_{M,\mathcal{B}_R}$ and induces the Poincare duality isomorphism

$$H^p(M;\mathbb{R}) = E(0, \Delta^p_{M,\mathcal{B}_A}) \approx E(0, \Delta^{m-p}_{M,\mathcal{B}_R}) = H^{m-p}(M, \partial M;\mathbb{R}).$$

Relative and absolute boundary conditions are important in index theory. For example, the *Euler-Poincare characteristic* is given analytically by:

$$\chi(M) = \Sigma_p(-1)^p \dim E(0, \Delta^p_{M,\mathcal{B}_A}), \text{ and}$$
$$\chi(M, \partial M) = \Sigma_p(-1)^p \dim E(0, \Delta^p_{M,\mathcal{B}_R}).$$

We shall need the following technical Lemma. Although it is well known, we give the proof to illustrate the techniques involved.

2.8.3 Lemma. *Let $\mathcal{B} = \mathcal{B}_D$, $\mathcal{B}_N$, $\mathcal{B}_A$, or $\mathcal{B}_R$.*

(1) $\Delta^p_{M,\mathcal{B}}$ *is self-adjoint.*

(2) *If $\mathcal{B} \neq \mathcal{B}_N$, $\Delta^p_{M,\mathcal{B}}$ is non-negative, i.e. $(\Delta^p_M\Phi, \Phi) \geq 0$ for any Φ in $C^\infty\Lambda^p(M)$ with $\mathcal{B}\Phi = 0$.*

Proof. We use Lemma 2.8.1 to see

$$\begin{aligned}&(\Delta^p_M\Phi, \Psi)_{L^2(M)} - (\Phi, \Delta^p_M\Psi)_{L^2(M)}\\ =&(\mathrm{int}(N^*_M)d_M\Phi, \Psi)_{L^2(\partial M)} - (\mathrm{ext}(N^*_M)\delta_M\Phi, \Psi)_{L^2(\partial M)}\\ &- (\Phi, \mathrm{int}(N^*_M)d_M\Psi)_{L^2(\partial M)} + (\Phi, \mathrm{ext}(N^*_M)\delta_M\Psi)_{L^2(\partial M)}.\end{aligned} \tag{2.8.a}$$

To establish assertion (1), we must show that if Φ and Ψ satisfy the boundary conditions $\mathcal{B}$ then

$$(\Delta^p\Phi, \Psi)_{L^2(M)} = (\Phi, \Delta^p\Psi)_{L^2(M)}. \tag{2.8.b}$$

We must also show that if we are given Ψ so that equation (2.8.b) holds for all Φ with $\mathcal{B}\Phi = 0$, then $\mathcal{B}\Psi = 0$.

Dirichlet boundary conditions: If Φ and Ψ satisfy Dirichlet boundary conditions, then the boundary terms in Lemma 2.8.1 (2) vanish and we have

$$(\Delta^p_M\Phi, \Psi)_{L^2(M)} = (d_M\Phi, d_M\Psi)_{L^2(M)} + (\delta_M\Phi, \delta_M\Psi)_{L^2(M)}. \tag{2.8.c}$$

Equation (2.8.c) is symmetric in Φ and Ψ; we interchange the roles of Φ and Ψ to see that (2.8.b) holds. Conversely, suppose that Ψ is given so that (2.8.b) holds for all Φ with $\Phi|_{\partial M} = 0$. We use equation (2.8.a) to see that

$$(\mathrm{int}(N^*_M)d_M\Phi, \Psi)_{L^2(\partial M)} - (\mathrm{ext}(N^*_M)\delta_M\Phi, \Psi)_{L^2(\partial M)} = 0 \tag{2.8.d}$$

for all Φ with $\Phi|_{\partial M} = 0$. Near the boundary of M, we decompose

$$\Psi = \Psi_1 + N^*_M \wedge \Psi_2 \text{ and } \Phi = \Phi_1 + N^*_M \wedge \Phi_2.$$

We assume $\Phi_i|_{\partial M} = 0$. Then equation (2.8.d) yields

$$(\partial_m\Phi_1, \Psi_1)_{L^2(\partial M)} + (\partial_m\Phi_2, \Psi_2)_{L^2(\partial M)} = 0.$$

Since we can specify the normal derivatives of Φ_i arbitrarily, this implies $\mathcal{B}_D\Psi = 0$ as desired. By taking $\Phi = \Psi$ in equation (2.8.c), we see $(\Delta^p_M\Phi, \Phi) \geq 0$ which establishes (2) for Dirichlet boundary conditions.

Absolute boundary conditions: Note that

$$(\mathrm{ext}(N_M^*)\delta_M\Phi,\Psi)=(\delta_M\Phi,\mathrm{int}(N_M^*)\Psi).$$

If Φ and Ψ satisfy absolute boundary conditions, then $\mathrm{int}(N_M^*)d_M\Phi$ and $\mathrm{int}(N_M^*)\Psi$ vanish on the boundary. Thus the boundary terms in Lemma 2.8.1 (2) vanish and equation (2.8.c) holds. As for Dirichlet boundary conditions, this implies equation (2.8.b) holds and shows that (2) holds. Conversely, suppose that Ψ is given so that (2.8.b) holds for all Φ with $\mathcal{B}_A\Phi=0$. We use equation (2.8.a) to see that

$$(\delta_M\Phi,\mathrm{int}(N_M^*)\Psi)_{L^2(\partial M)}+(\Phi,\mathrm{int}(N_M^*)d_M\Psi)_{L^2(\partial M)}=0. \tag{2.8.e}$$

Take adapted coordinate systems $x=(y,r)$ so $dr=N_M^*$ and so that $\partial_r=N_M$; r is the geodesic distance to the boundary. Near the boundary of M, let

$$\Phi:=\Phi_I^1(y)dy^I+r\Phi_J^2(y)N_M^*\wedge dy^J.$$

Then $r\Phi_J^2|_{\partial M}=0$ and $\partial_r\Phi_I^1|_{\partial M}=0$ so Φ satisfies absolute boundary conditions. Note that there exists an operator Q so that:

$$\delta_M\Phi|_{\partial M}=-\Phi_J^2(y)dy^J+Q(\Phi_I^1).$$

Define Φ by the equations:

$$\Phi_I^1dy^I:=\mathrm{int}(N_M^*)d_M\Psi|_{\partial M}\text{ and }-\Phi_J^2dy^J:=\mathrm{int}(N_M^*)\Psi|_{\partial M}-Q(\Phi_I^1).$$

We use equation (2.8.e) to show that $\mathcal{B}_A\Psi=0$ by computing:

$$\begin{aligned}0=&(-\Phi_J^2dy^J+Q(\Phi_I^1),\mathrm{int}(N_M^*)\Psi)_{L^2(\partial M)}+(\Phi_I^1dy^I,\mathrm{int}(N_M^*)d_M\Psi)_{L^2(\partial M)}\\=&||\,\mathrm{int}(N_M^*)\Psi||_{L^2(\partial M)}^2+||\,\mathrm{int}(N_M^*)d\Psi||_{L^2(\partial M)}^2.\end{aligned}$$

Relative boundary conditions: This case follows from absolute boundary conditions using the Hodge $\star$ operator.

Neumann boundary conditions: Let ';' denote multiple covariant differentiation with respect to a local orthonormal frame field. We adopt the Einstein convention and sum over repeated indices. We use the Weitzenböch formulas of Example 2.6.3 to express

$$\Delta_M^p\Phi=-\Phi_{;ii}+E\Phi$$

where E is a self-adjoint endomorphism of the exterior algebra given by the curvature tensor. We compute

$$(\Phi_{;ii},\Psi)_{L^2(M)}=(\Phi_{;i},\Psi_{;i})_{L^2(M)}-(\Phi_{;m},\Psi)_{L^2(\partial M)}$$

and we compute

$$\begin{aligned}&(\Delta_M^p\Phi,\Psi)_{L^2(M)}-(\Phi,\Delta_M^p\Psi)_{L^2(M)}\\=&(-\Phi_{;ii},\Psi)_{L^2(M)}+(\Phi,\Psi_{;ii})_{L^2(M)}\\=&-(\Phi_{;m},\Psi)_{L^2(\partial M)}+(\Phi,\Psi_{;m})_{L^2(\partial M)}.\end{aligned}$$

The boundary correction terms vanish in the final equation if both Φ and Ψ satisfy Neumann boundary conditions; conversely if these boundary correction terms vanish for all Φ satisfying Neumann boundary conditions, then Ψ satisfies Neumann boundary conditions. □

Theorem 2.3.1 generalizes to this setting to become:

2.8.4 Theorem. *Let M be a compact manifold with smooth boundary ∂M. Let $\mathcal{B}$ denote Dirichlet, Neumann, absolute, or relative boundary conditions. The eigenspaces $E(\lambda,\Delta^p_{M,\mathcal{B}})$ are finite dimensional spaces for all λ and we have an orthogonal direct sum*

$$L^2(\Lambda^p(M))=\oplus_\lambda E(\lambda,\Delta^p_{M,\mathcal{B}}).$$

There are only a finite number of negative eigenvalues. Order the eigenvalues so $\lambda_1\le\lambda_2\le\ldots$ and repeat the eigenvalues according to multiplicity. For any $\epsilon>0$, there exists an integer $n(\epsilon)$ so that

$$n^{\frac{2}{m}-\epsilon}\le\lambda_n\le n^{\frac{2}{m}+\epsilon}\text{ for } n\ge n(\epsilon).$$

Let ϕ be a smooth p form with $\mathcal{B}\phi=0$. Expand $\phi=\Sigma_n\phi_n$ in L^2 where the $\phi_n\in E(\lambda,\Delta^p_{M,\mathcal{B}})$. Then this series converges in the C^∞ topology.

2.8.5 Example. Let M be the interval $[0,\pi]$. The spectral resolution with Neumann boundary conditions is $\{\cos(nx),n^2\}_{n\ge0}$ and with Dirichlet boundary conditions is $\{\sin(nx),n^2\}_{n>0}$. On functions, absolute boundary conditions correspond to Neumann boundary conditions and relative boundary conditions correspond to Dirichlet boundary conditions; on 1 forms, the roles of these two boundary conditions are reversed. Thus

(1) $H^0(M;\mathbb{R})=E(0,\Delta^0_{M,B_N})=1\cdot\mathbb{R}$.

(2) $H^1(M;\mathbb{R})=E(0,\Delta^1_{M,B_D})=0$.

(3) $H^0(M,\partial M;\mathbb{R})=E(0,\Delta^0_{M,B_D})=0$.

(4) $H^1(M,\partial M;\mathbb{R})=E(0,\Delta^1_{M,B_N})=dx\cdot\mathbb{R}$

Theorem 2.6.6 generalizes to this setting as well. We only present the results for heat equation asymptotics with Dirichlet boundary conditions; the results are similar for Neumann boundary conditions. Recall the *second fundamental form* is given by:

$$L(X,Y)=(\nabla_X Y,\partial_N)$$

where X and Y are tangential vector fields and ∂_N is the inward unit normal. On the boundary, we let $\{e_i\}$ be a local orthonormal frame field so $e_m=\partial_N$. We let indices a, b, and c range from 1 to $m-1$ and index the corresponding orthonormal frame for the tangent bundle of the boundary.

2.8.6 Theorem. *Let M be a compact Riemannian manifold with smooth boundary. We impose Dirichlet boundary conditions. Let D be an operator of Laplace type on the space of smooth sections to a smooth vector bundle over M. Decompose $D=-\operatorname{Tr}(\nabla^2)-E$. Let ';' denote multiple covariant differentiation with respect to the connection ∇ and the Levi-Civita connection. Let Ω be the curvature of the connection ∇ and let R be the curvature of the Levi-Civita connection. We then have:*

(1) $a_0(D)=\int_M \dim(V)$.

(2) $a_1(D)=-\frac{\sqrt{4\pi}}{4}\int_{\partial M}\dim(V)$.

(3) $a_2(D)=\frac{1}{6}\{\int_M \operatorname{Tr}(6E+\mathcal{R})+\int_{\partial M}\operatorname{Tr}(2L_{aa})\}$.

(4) $a_3(D)=-\frac{\sqrt{4\pi}}{380}\int_{\partial M}\operatorname{Tr}(96E+16\mathcal{R}-8\rho_{mm}+7L_{aa}L_{bb}-10L_{ab}L_{ab})$.

(5) $$\begin{aligned}a_4(D)=\tfrac{1}{360}\{&\textstyle\int_M \operatorname{Tr}(60E_{;kk}+60\mathcal{R}E+180E^2+30\Omega^2+12\mathcal{R}_{;kk}+5\mathcal{R}^2\\&-2\rho^2+2|R|^2)+\textstyle\int_{\partial M}\operatorname{Tr}(-120E_{;m}-18\mathcal{R}_{;m}+120EL_{aa}+20\mathcal{R}L_{aa}\\&+4R_{amam}L_{bb}-12R_{ambm}L_{ab}+4R_{abcb}L_{ac}+24L_{aa:bb}+\tfrac{40}{21}L_{aa}L_{bb}L_{cc}\\&-\tfrac{88}{7}L_{ab}L_{ab}L_{cc}+\tfrac{320}{21}L_{ab}L_{bc}L_{ac})\}\end{aligned}$$

The situation for the spin Laplacian is slightly different; the Dirac operator does not admit local boundary conditions. Instead, we use the so-called *spectral boundary conditions* of Atiyah, Patodi, and Singer [7]. Suppose that M is spin and that the metric is product near the boundary. Expand

$$A_M^{\mathcal{C}}=\alpha(\partial_N+B)\text{ on }C^\infty(\mathcal{C}M)$$

where ∂_N is the inward unit normal, where α is a unitary bundle morphism, and where B is the associated Dirac operator on the boundary; B is a self-adjoint tangential differential operator. Let $\Pi_{\geq 0}$ be orthogonal projection on the span of the non-negative eigenspaces of B. We define

$$\mathcal{B}_{\geq 0}\phi=\Pi_{\geq 0}(\phi|_{\partial M}).$$

This defines an elliptic boundary value problem; the adjoint boundary operator $\mathcal{B}_{>0}$ is defined by $\Pi_{>0}$ which is projection on the positive spectrum of B. We let

$$D_{M,\mathcal{B}}^{\mathcal{C}}:=A_{M,\mathcal{B}_{>0}}^{\mathcal{C}}A_{M,\mathcal{B}_{\geq 0}}^{\mathcal{C}}$$

be the associated operator of Laplace type. We note that if the scalar curvature $\mathcal{R}$ is positive, then $\ker(B) = \{0\}$ by Theorem 2.6.5. Thus $A_M^{\mathcal{C}}$ is self-adjoint with this boundary condition in this setting. Theorems 2.3.1 and 2.6.5 generalize to this setting to become:

2.8.7 Theorem. *Let M be a compact spin manifold with smooth boundary. Assume the structures are product near the boundary. Let $\mathcal{B}$ denote spectral boundary conditions.*

(1) *The eigenspaces $E(\lambda, D_{M,\mathcal{B}}^{\mathcal{C}})$ are finite dimensional spaces for all λ and we have an orthogonal direct sum*

$$L^2(\mathcal{C}(M)) = \oplus_\lambda E(\lambda, D_{M,\mathcal{B}}^{\mathcal{C}}).$$

Order the eigenvalues $0 \le \lambda_1 \le \lambda_2 \le \ldots$ where each eigenvalues is repeated according to multiplicity. For any $\epsilon > 0$, there exists an integer $n(\epsilon)$ so that

$$n^{\frac{2}{m}-\epsilon} \le \lambda_n \le n^{\frac{2}{m}+\epsilon} \text{ for } n \ge n(\epsilon).$$

Let ϕ be a smooth p form with $\mathcal{B}\phi = 0$. Expand $\phi = \Sigma_n \phi_n$ in L^2 where the $\phi_n \in E(\lambda, D_{\mathcal{B}}^{\mathcal{C}})$. Then this series converges in the C^∞ topology.

(2) *If $\mathcal{R} > 0$, then $E(0, D_{\mathcal{B}}^{\mathcal{C}}) = 0$, i.e. there are no harmonic spinors.*

Proof. The first assertion follows from work of Grubb and Seeley [86, 87, 88] and we refer to them for further details. We prove the second assertion to illustrate the use of spectral boundary conditions; this was first established by Botvinnik and Gilkey [28]. Let $D = D_M^{\mathcal{C}}$ be the spinor Laplacian. Suppose that $Df = 0$ and that $\Pi_{\ge 0}(f|_{\partial M}) = 0$. We must show that $f = 0$. We generalize the proof of Theorem 2.6.5 to this setting. We integrate by parts and apply the Lichnerowicz formula discussed in §2.6 to see that

$$\begin{aligned} 0 &= \int_M (Df, f) = \int_M \{(-\operatorname{Tr}(\nabla^2) f, f) + \frac{1}{4}\mathcal{R}(f,f)\} \\ &= \int_M \{(\nabla f, \nabla f) + \frac{1}{4}\mathcal{R}(f,f)\} + \int_{\partial M} (\partial_N f, f). \end{aligned} \tag{2.8.f}$$

Since $(A_{M,\mathcal{B}}^{\mathcal{C}})^2 f = 0$, we have $A_{M,\mathcal{B}}^{\mathcal{C}} f = 0$ and consequently

$$\partial_N f|_{\partial M} = -B(f|_{\partial M}).$$

Since $\Pi_{\ge 0}(f|_{\partial M}) = 0$, we have

$$\int_{\partial M} (Bf, f) \le 0 \quad \text{so} \quad \int_{\partial M} (\partial_N f, f) \ge 0.$$

Since we assumed $\mathcal{R}$ was positive, all the terms appearing in equation (2.8.f) are non-negative. Thus we may conclude that $f = 0$ as desired. □

2.9 Riemannian submersions of manifolds with boundary

2.9.1 Definition. Let $\pi : Z \to Y$. We suppose Y and Z have non-empty boundary. We say that π is a Riemannian submersion in this setting if:

(1) We have π is a Riemannian submersion on the interior of Y.

(2) We have $\pi^{-1}\partial Y = \partial Z$.

(3) We have $\pi : \partial Z \to \partial Y$ is a Riemannian submersion.

Let $\{F_i\}$ be a local orthonormal frame field for the tangent bundle of Y on the boundary so that $F_m = N_Y$ is the inward unit normal and so that $\{F_a\}$ for $1 \le a \le m-1$ is a local orthonormal frame field for the tangent bundle of the boundary ∂Y. Let L^Y be the *second fundamental form* on Y;

$$L^Y_{ab} = L^Y(F_a, F_b) := g_Y(\nabla_{F_a} F_b, F_m);$$

the second fundamental form on Z is defined similarly. Let i_Y and i_Z be the inclusions of ∂Y and ∂Z in Y and Z. We have $\pi \circ i_Z = i_Y \circ \pi$. Let $F \in T^*Y$. Since π is a Riemannian submersion,

$$\pi^* \circ \operatorname{int}_Y(F) = \operatorname{int}_Z(\pi^* F) \circ \pi^*$$

and

$$\pi^* \circ \operatorname{ext}_Y(F) = \operatorname{ext}_Z(\pi^* F) \circ \pi^*.$$

The following Lemma summarizes some technical results that we shall need. Let Γ be the Christoffel symbols of the Levi-Civita connection.

2.9.2 Lemma. *Let $\pi : Z \to Y$ be a Riemannian submersion.*

(1) $L^Y_{ab} = L^Z_{ab}$ *and* $\Gamma^Z_{mai} = -2\omega_{ami} - L_{ai}$.

(2) $i_Y^* \operatorname{int}_Y(N_Y^*) *_Y = \epsilon(m,p) *_{\partial Y} i_Y^*$ *on* $C^\infty \Lambda^p(Y)$ *for* $\epsilon(m,p) = \pm 1$.

(3) $\mathcal{B}_R \Phi = 0 \iff \mathcal{B}_A \star_Y \Phi = 0 \iff i_Y^* \Phi = 0$ *and* $i_Y^* \delta_Y \Phi = 0$.

(4) *We have* $i_Z^*(\delta_Z \pi^* - \pi^* \delta_Y) = 0$ *if and only if:*

   a) *If $p = 0$, then there is no condition on θ. If $1 \le p < m$, then $\theta = 0$. If $1 \le p = m$, then $\theta_m = 0$.*

   b) *If $p = 0$ or if $p = 1$, then there is no condition on ω. If $1 < p < m$, then $\omega = 0$. If $2 \le p = m$, then $\omega_{amj} = 0 \ \forall\ a, j$.*

Proof. We prove the first assertion by computing

$$\begin{aligned}
&L^Y_{ab} = \Gamma^Y_{abm} = \Gamma^Z_{abm} = L^Z_{ab}, \text{ and}\\
&\Gamma^Z_{mai} = \Gamma^Z_{mai} - \Gamma^Z_{ami} - \Gamma^Z_{aim} = 2\omega_{mai} - \Gamma^Z_{aim} = -2\omega_{ami} - L_{ai}.
\end{aligned}$$

Assertion (2) is an easy calculation once the orientations involved are taken into account. We use assertion (2) to prove assertion (3) by computing

$$\begin{aligned}
&\mathcal{B}_R\Phi = 0 \iff \mathcal{B}_A \star_Y \Phi = 0\\
&\iff i_Y^* \operatorname{int}(N_Y^*) \star_Y \Phi = 0 \text{ and } i_Y^* \operatorname{int}(N_Y^*) d \star_Y \Phi = 0\\
&\iff \star_{\partial Y} i_Y^* \Phi = 0 \text{ and } \star_{\partial Y} i_Y^* \delta_Y \Phi = 0\\
&\iff i_Y^*\Phi = 0 \text{ and } i_Y^*\delta_Y\Phi = 0.
\end{aligned}$$

By Lemma 1.5.1,

$$i_Z^*(\delta_Z\pi^* - \pi^*\delta_Y) = i_Z^*(\operatorname{int}_Z(\theta) + \omega_{abi}\operatorname{ext}_Z(e^i)\operatorname{int}_Z(f^a)\operatorname{int}_Z(f^b))\pi^*.$$

The condition $i_Z^*(\delta_Z\pi^* - \pi^*\delta_Y) = 0$ decouples; it is satisfied if and only if we have the pair of equations

$$\begin{aligned}
&i_Z^* \operatorname{int}_Z(\theta)\pi^* = 0 \text{ and,}\\
&i_Z^*\omega_{abi}\operatorname{int}_Z(f^a)\operatorname{int}_Z(f^b))\pi^* = 0 \ \forall\ i.
\end{aligned}$$

If $p = 0$, then the tensors θ and ω play no role. If $p = 1$, then the tensor ω plays no role. If $p < m$, then both the normal and tangential components of the tensors ω and θ play a role; if $p = m$, then only the normal component of these tensors plays a role. $\square$

Let $\mathcal{B}^Y$ and $\mathcal{B}^Z$ denote the appropriate boundary conditions on Y and on Z.

2.9.3 Lemma. *Let $\pi : Z \to Y$ be a Riemannian submersion. Then*

(1) *If $\mathcal{B}_D^Y\Phi = 0$, then $\mathcal{B}_D^Z\pi^*\Phi = 0$.*
(2) *If $\mathcal{B}_A^Y\Phi = 0$, then $\mathcal{B}_A^Z\pi^*\Phi = 0$.*
(3) *Assume that $i_Z^*(\delta_Z\pi^* - \pi^*\delta_Y) = 0$. If $\mathcal{B}_R^Y\Phi = 0$, then $\mathcal{B}_R^Z\pi^*\Phi = 0$.*
(4) *If $p = 0$, Neumann boundary conditions are preserved. If $p > 0$, Neumann boundary conditions are preserved if and only if $\Gamma_{mai}^Z = 0 \ \forall\ i, a$.*

Proof. Assertion (1) is immediate. Suppose that Φ satisfies absolute boundary conditions. Then we argue:

$$\begin{aligned}
&i_Y^* \operatorname{int}_Y(N_Y^*)\Phi = 0 \text{ and } i_Y^* \operatorname{int}(N_Y^*) d_Y\Phi = 0,\\
\Rightarrow\ &\pi^* i_Y^* \operatorname{int}_Y(N_Y^*)\Phi = 0, \text{ and } \pi^* i_Y^* \operatorname{int}(N_Y^*) d_Y\Phi = 0,\\
\Rightarrow\ &i_Z^*\pi^* \operatorname{int}_Y(N_Y^*)\Phi = 0, \text{ and } i_Z^*\pi^* \operatorname{int}(N_Y^*) d_Y\Phi = 0,\\
\Rightarrow\ &i_Z^* \operatorname{int}_Z(N_Z^*)\pi^*\Phi = 0 \text{ and } i_Z^* \operatorname{int}_Z(N_Z^*) d_Z\pi^*\Phi = 0,\\
\Rightarrow\ &\pi^*\Phi \text{ satisfies absolute boundary conditions on } Z.
\end{aligned}$$

This proves assertion (2). If we assume $\delta_Z\pi^* = \pi^*\delta_Y$, the proof of assertion (3) is the same. We note that

$$\nabla_{N_Z}\pi^*\Phi - \pi^*\nabla_{N_Y}\Phi = \operatorname{ext}_Z(e^i)\operatorname{int}_Z(f^a)\Gamma^Z_{mai}\pi^*\Phi.$$

If $p = 0$, this vanishes automatically. If $p > 0$, this vanishes if and only if Γ^Z_{mai} vanishes on the boundary of Z. $\square$

We conclude this subsection by showing that Lemma 2.8.3 (2) is sharp; if $p \geq 1$, then the Neumann Laplacian need not be positive definite.

2.9.4 Lemma. *Let $0 < p$ and let $\lambda, \mu \in \mathbb{R}$ be given. There exists a compact Riemannian manifold Y with smooth boundary, there exists a Riemannian submersion $\pi : Z \to Y$, and there exists $0 \neq \Phi \in E(\lambda, \Delta^p_{Y,\mathcal{B}_N})$ so that $\pi^*\Phi \in E(\mu, \Delta^p_{Z,\mathcal{B}_N})$.*

Proof. We suppose first that $p = 1$, $\lambda = 0$, and $m = 2$. Let $Y := [0,1]$ with parameter y and let $\Phi := dy$; Φ satisfies Neumann boundary conditions and $\Delta^1_Y\Phi = 0$. Let $Z := [0,1] \times S^1$ and let t be the usual periodic parameter on the circle. We consider a metric of the form

$$ds^2 := dy^2 + e^{2f(y)}dt^2$$

where $f(y) := -\frac{\mu}{2}y^2$. By Lemma 1.7.2, we have that $\theta = -df = \mu y dy$. Since $\dim(Y) = 1$, the horizontal distribution $\mathcal{H}$ is integrable and $\omega = 0$. Since $\Delta_Y\Phi = 0$, Lemma 1.5.1 shows that

$$\Delta_Z\pi^*\Phi = d_Z\operatorname{int}(\theta)\pi^*\Phi = \mu\pi^*dy = \mu\pi^*\Phi.$$

Consequently $\pi^*\Phi \in E(\mu, \Delta^1_{Z,\mathcal{B}_N})$. For fixed t, the curves $y \to (y,t)$ are unit speed geodesics which are normal to the boundary. We check dy satisfies Neumann boundary conditions by computing:

$$\nabla_{\partial y}\partial_y = 0 \text{ and } \nabla_{\partial y}\partial_t = \frac{1}{2}\frac{\partial f}{\partial y}\partial_t.$$

We verify that $\nabla_{\partial y}dy = 0$ by computing

$$(\nabla_{\partial y}dy, \partial_y) = \partial_y(dy, \partial_y) - (dy, \nabla_{\partial_y}\partial_y) = 0$$

and

$$(\nabla_{\partial y}dy, \partial_t) = \partial_y(dy, \partial_t) - (dy, \nabla_{\partial_y}\partial_t) = 0.$$

To deal with the general case, we take a Riemannian product. Let $p \geq 1$ and let $\bar\lambda \in \mathbb{R}$ and $\bar\mu \in \mathbb{R}$ be given. Let $\mu = \bar\mu - \bar\lambda$. As above, we construct a metric on

Z so that $\pi^* dy \in E(\mu, \Delta^1_{Z,\mathcal{B}_N})$. Let $\mathbb{T}^{p-1}$ be the flat torus of dimension $p-1$. By rescaling the metric on $\mathbb{T}^{p-1}$, we may find $0 \neq \Phi \in E(\lambda, \Delta_{\mathbb{T}})$. Let

$$\bar{Y} := Y \times \mathbb{T}^{p-1}, \ \bar{Z} := Z \times \mathbb{T}^{p-1}, \text{ and } \bar{\pi}(z,t) = (\pi(z), t).$$

We then have

$$\Delta^p_{\bar{Y}}(dy \wedge \Phi) = (\Delta^1_Y dy) \wedge \Phi + dy \wedge \Delta^{p-1}_{\mathbb{T}} \Phi$$

and

$$\Delta^p_{\bar{Z}} \bar{\pi}^*(dy \wedge \Phi) = (\Delta^1_Z \pi^* dy) \wedge \Phi + (\pi^* dy) \wedge \Delta^{p-1}_{\mathbb{T}} \Phi.$$

Since $dy \wedge \Phi$ and $\pi^* dy \wedge \Phi$ satisfy Neumann boundary conditions on $\bar{Y}$ and $\bar{Z}$, we have

$$dy \wedge \Phi \in E(\bar{\lambda}, \Delta^p_{\bar{Y},\mathcal{B}_N}) \text{ and } \bar{\pi}^*(dy \wedge \Phi) \in E(\bar{\mu}, \Delta^p_{\bar{Z},\mathcal{B}_N}). \qquad \square$$

Chapter Three:
Rigidity of Eigenvalues

3.1 Introduction

Let $\pi : Z \to Y$ be a Riemannian submersion. Let $E(\lambda, \Delta_Y^p)$ and $E(\lambda, \Delta_Z^p)$ be the eigenspaces of the p form valued Laplacians on Y and on Z. In this chapter, we study the relationship between $\pi^* E(\lambda, \Delta_Y^p)$ and $E(\mu, \Delta_Z^p)$. In §3.2 through §3.7, we work in the category of closed manifolds; in §3.8 we turn to manifolds with boundary.

3.1.1 Definition.

(1) We say that an eigenform Φ is preserved if Φ belongs to $E(\lambda, \Delta_Y^p)$ and if the pullback $\pi^*\Phi$ belongs to $E(\mu, \Delta_Z^p)$.

(2) Suppose that an eigenform Φ is preserved. We say the eigenvalue changes if $\mu \neq \lambda$. Otherwise we say that the eigenvalue is preserved.

In §3.2, we study the scalar case, i.e. $p = 0$. Let $\Phi \in E(\lambda, \Delta_Y^0)$ and suppose $\pi^*\Phi \in E(\mu, \Delta_Z^0)$. We show in Theorem 3.2.1 that $\lambda = \mu$; an eigenvalue can not change in this situation. We show in Theorem 3.2.2 that π^* intertwines Δ_Y^0 and Δ_Z^0, or equivalently that π^* preserves all the eigenfunctions, if and only if the mean curvature θ vanishes.

In §3.3 we study the Bochner Laplacian which was defined in §2.6. Many of the results for the scalar case extend naturally to this setting. For example, in Theorem 3.3.1, we will show that if the pull back of an eigensection for a Bochner Laplacian on Y is an eigensection for the corresponding Bochner Laplacian on Z, then the eigenvalue is preserved.

In §3.4, we study the case $p > 0$. In contrast to the scalar valued case, eigenvalues can change. However, we shall show in Theorem 3.4.1 that they can only increase, i.e. if $\Phi \in E(\lambda, \Delta_Y^p)$ and if $\pi^*\Phi \in E(\mu, \Delta_Z^p)$, then $\lambda \le \mu$. In the next chapter in Theorem 4.2.2, we will show that this theorem is sharp by showing eigenvalues can increase if $p \ge 2$; we do not know if this result is sharp if $p = 1$ and if the boundary of M is empty. However, in Theorem 3.6.1, we show eigenvalues are rigid if $p = 1$ and if $\theta = 0$. In Theorem 3.4.2, we show that π^* preserves all the eigenfunctions if and only if both the mean curvature covector θ and the integrability tensor ω vanish.

In §3.5, we turn to the complex setting and prove analogous results for the complex Laplacian $\Delta^{(p,q)}$. We show in Theorem 3.5.1 that eigenvalues can only increase. We shall also show that if $q = 0$, then eigenvalues are rigid. In the next chapter in Theorem 4.4.1 we will show that this result is sharp by showing eigenvalues can increase if $q \ge 1$ and $p+q \ge 2$; we do not know if this result is sharp if $(p, q) = (0, 1)$. In Theorem 3.5.2, we give necessary and sufficient conditions that all the eigenforms are preserved once the bidegree (p, q) is fixed.

In §3.6, we turn our attention to examples where eigenvalues are rigid. We show in Theorem 3.6.1 that eigenvalues are rigid for SL submersions and for sphere bundles with fiber dimension at least 2. We also show that eigenvalues are rigid for principal G bundles where $H^1(G; \mathbb{R}) = 0$. In the next chapter we will show eigenvalues can change for a principal G bundle if $H^1(G; \mathbb{R}) \ne 0$.

In §3.7, we study the analogous theorems in the spin setting. The generalization of Theorems 3.4.2 and 3.5.2 is a bit different. We show that if π_α^* preserves all the eigen spinors, then the eigenvalue is shifted by a constant. This happens if and only if the integrability tensor $\omega = 0$. Since we are working with principal bundles and principal bundle metrics, the mean curvature covector θ always vanishes by assumption.

In §3.8, we study manifolds with boundary and extend Theorems 3.2.1, 3.2.2, 3.4.1, and 3.4.2 to this setting with Dirichlet, absolute, or relative boundary conditions. The observation that the operators Δ^p and $\Delta^{(p,q)}$ are non-negative operators plays a crucial role in the proof of Theorem 3.4.1; as noted in §2.9, the Neumann Laplacian for a manifold with boundary is *not* a non-negative operator and eigenvalues can decrease in this setting.

In §3.10, we discuss the heat content asymptotics. Let M be a compact manifold with smooth boundary. We impose Dirichlet boundary conditions. Let the initial temperature of M be 1 and let $h_M(t)$ be the total heat energy content of M for positive time t. As $t \downarrow 0$, $h_M(t)$ has an asymptotic expansion where the coefficients are locally computable. Let $\pi : Z \to Y$ be a Riemannian submersion. If $\theta = 0$, the volume of the fibers is constant by Lemma 1.17.2. We use Theorem 3.8.1 to show that if $\theta = 0$, then the heat content asymptotics on Z and on Y are related by the identity

$$h_Z(t) = h_Y(t) \cdot vol(F).$$

Note that the heat equation asymptotics defined in §2.6 and §2.8 do not satisfy this property.

3.2 The scalar Laplacian

The eigenvalues of the scalar Laplacian are rigid.

3.2.1 Theorem. *Let $\pi : Z \to Y$ be a Riemannian submersion. If $\Phi \in E(\lambda, \Delta_Y^0)$ does not vanish identically and if $\pi^*\Phi \in E(\mu, \Delta_Z^0)$, then $\lambda = \mu$.*

Proof. Let $0 \neq \Phi \in E(\lambda, \Delta_Y^0)$ and let $\phi = \pi^*\Phi$. Suppose $\phi \in E(\mu, \Delta_Z^0)$. By Lemma 1.5.1,

$$(\mu - \lambda)\phi = \mathrm{int}_Z(\theta)\pi^* d_Y \Phi.$$

Choose y_0 so $\Phi(y_0)$ is maximal. Then $d_Y\Phi(y_0) = 0$ so $(\mu - \lambda)\phi(z_0) = 0$ where $\pi z_0 = y_0$. By replacing Φ by $-\Phi$ if need be, we may assume the maximal value of Φ is positive and conclude $\lambda = \mu$. $\square$

This shows that a single eigenvalue can not change. We can also give necessary and sufficient conditions that all the eigenfunctions are preserved. We recall the definition of the unnormalized mean curvature θ given in Definition 1.2.3:

$$\theta := -g_Z([e_i, f_a], e_i)f^a = {}^Z\Gamma_{iia}f^a \in C^\infty(\mathcal{H}).$$

3.2.2 Theorem. *Let $\pi : Z \to Y$ be a Riemannian submersion. The following conditions are equivalent:*

(1) *We have $\Delta_Z^0\pi^* = \pi^*\Delta_Y^0$.*
(2) *For all λ, we have $\pi^*E(\lambda, \Delta_Y^0) \subset E(\lambda, \Delta_Z^0)$.*
(3) *We have $\theta = 0$.*

Proof. The equivalence of assertions (1) and (2) follows from Theorem 2.3.1 which gives a discrete spectral resolution for Δ_Y^0. If $\theta = 0$, then Lemma 1.5.1 implies $\Delta_Z^0\pi^* = \pi^*\Delta_Y^0$. Conversely, if this identity holds, then $\mathrm{int}_Z(\theta)\pi^* d_Y = 0$. Since θ is a horizontal co-vector, this implies $\theta = 0$. $\square$

3.3 The Bochner Laplacian

The Bochner Laplacian was introduced in §2.6. Let ∇ be a Riemannian connection on a complex vector bundle V over a Riemannian manifold M without boundary. We use the connection on V and the Levi-Civita connection on the tangent bundle TM to covariantly differentiate tensors of all types; let '$f_{;uv}$' denote the components of the second covariant derivative of a tensor field f. The Bochner Laplacian is defined by the formula:

$$D_\nabla f = -f_{;ii} = -\operatorname{Tr}\nabla^2 f.$$

This is a self-adjoint elliptic operator of Laplace type on $C^\infty(V)$. For example, if we take V to be the trivial vector bundle $M \times \mathbb{C}$ and if we take ∇ to be the trivial connection, the associated Bochner Laplacian is the scalar Laplacian Δ_M^0. In this section, we generalize the results of §3.2 to this setting.

Let $\pi : Z \rightarrow Y$ be a Riemannian submersion with fiber X and let D_Y be a Bochner Laplacian on a vector bundle V over Y. We can give the *pull back bundle* $\pi^* V$ over Z the pull back connection and pull back metric. We use these structures to define the Bochner Laplacian D_Z. Pull back induces a natural map

$$\pi^* : C^\infty(V_Y) \rightarrow C^\infty(V_Z).$$

The following result generalizes Theorem 3.2.1 to this setting.

3.3.1 Theorem. *Let $\pi : Z \rightarrow Y$ be a Riemannian submersion. Let D_Y be a Bochner Laplacian over Y and let D_Z be the induced Bochner Laplacian over Z.*

(1) *We have $D_Z \pi^* - \pi^* D_Y = \operatorname{int}_Z(\theta)\pi^* \nabla_Y$.*

(2) *If $\Phi \in E(\lambda, D_Y)$ is non-trivial and if $\pi^*\Phi \in E(\mu, D_Z)$, then $\lambda = \mu$.*

(3) *The following assertions are equivalent:*
   3a) *We have $D_Z \pi^* = \pi^* D_Y$.*
   3b) *For all λ, we have $\pi^* E(\lambda, D_Y) \subset E(\lambda, D_Z)$.*
   3c) *We have $\theta = 0$.*

Proof. Since the calculations are local, we may assume V is trivial. Let $\{F_a\}$ be a local orthonormal frame for TY. We expand $\nabla_Y \Phi$ and $\nabla_Y^2 \Phi$ in the form:

$$\nabla_Y \Phi := \Phi_{;a} \otimes F^a \text{ and } \nabla_Y^2 \Phi := \Phi_{;ab} \otimes F^a \otimes F^b.$$

Decompose $\Phi \in C^\infty(V)$ into its components; thus we have $\Phi = (\Phi_1, ..., \Phi_r)$. We define the vector valued derivative of Φ by differentiating the components; this means that

$$F_a(\Phi) := (F_a(\Phi_1), ..., F_a(\Phi_r)).$$

This corresponds to taking the flat connection defined locally by the frame in question. Let ${}^Y\mathcal{A}$ be the connection 1-form of ∇_Y; we may expand ${}^Y\mathcal{A} := {}^Y\mathcal{A}_a F^a$ where ${}^Y\mathcal{A}_a$ is an $r \times r$ matrix. We may then express the components of $\nabla\Phi$ and $\nabla^2\Phi$ in the form:

$$\Phi_{;a} = F_a \Phi + {}^Y\mathcal{A}_a \Phi, \text{ and } \Phi_{;ab} = F_b \Phi_{;a} + {}^Y\mathcal{A}_b \Phi_{;a} + {}^Y\Gamma_{bca}\Phi_{;c}.$$

This then leads to the following expression for the Bochner Laplacian:

$$D_Y \Phi = -(F_a \Phi_{;a} + {}^Y\mathcal{A}_a \Phi_{;a} + {}^Y\Gamma_{aca}\Phi_{;c}).$$

Let $\phi = \pi^*\Phi$. Note that ${}^Z\mathcal{A} = \pi^{*Y}\mathcal{A}$. Thus $\nabla_Z\phi = \pi^*\nabla_Y\Phi$. Since the *vertical covariant derivatives* $\phi_{;i} = 0$ and since by Lemma 1.3.2, ${}^Z\Gamma_{abc} = \pi^{*Y}\Gamma_{abc}$, we prove assertion (1) by computing:

$$\begin{aligned}
&D_Z\phi = -(f_a\phi_{;a} + {}^Z\mathcal{A}_a\phi_{;a} + {}^Z\Gamma_{aca}\phi_{;c} + {}^Z\Gamma_{ici}\phi_{;c}),\\
&(D_Z\pi^* - \pi^*D_Y)\Phi = -{}^Z\Gamma_{ici}\phi_{;c} = \mathrm{int}_Z(\theta)(\nabla_Z\phi) = \mathrm{int}_Z(\theta)\pi^*\nabla_Y\Phi.
\end{aligned}$$

We apply essentially the same argument as that which was used to prove Theorem 3.2.1 in our proof of assertion (2). Let $0 \neq \Phi \in E(\lambda, D_Y)$. Suppose that $\pi^*\Phi \in E(\mu, D_Z)$. Let $\phi := \pi^*\Phi$. We use the first assertion to see that

$$(\mu - \lambda)\phi = \mathrm{int}_Z(\theta)\pi^*\nabla_Y\Phi.$$

Since ∇ is a Riemannian connection, we may take the inner product with ϕ to see

$$\begin{aligned}
(\mu - \lambda)|\phi|^2 &= \mathrm{int}_Z(\theta)\pi^*(\nabla_Y\Phi, \Phi) = \tfrac{1}{2}\,\mathrm{int}_Z(\theta)\pi^*d|\Phi|^2\\
&= \tfrac{1}{2}(\theta, \pi^*d|\Phi|^2).
\end{aligned}$$

Choose y_0 so $|\Phi|^2(y_0)$ is maximal. Choose z_0 so $\pi z_0 = y_0$. Then $d|\Phi|^2(y_0) = 0$ so

$$(\mu - \lambda)|\phi|^2(z_0) = 0.$$

Since $|\phi|^2(z_0) = |\Phi|^2(y_0) \neq 0$, we conclude $\lambda = \mu$.

The equivalence of assertions (3-a) and (3-b) follows from the discrete spectral resolution for D_Y; the fact that assertion (3-c) implies assertion (3-a) follows from assertion (1). Suppose that assertion (3-a) holds so $D_Z\pi^* = \pi^*D_Y$. Then assertion (1) shows that $\mathrm{int}_Z(\theta)\pi^*\nabla_Y = 0$. Since θ is a horizontal differential form, this equality on the operator level implies that $\theta = 0$. $\square$

3.4 The form valued Laplacian

We shall show in Theorem 4.2.2 that eigenvalues can change for the form valued Laplacian. However, Theorem 3.2.1 does have at least a partial generalization:

3.4.1 Theorem. *Let $\pi : Z \to Y$ be a Riemannian submersion. If $\Phi \in E(\lambda, \Delta_Y^p)$ is non-trivial and if $\pi^*\Phi \in E(\mu, \Delta_Z^p)$, then $\lambda \le \mu$.*

Proof. We use the fiber product defined in §1.8. Let $\pi : Z \to Y$ be a Riemannian submersion. Let $0 \neq \Phi \in E(\lambda, \Delta_Y^p)$ and let $\pi^*\Phi \in E(\lambda + \epsilon, \Delta_Z^p)$. Let $Z(0) = Z$ and inductively let

$$Z(n) := W(Z(n-1), Z(n-1))$$

be the fiber product of $Z(n-1)$ with itself. Let $\pi_n : Z(n) \to Y$ be the associated projection. By Lemma 1.8.3,

$$\pi_n^*\Phi \in E(\lambda + 2^n\epsilon, \Delta^p_{Z(n)}).$$

Since the Laplacian on $Z(n)$ is a non-negative operator, $\lambda + 2^n\epsilon \geq 0$. Since this holds for all n, $\epsilon \geq 0$ as desired. $\square$

We can also give necessary and sufficient conditions that all the eigenforms are preserved. Since eigenvalues can change, the statement is just a bit more complicated than in the case $p = 0$.

3.4.2 Theorem. *Let $\pi : Z \to Y$ be a Riemannian submersion. Fix an index $1 \leq p \leq \dim(Y)$. The following conditions are equivalent:*

(1) *We have $\Delta^p_Z\pi^* = \pi^*\Delta^p_Y$.*

(2) *For all λ, we have $\pi^*E(\lambda, \Delta^p_Y) \subset E(\lambda, \Delta^p_Z)$.*

(3) *For all λ, there exists $\mu = \mu(\lambda)$ so $\pi^*E(\lambda, \Delta^p_Y) \subset E(\mu, \Delta^p_Z)$.*

(4) *We have $\theta = 0$ and $\omega = 0$.*

3.4.3 Remark: Note that if condition (2) holds for any p with $0 < p \leq \dim(M)$, then assertion (2) holds for all p with $0 < p \leq \dim(M)$.

Proof. It is immediate that assertion (1) implies assertion (2), that assertion (2) implies assertion (3), and that assertion (4) implies assertion (1). Thus to complete the proof, we must show that assertion (3) implies assertion (4).

We suppose that assertion (3) holds. This means that for any $\lambda \in \mathbb{R}$ there exists $\epsilon(\lambda)$ so that

$$\pi^*E(\lambda, \Delta^p_Y) \subset E(\lambda + \epsilon(\lambda), \Delta^p_Z).$$

Let $\rho_{\mathcal{H}}$ be orthogonal projection from $\Lambda^p(Z)$ to $\Lambda^p(\mathcal{H})$. If $\Phi \in E(\lambda, \Delta^p_Y)$, we use Lemma 1.5.1 to see

$$\begin{aligned} 0 =&\epsilon(\lambda)(1-\rho_{\mathcal{H}})\pi^*\Phi \\ =&(1-\rho_{\mathcal{H}})(d_Z(\mathrm{int}_Z(\theta)+\mathcal{E}) + (\mathrm{int}_Z(\theta)+\mathcal{E})d_Z)\pi^*\Phi. \end{aligned}$$

Since the span of the eigenspaces $E(\lambda, \Delta^p_Y)$ is dense in $C^\infty\Lambda^p(Y)$, we have

$$(1-\rho_{\mathcal{H}})(d_Z(\mathrm{int}_Z(\theta)+\mathcal{E}) + (\mathrm{int}_Z(\theta)+\mathcal{E})d_Z)\pi^* = 0 \text{ on } C^\infty\Lambda^p(Y). \tag{3.4.a}$$

Fix a point $z_0 \in Z$ and let $y_0 = \pi z_0$. Choose $F \in C^\infty Y$ so that $F(y_0) = 0$. Let $\xi := dF(y_0)$. Since $\mathrm{int}_Z(\theta) + \mathcal{E}$ is a 0^{th} order operator, we apply equation (3.4.a) to $F\Phi$ and evaluate at z_0 to see that

$$\begin{aligned} 0 =&(1-\rho_{\mathcal{H}})\{\mathrm{ext}_Z(\pi^*\xi)(\mathrm{int}_Z(\theta)+\mathcal{E}) \\ &+ (\mathrm{int}_Z(\theta)+\mathcal{E})\,\mathrm{ext}_Z(\pi^*\xi)\}\pi^*(\Phi(y_0)). \end{aligned}$$

Note that

$$0 = (1-\rho_{\mathcal{H}})\{\mathrm{ext}_Z(\pi^*\xi)\,\mathrm{int}_Z(\theta) + \mathrm{int}_Z(\theta)\,\mathrm{ext}_Z(\pi^*\xi)\}\pi^*.$$

Since $\mathcal{E}$ always introduces a vertical covector, we conclude

$$0 = \{\mathrm{ext}_Z(\pi^*\xi)\mathcal{E} + \mathcal{E}\,\mathrm{ext}_Z(\pi^*\xi)\}\pi^*.$$

To simplify the notation, we temporarily define

$$E^a := \mathrm{ext}_Z(f^a),\ E^i := \mathrm{ext}_Z(e^i),\ \text{and } I^a := \mathrm{int}_Z(f^a).$$

Fix c and choose ξ so that $\pi^*\xi = f^c$. We compute:

$$\begin{aligned}0 =&\omega_{abi}\{E^cE^iI^aI^b + E^iI^aI^bE^c\} = \omega_{abi}E^i\{-E^cI^aI^b + I^aI^bE^c\}\\ =&\omega_{abi}E^i\{I^aE^cI^b + I^aI^bE^c - \delta_{ac}I^b\}\\ =&\omega_{abi}E^i\{-I^aI^bE^c + I^aI^bE^c - \delta_{ac}I^b + \delta_{bc}I^a\} = -2\omega_{cbi}E^iI^b.\end{aligned}$$

Since $p \geq 1$, this implies $\omega = 0$ and hence the horizontal distribution $\mathcal{H}$ is integrable.

We now apply the results of §1.7. Let d_X denote exterior differentiation along the fiber. We set $\mathcal{E} = 0$ and use equation (3.4.a) to see

$$0 = d_X\,\mathrm{int}_Z(\theta)\pi^* \text{ on } C^\infty\Lambda^p(Y).$$

This implies that the mean curvature covector θ is constant on the fibers so we may express $\theta = \pi^*\Theta$ as the pull back of a globally defined 1-form on the base. Since the horizontal distribution $\mathcal{H}$ is integrable, we use Lemma 1.7.2 to give a local decomposition of Z so that we have

$$\theta = \pi^*\Theta = -d_Y\ln(g_X).$$

Let $\psi(y)$ be the volume of the fibers. Let $d\nu_x^e$ be the Euclidean measure. Then

$$\begin{aligned}d_Y\psi(y) =&d_Y\int_X g_X(x,y)d\nu_x^e = \int_X (g_Xg_X^{-1}d_Yg_X)(x,y)d\nu_x^e\\ =&-\int_X g_X(x,y)\theta(x,y)d\nu_x^e = -\Theta(y)\int_X g_X(x,y)d\nu_x^e\\ =&-\Theta(y)\psi(y).\end{aligned}$$

Thus the globally defined function ψ, which gives the volume of the fibers, defines θ, i.e. we have

$$\theta = -\pi^*d_Y\ln\psi.$$

We define a conformal variation of the metric on the vertical distribution and leave the metric on the horizontal distribution unchanged:

$$g(t)_Z = \psi^{2t} ds^2_{\mathcal{V}} + ds^2_{\mathcal{H}}.$$

Then $\pi : Z(t) \rightarrow Y$ is a Riemannian submersion with integrable horizontal distribution. We use Lemma 1.7.2 to see that

$$\theta(t) = (1 + t \dim(X))\theta.$$

Consequently we have

$$\begin{aligned}\Delta^p_{Z(t)}\pi^* - \pi^*\Delta^p_Y =&(1 + t \dim(X))(d_Z \operatorname{int}_Z(\theta) + \operatorname{int}_Z(\theta)d_Z)\pi^* \\ =&(1 + t \dim(X))(\Delta^p_Z\pi^* - \pi^*\Delta^p_Y).\end{aligned}$$

This shows that

$$\pi^* E(\lambda, \Delta^p_Y) \subset E(\lambda + (1 + t \dim(X))\epsilon(\lambda), \Delta^p_{Z(t)}).$$

Since the Laplacian is a non-negative operator, we have

$$\lambda + (1 + t \dim(X))\epsilon(\lambda) \geq 0.$$

Since this identity holds for arbitrary t, $\epsilon(\lambda) = 0$. This shows that

$$(d_Z \operatorname{int}_Z(\theta) + \operatorname{int}_Z(\theta)d_Z)\pi^* = 0.$$

This permits us to conclude that $\theta = 0$. $\square$

3.4.4 Remark. This is exactly where the argument fails when we consider the Neumann Laplacian for forms of degree at most 1. This operator need not be non-negative and eigenvalues can decrease. We refer to Lemma 2.9.4 for further details.

3.5 The complex Laplacian

In this section, we extend Theorems 3.2.1 and 3.3.1 to the holomorphic setting. We refer to §A.5 for additional information. In §2.7.1, we defined the notion of a *Hermitian submersion*. This meant that π is a Riemannian submersion from Z to Y, that Z and Y are complex manifolds, that π is complex analytic, and that the metrics on Z and on Y are Hermitian. We refer to work of Johnson [105] and Watson [194] for a discussion of some of the geometry which is involved;

these authors also consider the almost complex and the Kaehler categories. We complexify π^* to define

$$\pi^* : C^\infty\Lambda^{p,q}(Y) \to C^\infty\Lambda^{p,q}(Z).$$

We then have the relations

$$\pi^*\pi_Y^{p,q} = \pi_Z^{p,q}\pi^* \text{ and } \pi^*\bar\partial_Y = \bar\partial_Z\pi^*.$$

We extend interior multiplication, exterior multiplication, and ω to be complex linear. Let J be the almost complex structure. We then have $J\mathcal{H} \subset \mathcal{H}$. If ω is a 2 form, we define:

$$J^*\omega(\xi_1,\xi_2) := \omega(J\xi_1, J\xi_2).$$

3.5.1 Theorem. *Let $\pi : Z \to Y$ be a Hermitian submersion.*

(1) *If $\Phi \in E(\lambda, \Delta_Y^{p,q})$ is non-trivial and if $\pi^*\Phi \in E(\mu, \Delta_Z^{p,q})$, then $\lambda \le \mu$.*

(2) *If $\Phi \in E(\lambda, \Delta_Y^{p,0})$ is non-trivial and if $\pi^*\Phi \in E(\mu, \Delta_Z^{p,0})$, then $\lambda = \mu$.*

Proof. The proof given of Theorem 3.4.1 extends without change to the complex setting to establish the first assertion. The proof of assertion (2) is quite different in the complex case however. Let

$$0 \ne \Phi \in E(\lambda, \Delta_Y^{p,0}) \text{ and } \pi^*\Phi \in E(\lambda+\varepsilon, \Delta_Z^{p,0}).$$

Since $\Delta^{p,0} = \bar\partial^*\bar\partial$, we use Lemma 2.7.2 to see that

$$\varepsilon\pi^*\Phi = \pi_Z^{p,0}\Xi\pi^*\bar\partial_Y\Phi.$$

Since $\mathcal{E}$ has a non-trivial vertical component,

$$0 = \pi_Z^{p,0}\mathcal{E}\pi^*\bar\partial_Y\Phi \text{ so } \varepsilon\pi^*\Phi = \pi_Z^{p,0}\operatorname{int}_Z(\theta)\pi^*\bar\partial_Y\Phi.$$

We apply π_* to see that

$$\varepsilon\Phi = \pi_Y^{p,0}\operatorname{int}_Y(\Theta)\bar\partial_Y\Phi.$$

We define a variation of the metric which leaves the metric on the horizontal distribution unchanged and is a conformal deformation on the vertical distribution:

$$g(t)_Z := V^{2t}ds^2_{\mathcal{V}} + ds^2_{\mathcal{H}}$$

Then $\pi : Z(t) \to Y$ is a Hermitian submersion and $\mathcal{E}$ transforms conformally. We use Lemma 1.7.2 to see $\theta(t) = (1 + t\dim(X))\theta$ and

$$\pi^*\Phi \in E(\lambda + (1+t\dim(X))\varepsilon, \Delta^{p,0}_{Z(t)}).$$

Thus by assertion (1), $\lambda + (1+t\dim(X))\varepsilon \ge 0$ for all $t \in \mathbb{R}$. This shows $\varepsilon = 0$. $\square$

We can also generalize Theorem 3.4.2 to this setting. The proof is quite different and there are more cases to be considered.

3.5.2 Theorem. *Fix (p,q) with $0 \le p,q \le \dim_{\mathbb{C}} Y$. The following conditions are equivalent:*

(1) $\Delta_Z^{p,q}\pi^* = \pi^*\Delta_Y^{p,q}$.

(2) $\pi^* E(\lambda, \Delta_Y^{p,q}) \subset E(\lambda, \Delta_Z^{p,q})$.

(3) *For all λ, there exists $\mu = \mu(\lambda)$ so $\pi^* E(\lambda, \Delta_Y^{p,q}) \subset E(\mu, \Delta_Z^{p,q})$.*

(4) *The fibers of π are minimal and:*
  4-a) *if $p = 0$ and if $q = 0$, there is no condition on ω.*
  4-b) *if $p > 0$ and if $q = 0$, then $J^*\omega = -\omega$.*
  4-c) *if $p = 0$ and if $q > 0$, then $J^*\omega = \omega$ i.e. $\mathcal{H}_{1,0}$ is integrable.*
  4-d) *if $p > 0$ and if $q > 0$, then $\omega = 0$ i.e. $\mathcal{H}$ is integrable.*

3.5.3 Remark. In §4.5, we will provide examples of Riemannian submersions where $\omega \neq 0$ and $J^*\omega = \pm\omega$. Thus the conditions in assertion (4) are distinct and non-vacuous.

The remainder of this section is devoted to the proof of Theorem 3.5.2. There are a number of technical Lemmas to be established. Let π be a Hermitian submersion. Then the horizontal and vertical distributions $\mathcal{H}$ and $\mathcal{V}$ are invariant under the almost complex structure J. The canonical decomposition of

$$TZ \otimes \mathbb{C} = TZ_{1,0} \oplus TZ_{0,1}$$

therefore induces a decomposition

$$\mathcal{H} \otimes \mathbb{C} = \mathcal{H}_{1,0} \oplus \mathcal{H}_{0,1} \text{ and } \mathcal{V} \otimes \mathbb{C} = \mathcal{V}_{1,0} \oplus \mathcal{V}_{0,1}.$$

Choose a local orthonormal frame field for the horizontal distribution $\mathcal{H}$ of the form $\{f_1, ..., f_\nu, Jf_1, ..., Jf_\nu\}$ where $\nu = \dim_{\mathbb{C}} Y$. The corresponding dual coframe field for $\mathcal{H}^*$ is then given by

$$\{f^1, ..., f^\nu, -Jf^1, ..., -Jf^\nu\}.$$

Let

$$\zeta_\alpha := \tfrac{1}{2}(f_\alpha - \sqrt{-1}Jf_\alpha) \text{ and } \zeta^\alpha := (f^\alpha - \sqrt{-1}Jf^\alpha).$$

The collections of complex tangent vectors $\{\zeta_\alpha\}$ and $\{\bar\zeta_\alpha\}$ are frames for $\mathcal{H}_{1,0}$ and $\mathcal{H}_{0,1}$ and the corresponding collections of complex cotangent vectors $\{\zeta^\alpha\}$ and $\{\bar\zeta^\alpha\}$ are the dual frames for $\Lambda^{1,0}(\mathcal{H}^*)$ and $\Lambda^{0,1}(\mathcal{H}^*)$. Interior multiplication by ζ^α lowers the bi degree by $(0,1)$; interior multiplication by $\bar\zeta^\alpha$ lowers the bi degree by $(1,0)$. We let indices α and β range from 1 to $\dim_{\mathbb{C}} Y$ and sum over repeated indices.

3.5.4 Lemma. *Let $\pi : Z \to Y$ be a Hermitian submersion.*

(1) *We have $\omega(\zeta_\alpha, \zeta_\beta) \in \Lambda^{0,1}(\mathcal{V}^*)$.*

(2) *We have $\pi_Z^{p,q}\mathcal{E}\pi^* = \operatorname{ext}_Z(\omega(\zeta_\alpha, \zeta_\beta))\operatorname{int}_Z(\zeta^\alpha)\operatorname{int}_Z(\zeta^\beta)$*
$+2\operatorname{ext}_Z(\pi_Z^{1,0}\omega(\zeta_\alpha, \bar\zeta_\beta))\operatorname{int}_Z(\zeta^\alpha)\operatorname{int}_Z(\bar\zeta^\beta)$ *on $\Lambda^{p,q+1}(Y)$.*

(3) *We have $\omega(\zeta_\alpha, \zeta_\beta) = 0$ for all α and β if and only if we have $J^*\omega = \omega$.*

(4) *We have $\omega(\zeta_\alpha, \bar\zeta_\beta) = 0$ for all α and β if and only if we have $J^*\omega = -\omega$.*

Proof. Since Z is a complex manifold, the almost complex structure J is integrable and $[\zeta_\alpha, \zeta_\beta] \in TZ_{1,0}$. Since the horizontal and vertical distributions $\mathcal{H}$ and $\mathcal{V}$ are J invariant, $\rho_{\mathcal{V}}[\zeta_\alpha, \zeta_\beta] \in \mathcal{V}_{1,0}$. Let $\tilde g_Z$ be the extension of g_Z to be complex bilinear; ω is the dual of $\rho_{\mathcal{V}}[\cdot,\cdot]$ with respect to $\tilde g_Z$. The first assertion now follows since the dual of $\mathcal{V}_{1,0}$ with respect to $\tilde g_Z$ is $\Lambda^{0,1}(\mathcal{V}^*)$. To prove the second assertion, we compute:

$$\begin{aligned}
&(3.5.a) \qquad \mathcal{E} = \operatorname{ext}_Z(\omega(\zeta_\alpha, \zeta_\beta))\operatorname{int}_Z(\zeta^\alpha)\operatorname{int}_Z(\zeta^\beta)\\
&(3.5.b) \qquad + \operatorname{ext}_Z(\omega(\bar\zeta_\alpha, \bar\zeta_\beta))\operatorname{int}_Z(\bar\zeta^\alpha)\operatorname{int}_Z(\bar\zeta^\beta)\\
&(3.5.c) \qquad + \operatorname{ext}_Z(\omega(\zeta_\alpha, \bar\zeta_\beta))\operatorname{int}_Z(\zeta^\alpha)\operatorname{int}_Z(\bar\zeta^\beta)\\
&(3.5.d) \qquad + \operatorname{ext}_Z(\omega(\bar\zeta_\alpha, \zeta_\beta))\operatorname{int}_Z(\bar\zeta^\alpha)\operatorname{int}_Z(\zeta^\beta).
\end{aligned}$$

The terms in (3.5.a) lower the horizontal bi degree by (0,2), the terms in (3.5.b) lower the horizontal bi degree by (2,0), and the terms in (3.5.c) and (3.5.d) lower the horizontal bi degree by (1,1); the symmetries involved permit us to combine these two terms. Thus in (3.5.a) we must use exterior multiplication by $\pi_Z^{0,1}\omega(\zeta_\alpha, \zeta_\beta)$; in (3.5.c) and in (3.5.d) we must use exterior multiplication by $\pi_Z^{1,0}\omega(\zeta_\alpha\bar\zeta_\beta)$; (3.5.b) plays no role. By the first assertion, we may replace $\pi_Z^{0,1}\omega(\zeta_\alpha, \zeta_\beta)$ by $\omega(\zeta_\alpha, \zeta_\beta)$; this proves the second assertion. The final assertions are immediate consequences of the definition. □

Suppose that assertion (4) of Theorem 3.5.2 holds. We apply Lemma 2.7.2. Since $\theta = 0$, Ξ is determined by $\mathcal{E}$. If $p = 0$ and if $q = 0$, $\mathcal{E}$ acts trivially on $\Lambda^{0,1}(Z)$ and assertion (1) holds. If $p > 0$ and if $q > 0$, we assume $\omega = 0$ and $\mathcal{E} = 0$. If $p > 0$ and if $q = 0$, we need only consider the action of $\pi_Z^{p,0}\mathcal{E}\pi^*$ on $\Lambda^{p,1}(Y)$. Thus only the terms in (3.5.c) and (3.5.d) above are relevant and these vanish since we assumed $J^*\omega = -\omega$. If $p = 0$ and $q > 0$, then we need only consider the action of $\pi_Z^{0,q}\mathcal{E}\pi^*$ on $\Lambda^{0,q+1}(Y)$. Thus only the terms in (3.5.a) are relevant. These vanish since we assumed $J^*\omega = \omega$. This shows assertion (4) implies assertion (1). It is immediate that assertion (1) implies assertion (2) and that assertion (2) implies assertion (3). The remainder of this section is devoted to the proof that assertion (3) implies assertion (4); Theorem 3.5.1 will play a crucial role in the proof.

We begin with a technical Lemma in the theory of PDE's.

3.5.5 Lemma.

(1) *Let R be any linear operator on $C^\infty\Lambda^{p,q}(Y)$ so that $(R\Phi,\Phi)_{L^2}=0$ for all Φ in $C^\infty\Lambda^{p,q}(Y)$. Then $R=0$.*

(2) *Let P be a 1^{th} order partial differential operator on $C^\infty\Lambda^{p,q}(Y)$. Suppose that P is non-negative, i.e. $(P\Phi,\Phi)_{L^2}\ge 0$ for all $\Phi\in C^\infty\Lambda^{p,q}(Y)$. Then P is a 0^{th} order operator, i.e. if $\Phi(y_0)=0$, then $P\Phi(y_0)=0$.*

Proof. Let ε be a real parameter. Since $(R(\Phi_1+\varepsilon\Phi_2),\Phi_1+\varepsilon\Phi_2)=0$ for all ε, we have

$$(R\Phi_1,\Phi_2)+(R\Phi_2,\Phi_1)=0.$$

We replace ε by $\sqrt{-1}\varepsilon$ to see that

$$(R\Phi_1,\Phi_2)-(R\Phi_2,\Phi_1)=0.$$

This shows that $(R\Phi_1,\Phi_2)=0$ for all Φ_i; we take $\Phi_2=R\Phi_1$ to see $R=0$.

We use the method of *stationary phase* to prove the second assertion. Decompose the operator P in the form

$$P=\Sigma_a P^a\partial_a^y+Q.$$

We must show that $P^a=0$ for all a. Let $\Psi\in C^\infty(Y)$ and let $\Phi_0\in C^\infty\Lambda^{p,q}(Y)$. We define

$$\Phi(t):=e^{\sqrt{-1}t\Psi}\Phi_0 \text{ and } R(\Psi):=\Sigma_a\partial_a^y(\Psi)P^a.$$

We then may compute:

$$P\Phi(t)=e^{\sqrt{-1}t\Psi}P\Phi_0+\Sigma_a\sqrt{-1}t\partial_a^y(\Psi)P^ae^{\sqrt{-1}t\Psi}\Phi_0.$$

This implies that

$$(P\Phi(t),\Phi(t))_{L^2}=(P\Phi_0,\Phi_0)_{L^2}+t\sqrt{-1}(R(\Psi)\Phi_0,\Phi_0)_{L^2}\ge 0.$$

This inequality holds for all t so $(R(\Psi)\Phi_0,\Phi_0)_{L^2}=0$ and thus $R(\Psi)=0$ for all Ψ. This implies $P^a=0$ for all a. □

3.5.6 Remark. This Lemma fails in the real setting. Let $M=S^1$ and let $P=\partial_\theta$. Then $(Pf,f)_{L^2}=0$ for any real smooth function on S^1.

We now recall some facts concerning *integration over the fiber* and *push forward*; integration over the fiber was first discussed in §1.17. We adopt the following notational conventions. Let

$$X(y):=\pi^{-1}(y)$$

be the fiber of π over a point $y \in Y$, let $m := \dim_{\mathbb{R}} Y$, let $n := \dim_{\mathbb{R}} Z$, and let $\nu_X := e^1 \wedge ... \wedge e^{n-m}$. Then

$$dvol_Z = \nu_X \wedge \pi^* dvol_Y$$

and the restriction of ν_X to $X(y)$ is the Riemannian volume element of the fiber. Let

$$V(y) := \int_{x \in X(y)} \nu_X(x, y)$$

be the volume of the fiber $X(y)$. We *average over the fibers* to define push forward

$$\pi_* : C^\infty \Lambda^p(Z) \to C^\infty \Lambda^p(Y)$$

as follows. Let $\phi \in C^\infty \Lambda^p(Z)$ and let $F_1, ..., F_p$ be tangent vectors at $y \in Y$. Let f_1,..., f_p be the corresponding horizontal lifts. We define

$$(\pi_* \phi)(F_1, ..., F_p) := V(y)^{-1} \int_{x \in X(y)} \phi(f_1, ..., f_p)(x, y) \nu_X(x, y).$$

Alternatively, let $\pi^{\mathcal{H}}$ be orthogonal projection of $\Lambda^p(Z)$ on $\pi^* \Lambda^p(Y)$. Decompose $\pi^{\mathcal{H}} \phi = \Sigma_{|A|=p} c_A(x, y) \pi^* dy^A$. Then

$$\pi_* \phi = \sum_{|A|=p} \{V(y)^{-1} \int_{x \in X(y)} c_A(x, y) \nu_X(x, y)\} dy^A.$$

It is immediate from the definition that $\pi_* \pi^*$ is the identity on $C^\infty \Lambda^p(Y)$.

We may decompose any real covector ξ into complex covectors of degrees $(1, 0)$ and $(0, 1)$ to express

$$\xi = \xi^{1,0} + \xi^{0,1}; \text{ note that: } \bar{\xi}^{1,0} = \xi^{0,1}.$$

3.5.7 Lemma. *Let $\pi : Z \to Y$ be a Hermitian submersion. Fix (p, q) and assume that for all λ, there exists $\epsilon(\lambda)$ so that*

$$\pi^* E(\lambda, \Delta_Y^{p,q}) \subseteq E(\lambda + \varepsilon(\lambda), \Delta_Z^{p,q}).$$

Then for any $\xi \in \mathcal{H}^*$ and for any $\Phi \in \Lambda^{p,q}(Y)$, we have

$$0 = \pi_Z^{p,q}(\operatorname{ext}_Z(\xi^{0,1})\mathcal{E} + \mathcal{E} \operatorname{ext}_Z(\xi^{0,1}))\pi^* \Phi, \text{ and}$$
$$0 = \pi_Z^{p,q}(\operatorname{ext}_Z(\xi^{0,1}) \operatorname{int}_Z(\theta) + \operatorname{int}_Z(\theta) \operatorname{ext}_Z(\xi^{0,1}))\pi^* \Phi.$$

Proof. We define a 1^{th} order differential operator on $C^\infty\Lambda^{p,q}(Y)$ by:

$$P := \pi_*\pi_Z^{p,q}\{\bar\partial_Z\Xi + \Xi\bar\partial_Z\}\pi^*.$$

Let $\Phi_\lambda \in E(\lambda, \Delta_Y^{p,q})$. By Lemma 2.7.2,

$$\varepsilon(\lambda)\pi^*\Phi_\lambda = \pi_Z^{p,q}\{\bar\partial_Z\Xi + \Xi\bar\partial_Z\}\pi^*\Phi_\lambda.$$

Note that

$$\pi_*\pi^* P\Phi_\lambda = \varepsilon(\lambda)\Phi_\lambda.$$

Thus $\{E(\lambda, \Delta_Y^{p,q})\}$ are eigenspaces of P. Since these eigenspaces are orthogonal and the eigenvalues are real, we see that P is self-adjoint. By Theorem 3.5.1, $\varepsilon(\lambda) \geq 0$ so P is a non-negative first order self-adjoint differential operator. Thus P has order 0. If $\Phi \in C^\infty\Lambda^{p,q}(Y)$, we may expand

$$\Phi = \Sigma_\lambda\Phi_\lambda \text{ for } \Phi_\lambda \in E(\lambda, \Delta_Y^{p,q}).$$

We use Theorem 2.3.1 to see that this series converges in the C^∞ topology. Then $P\Phi = \Sigma_\lambda\varepsilon(\lambda)\Phi_\lambda$ so

$$(\Delta_Z^{p,q}\pi^* - \pi^*\Delta_Y^{p,q})\Phi = \pi_Z^{p,q}(\bar\partial_Z\Xi + \Xi\bar\partial_Z)\pi^*\Phi = \Sigma_\lambda\pi^*\varepsilon(\lambda)\Phi_\lambda = \pi^*P(\Phi).$$

Since P is a 0^{th} order operator, $P(F\Phi) = FP(\Phi)$ for any $F \in C^\infty(Y)$ so the derivatives of F do not enter into this equation. This implies

$$\pi_Z^{p,q}(\operatorname{ext}_Z(\pi^*\bar\partial_Y F)\Xi + \Xi\operatorname{ext}_Z(\pi^*\bar\partial_Y F))\pi^*\Phi = 0. \tag{3.5.e}$$

Recall that $\Xi = \operatorname{int}_Z(\theta) + \mathcal{E}$ where $\operatorname{int}_Z(\theta)$ does not involve any vertical covectors and where $\mathcal{E}$ does involve vertical covectors. Thus equation (3.5.e) decouples into two separate equations involving $\operatorname{int}_Z(\theta)$ and $\mathcal{E}$ separately. If ξ is a horizontal covector at z_0, we can choose F so $\pi^*d_Y F(z_0) = \xi$. Then $\pi^*\bar\partial_Y F(z_0) = \xi^{0,1}$ and the Lemma follows. $\square$

Proof of Theorem 3.5.2. We must show assertion (3) implies assertion (4). Recall that

$$\begin{aligned}
&\operatorname{int}_Z(\xi^1)\operatorname{int}_Z(\xi^2) + \operatorname{int}_Z(\xi^2)\operatorname{int}_Z(\xi^1) = 0,\\
&\operatorname{ext}_Z(\xi^1)\operatorname{ext}_Z(\xi^2) + \operatorname{ext}_Z(\xi^2)\operatorname{ext}_Z(\xi^1) = 0, \text{ and}\\
&\operatorname{int}_Z(\xi^1)\operatorname{ext}_Z(\xi^2) + \operatorname{ext}_Z(\xi^2)\operatorname{int}_Z(\xi^1) = g(\xi^1, \xi^2).
\end{aligned}$$

Recall that $\tilde g_Z$ is the extension of g_Z to be complex bilinear. Suppose the assumptions of Theorem 3.5.2 (3) hold. To simplify notation, let $\eta = \xi^{0,1}$, let

$E^\eta := \mathrm{ext}_Z(\eta)$, let $I^\theta := \mathrm{int}_Z(\theta)$, let $E^i := \mathrm{ext}_Z(e^i)$, and let $I^a := \mathrm{int}_Z(f^a)$. By Lemma 3.5.7,

$$0 = \pi_Z^{p,q}(E^\eta I^\theta + I^\theta E^\eta) = \pi_Z^{p,q}\tilde{g}_Z(\theta,\eta);$$

This implies $\theta = 0$ since θ is horizontal. We also compute:

$$\begin{aligned}0 =&\omega_{abi}\pi_Z^{p,q}\{E^\eta E^i I^a I^b + E^i I^a I^b E^\eta\}\pi^* \\ =&\omega_{abi}\pi_Z^{p,q}\{-E^i E^\eta I^a I^b + E^i I^a I^b E^\eta\}\pi^* \\ =&\omega_{abi}\pi_Z^{p,q}\{E^i I^a E^\eta I^b + E^i I^a I^b E^\eta - \tilde{g}_Z(\eta, f^a)E^i I^b\}\pi^* \\ =&\omega_{abi}\pi_Z^{p,q}\{-\tilde{g}_Z(\eta, f^a)E^i I^b + \tilde{g}_Z(\eta, f^b)E^i I^a\}\pi^*\end{aligned} \tag{3.5.f}$$

on $\Lambda^{p,q}(Y)$. We take $\eta = \bar{\zeta}^\alpha$. The dual of $\bar{\zeta}^\alpha$ with respect to $\tilde{g}_Z$ is ζ_α. Thus by equation (3.5.f),

$$0 = \mathrm{ext}_Z(\pi^{0,1}\omega(\zeta_\alpha,\zeta_\beta))\,\mathrm{int}_Z(\zeta^\beta) + \mathrm{ext}_Z(\pi^{1,0}\omega(\zeta_\alpha,\bar{\zeta}_\beta))\,\mathrm{int}_Z(\bar{\zeta}^\beta)$$

on $\Lambda^{p,q}(\mathcal{H}^*)$ for all α and β. These equations decouple and we have:

$$0 = \mathrm{ext}_Z(\pi^{0,1}\omega(\zeta_\alpha,\zeta_\beta))\,\mathrm{int}_Z(\zeta^\beta) \tag{3.5.g}$$

$$0 = \mathrm{ext}_Z(\pi^{1,0}\omega(\zeta_\alpha,\bar{\zeta}_\beta))\,\mathrm{int}_Z(\bar{\zeta}^\beta). \tag{3.5.h}$$

If $p = 0$, then we can draw no conclusion from equation (3.5.h); if $q = 0$, then we can draw no conclusion from equation (3.5.g). If $p > 0$, then equation (3.5.h) shows $\pi^{1,0}\omega(\zeta_\alpha,\bar{\zeta}_\beta) = 0$ for all α and β; by Lemma 3.5.4, this implies $J^*\omega = -\omega$. If $q > 0$, then equation (3.5.g) shows $\pi^{0,1}\omega(\zeta_\alpha,\zeta_\beta) = 0$ and hence $\omega(\zeta_\alpha,\zeta_\beta) = 0$ for all α and β; by Lemma 3.5.4, this implies $J^*\omega = \omega$. If $p > 0$ and $q > 0$, we combine these two identities to see $\omega = 0$. This shows the conditions of (4) are satisfied. $\square$

In §4.5, we will give examples where $\omega \neq 0$ and $J^*\omega = \pm\omega$. We postpone the discussion until that point to avoid interrupting the flow of our discussion.

3.6 Other settings where eigenvalues are rigid

Generically, the pull back of an eigenform will not be an eigenform; it is quite a special situation when $0 \neq \Phi \in E(\lambda, \Delta_Y^p)$ and $\pi^*\Phi \in E(\mu, \Delta_Z^p)$. We say *eigenvalues change* if $\lambda \neq \mu$. Theorem 3.2.1 shows that eigenvalues can not change if $p = 0$. Furthermore, if $p > 0$ and if all the eigenforms are preserved, then eigenvalues can not change. There are other circumstances under which even a single eigenvalue can not change; the eigenvalues are rigid. Let $H^k(M;\mathbb{R})$ denote the de Rham cohomology groups of a manifold M.

3.6.1 Theorem. *Let $\pi : Z \rightarrow Y$ be a Riemannian submersion. We suppose given $0 \neq \Phi \in E(\lambda, \Delta_Y^p)$ so that $\pi^*\Phi \in E(\mu, \Delta_Z^p)$.*

(1) *If $\pi : Z \rightarrow Y$ is flat with structure group SL, then $\lambda = \mu$*
(2) *If $\pi : Z \rightarrow Y$ is a principal G bundle with $H^1(G; \mathbb{R}) = 0$, then $\lambda = \mu$.*
(3) *If $\pi : Z \rightarrow Y$ is a sphere bundle of fiber dimension $r \geq 2$, then $\lambda = \mu$.*
(4) *If the fibers of π are minimal and if $p = 1$, then $\lambda = \mu$.*

3.6.2 Flat Riemannian submersions with structure group SL. Let π be a Riemannian submersion from Z to Y with integrable horizontal distribution. Assume there exists a measure ν on the fibers so the Lie derivative $\mathcal{L}_{\mathfrak{H}F}\nu = 0$ where $\mathfrak{H}F$ is the horizontal lift of a vector field on Y. Expand $dvol_Z = e^\alpha \nu \pi^* dvol_Y$ to define a smooth function $\alpha \in C^\infty(Z)$. We apply Lemma 1.7.2 to choose a local decomposition of Z so that

$$ds_Z^2 = g_{ij}(x,y)dx^i \circ dx^j + h_{ab}(y)dy^a \circ dy^b.$$

Expand $\nu = e^{\beta(x,y)}\nu_x^e$ where ν_x^e is Euclidean measure. Since $\mathcal{L}_{\partial_a^y}\nu = 0$ and since $\mathcal{L}_{\partial_a^y}\nu_x^e = 0$, β is independent of y. We expand

$$dvol_Z = g_X \nu_x^e \pi^* dvol_Y = e^{\beta+\alpha}\nu_x^e \pi^* dvol_Y.$$

This shows $g_X = e^{\beta+\alpha}$ so $\theta = -d_Y \ln(g_X) = -d_Y(\beta + \alpha) = -d_Y(\alpha)$. Suppose that

$$0 \neq \Phi \in E(\lambda, \Delta_Y^p) \text{ and } \pi^*\Phi \in E(\lambda + \epsilon, \Delta_Z^p).$$

As in the proof of Theorem 3.4.2, we consider the *canonical variation*

$$ds_{Z(t)}^2 = e^{2t\alpha} ds_{\mathcal{V}}^2 + ds_{\mathcal{H}}^2$$

to see

$$\pi^*\Phi \in E(\lambda + \epsilon(1 + t \dim X), \Delta_{Z(t)}^p).$$

This shows $\epsilon = 0$ and completes the proof of Theorem 3.6.1 (1). $\square$

3.6.3 Principal G bundles. Let G be a compact Lie group with a bi-invariant metric. Assume $H^1(G; \mathbb{R}) = 0$. Let $\pi : P \rightarrow Y$ be a principal Riemannian G bundle. Let $\xi \in T_e(G)$ and let $g(t)$ be the 1-parameter subgroup of G with $\dot{g}(0) = \xi$. Multiplication by $g(t)$ defines a flow on P which is an isometry. Let ξ be the associated *Killing vector field*; ξ has constant length since the fibers have the bi-invariant metric. Consequently the integral curves of ξ are geodesics; this implies that the fibers of π are totally geodesic so $\theta = 0$ and only the curvature enters. Let $0 \neq \Phi \in E(\lambda, \Delta_Y^p)$ and let $\pi^*\Phi \in E(\mu, \Delta_P^p)$. By replacing Φ by $d_Y\Phi$ if necessary, we may assume without loss of generality that $d_Y\Phi = 0$. Expand

$$\Phi = \Sigma_{|A|=p}\Phi_A F^A \text{ and } \mathcal{E}\pi^*\Phi = \Sigma_{|A|=p-2}\operatorname{ext}_P(\beta_A)f^A.$$

The β_A are vertical co-vectors which are G invariant. Since $d_Y\Phi = 0$ and $\theta = 0$,

$$(\mu - \lambda)\pi^*\Phi = \Delta_P^p\pi^* - \pi^*\Delta_Y^p = d_P\mathcal{E}\pi^*\Phi$$

has no vertical dependence. Thus the *vertical derivative* of the restriction of β_A to the fibers vanishes. Since $H^1(G;\mathbb{R}) = 0$ and since the β_A are G invariant, this implies the restriction of the β_A to the fibers vanishes and hence $\beta_A = 0$ for all multi-indexes A. This implies $\lambda = \mu$ and completes the proof of Theorem 3.6.1 (2). $\square$

3.6.4 The geometry of sphere bundles. Let V be a real vector bundle of rank $r \geq 3$ over Y. We assume V is equipped with a fiber metric and let $S(V)$ be the unit sphere bundle. We use the Riemannian connection ${}^V\nabla$ on the bundle V to split $T(V) = \mathcal{V} \oplus \mathcal{H}$ into the vertical and horizontal distributions. We use this splitting to define a Riemannian metric g_V so $\pi : V \to Y$ is a Riemannian submersion. The restriction of π to $S(V)$ defines a projection $\pi_S : S(V) \to Y$ which is a Riemannian submersion. We apply Lemma 1.15.2. Let $0 \neq \Phi \in E(\lambda, \Delta_Y^p)$ and let $\pi^*\Phi \in E(\mu, \Delta_S^p)$. Since $\theta = 0$,

$$(\lambda - \mu)\pi^*\Phi = (d_S\mathcal{E} + \mathcal{E}d_S)\pi^*\Phi.$$

By replacing Φ by $d_Y\Phi$ if necessary, we may assume without loss of generality that $d_Y\Phi = 0$. Let $d_{\mathcal{V}} := \rho_{\mathcal{V}}d_S$ be the vertical exterior derivative. We apply $(1 - \rho_{\mathcal{H}})$ to see

$$\operatorname{ext}_S(d_{\mathcal{V}}(x^j dx^i))R_{abij}\operatorname{int}_S(f^a)\operatorname{int}_S(f^b)\pi^*\Phi = 0. \tag{3.6.a}$$

If we normalize the coordinates so $A(y_0) = 0$, then $d_{\mathcal{V}}(x^j dx^i) = dx^j \wedge dx^i$. Since the fiber dimension of V is at least 3, the dimension of the fiber spheres is at least 2 and equation (3.6.a) implies that

$$R_{abij}\operatorname{int}_S(f^a)\operatorname{int}_S(f^b)\pi^*\Phi = 0$$

for all i, j. This shows $\mathcal{E}\pi^*\Phi = 0$ and hence $\lambda = \mu$. This completes the proof of Theorem 3.6.1 (3). $\square$

3.6.5 One forms. Let $0 \neq \Phi \in E(\lambda, \Delta_Y^1)$, and let $\pi^*\Phi \in E(\mu, \Delta_Z^1)$. Suppose that $\theta = 0$. Then

$$(\lambda - \mu)\pi^*\Phi = (d_Z\mathcal{E} + \mathcal{E}d_Z)\pi^*\Phi = \mathcal{E}d_Z\pi^*\Phi.$$

Since $\mathcal{E}d_Z\pi^*\Phi$ has vertical dependence, it must vanish. This shows $\lambda = \mu$ and completes the proof of Theorem 3.6.1 (4). $\square$

3.6.6 Harmonic forms. The eigenvalue $\lambda = 0$ is distinguished; the *Hodge de Rham* theorem permits us to identify $E(0, \Delta_M^p)$ with the cohomology groups $H^p(M; \mathbb{R})$. The crucial point in the following Theorem is that there is no condition on the metric; we do *not* assume the submersion is Riemannian in the following theorem.

3.6.7 Theorem. *Let $\pi : Z \to Y$ be a submersion. Suppose that $0 \neq \Phi \in E(0, \Delta_Y^p)$ is non-trivial. Suppose that $\pi^*\Phi \in E(\mu, \Delta_Z^p)$. Then*

(1) *If $p = 1$, then $\mu = 0$.*
(2) *If the fiber of π is an even dimensional sphere, then $\mu = 0$.*

Proof. We use methods of algebraic topology to prove Theorem 3.6.7; we refer to Spanier [178] for details concerning the results which we will use. Let M be a connected manifold. The Abelianization of the fundamental group $\pi_1(M)$ is first integer homology group $H_1(M; \mathbb{Z})$. By the *universal coefficient theorem*,

$$H^1(M; \mathbb{R}) = Hom(H_1(M; \mathbb{Z}); \mathbb{R}).$$

Consequently

$$H^1(M; \mathbb{R}) = Hom(\pi_1(M); \mathbb{R}).$$

We can interpret this isomorphism geometrically as follows. Let $[\Phi] \in H^1(M; \mathbb{R})$ represent a de Rham cohomology class and let $[\alpha] \in \pi_1(M)$ represent an element of the fundamental group. Then $(\Phi, \alpha) \to \int_\alpha \Phi$ extends to a well defined map

$$\mathcal{I} : H^1(M; \mathbb{R}) \times \pi_1(M) \to \mathbb{R};$$

the map $[\Phi] \to \mathcal{I}([\Phi], \cdot)$ provides the isomorphism between the cohomology group $H^1(M; \mathbb{R})$ and the dual group $Hom(\pi_1(M); \mathbb{R})$.

Let $\pi : Z \to Y$ be a fiber bundle; we impose no restrictions on the metric. Suppose that $0 \neq \Phi \in E(0, \Delta_Y^1)$; by the Hodge decomposition theorem, $[\Phi]$ is a non-trivial element in $H^1(Y; \mathbb{R})$. If $\int_\alpha \Phi = 0$ for all closed paths α, we can define the potential function or primitive

$$F(y) := \int_\beta \Phi$$

where β is *any* path from y_0 to y. Then $dF = \Phi$ so $[\Phi]$ is trivial in $H^1(Y; \mathbb{R})$.

This contradiction shows that there exists a closed curve α in Y so that $\int_\alpha \Phi \neq 0$. For $x \in X$, let $\mathfrak{H}_x\alpha$ be the horizontal lift of α to Z with $\mathfrak{H}_x\alpha(0) = x$. Let the equivalence class $[x]$ of a point x of X denote the arc component in $\pi_0(X)$ to which x belongs. The map

$$x \to [\mathfrak{H}_x\alpha(1)]$$

extends to a well defined bijective map $\delta_{[\alpha]}$ of $\pi_0(X)$ which only depends on the class $[\alpha]$ in $\pi_1(Y)$. Since Z is compact, $\pi_0(X)$ is finite. Thus there exists an integer ν so $\delta^\nu_{[\alpha]}$ is the identity. Let $[\bar\alpha] = \nu[\alpha]$ in $\pi_1(Y)$. Since $\delta_{[\bar\alpha]} = \delta^\nu_{[\alpha]} = id$, x and $\mathfrak{H}_x\bar\alpha(1)$ are in the same arc component of X. Choose a curve β from $\mathfrak{H}_x\bar\alpha(1)$ to x. Let γ be the concatenation of $\mathfrak{H}_x\bar\alpha$ and some curve β from $\mathfrak{H}_x\bar\alpha(1)$ to x which lies in X. Then γ is a closed curve in Z and $[\pi\gamma] = [\alpha]$ in $\nu\alpha$ in $\pi_1(Y)$. Then $[\pi^*\Phi]$ is a non trivial cohomology class in $H^1(Z;\mathbb{R})$ since

$$\int_\gamma \pi^*\Phi = \int_{\pi\gamma}\Phi = \nu\int_\alpha \Phi \neq 0.$$

Suppose $\pi^*\Phi \in E(\mu, \Delta^1_Z)$. Since Φ is harmonic, $d_Y\Phi = 0$ so $d_Z\pi^*\Phi = 0$. Suppose $\mu \neq 0$. Since $\Delta_Z\pi^*\Phi = \mu\pi^*\Phi$ and since $d_Z\pi^*\Phi = 0$,

$$\pi^*\Phi = \mu^{-1}d_Z\delta_Z\pi^*\Phi$$

belongs to the image of d so $[\pi^*\Phi]$ is trivial in the de Rham cohomology group $H^1(Z;\mathbb{R})$. This contradiction completes the proof of assertion (1).

Let $\pi : S \to Y$ be a sphere bundle with fiber dimension $\nu - 1$ where ν is odd. Let $0 \neq \Phi \in E(0, \Delta^p_Y)$ and $\pi^*\Phi \in E(\mu, \Delta^p_Z)$. As in the proof given of assertion (1), we suppose $\mu \neq 0$ so the argument given above shows that

$$\pi^* : H^p(Y;\mathbb{R}) \to H^p(S;\mathbb{R})$$

is not injective. We establish the required contradiction by showing on the other hand that π^* is injective.

By passing to a suitable double cover $\mathbb{Z}_2 \to \tilde Y \to Y$ and by considering the associated sphere bundle $\pi : \tilde S \to \tilde Y$ if necessary, we may assume without loss of generality that the bundle S is orientable. The Gysin sequence gives rise to a long exact sequence

$$...H^{p-\nu}(Y;\mathbb{R}) \xrightarrow{\cup e} H^p(Y;\mathbb{R}) \xrightarrow{\pi^*} H^p(S;\mathbb{R}) \xrightarrow{\epsilon} H^{p-\nu+1}(Y;\mathbb{R})...$$

In this equation, $\cup e$ denotes cup product with the Euler form, π^* is the pull back, and ϵ is the connecting homomorphism. The crucial point here is that the *Euler form* e vanishes with $\mathbb{R}$ coefficients. Thus π^* is injective. This contradiction completes the proof of the second assertion. $\square$

3.7 The spin Laplacian

We can generalize Theorem 3.4.1 to the context of *real* spinors; as we shall see in the next chapter, this result fails if we consider *complex* spinors.

3.7.1 Theorem. *Let $\pi : P \to Y$ be a principal bundle with structure group G. Let $0 \neq \Phi \in E(\lambda, D_Y^{\mathcal{C}})$ be a real eigen spinor on Y and let α be a real spinor on G with $|\alpha|_{L^2(\mathcal{C}G)} = 1$. If $\pi_\alpha^* \Phi \in E(\mu, D_P^{\mathcal{C}})$, then*

$$\lambda + |A_G^{\mathcal{C}}\alpha|^2_{L^2(\mathcal{C}G)} + |\mathcal{E}\pi_\alpha^*\Phi|^2_{L^2(\mathcal{C}P)} \leq \mu.$$

Proof. We may suppose $|\Phi|_{L^2(\mathcal{C}Y)} = 1$ so that $|\pi_\alpha^*\Phi|_{L^2(\mathcal{C}P)} = 1$. By Lemma 2.7.3,

$$\begin{aligned}
\mu =&(D_P^{\mathcal{C}}\pi_\alpha^*\Phi, \pi_\alpha^*\Phi)_{L^2(\mathcal{C}P)} = (A_P^{\mathcal{C}}\pi_\alpha^*\Phi, A_P^{\mathcal{C}}\pi_\alpha^*\Phi)_{L^2(\mathcal{C}P)} \\
=&|A_Y^{\mathcal{C}}\Phi|^2_{L^2(\mathcal{C}Y)} + |A_G^{\mathcal{C}}\alpha|^2_{L^2(\mathcal{C}G)} + |\mathcal{E}\pi_\alpha^*\Phi|^2_{L^2(\mathcal{C}P)} \qquad (3.7.a)\\
&+ 2(\mathcal{E}(\Phi \otimes \alpha), A_Y^{\mathcal{C}}\Phi \otimes \alpha + \chi\Phi \otimes A_G^{\mathcal{C}}\alpha)_{L^2(\mathcal{C}P)}. \qquad (3.7.b)
\end{aligned}$$

The terms in (3.7.a) give rise to the desired estimate; we complete the proof by showing the terms in (3.7.b) give zero. We compute:

$$\begin{aligned}
(3.7.b) =&\frac{1}{2}\int_Y \omega_{abi}(c_Y(F^a)c_Y(F^b)\chi\Phi, A_Y^{\mathcal{C}}\Phi)dy \int_G (c_{\mathfrak{g}}(\mathfrak{g}^i)\alpha, \alpha)dg \\
&+ \frac{1}{2}\int_Y \omega_{abi}(c_Y(F^a)c_Y(F^b)\chi\Phi, \Phi)dy \int_G (c_{\mathfrak{g}}(\mathfrak{g}^i)\alpha, A_G^{\mathcal{C}}\alpha)dg.
\end{aligned}$$

Since $c_{\mathfrak{g}}(\mathfrak{g}^i)^* = -c_{\mathfrak{g}}(\mathfrak{g}^i)$,

$$(c_{\mathfrak{g}}(\mathfrak{g}^i)\alpha, \alpha) = (\alpha, -c_{\mathfrak{g}}(\mathfrak{g}^i)\alpha) = -(c_{\mathfrak{g}}(\mathfrak{g}^i)\alpha, \alpha) = 0.$$

Since $(c_Y(F^a)c_Y(F^b)\chi)^* = -c_Y(F^a)c_Y(F^b)\chi$, the second term also vanishes. □

We conclude this section by generalizing Theorem 3.4.2. We use complex spinors to define the pull back; Lemma 2.7.3 extends to this setting without change.

3.7.2 Theorem. *Let $\pi : P \to Y$ be a principal bundle. Let $\alpha \in \mathcal{C}^c\mathfrak{g}$ and let $\epsilon \in \mathbb{R}$. The following conditions are equivalent:*

(1) $D_P^{\mathcal{C}^c}\pi_\alpha^* = \pi_\alpha^*(D_Y^{\mathcal{C}^c} + \epsilon)$.

(2) $\forall\lambda \geq 0$, $\exists\mu(\lambda) \geq 0$ *so* $\pi^*E(\lambda, D_Y^{\mathcal{C}^c}) \subset E(\mu(\lambda), D_P^{\mathcal{C}^c})$.

(3) *The horizontal distribution defined by π is integrable and $\alpha \in E(\epsilon, D_G^{\mathcal{C}^c})$.*

Proof. The implication (3) $\Rightarrow$ (1) follows from Lemma 2.7.3 and the commutation relation $\chi D_Y^{\mathcal{C}} + D_Y^{\mathcal{C}}\chi = 0$. Assertion (1) $\Rightarrow$ (2) is immediate. Suppose that assertion (2) holds. As we are working with the complexification of real operators, we have

$$E(\lambda, D_Y^{\mathcal{C}^c}) = E(\lambda, D_Y^{\mathcal{C}}) \otimes_{\mathbb{R}} \mathbb{C} \text{ and,}$$
$$E(\epsilon, D_G^{\mathcal{C}^c}) = E(\epsilon, D_G^{\mathcal{C}}) \otimes_{\mathbb{R}} \mathbb{C}.$$

Thus we can restrict to real Φ. Decompose $\alpha = \alpha_1 + \sqrt{-1}\alpha_2$ where the α_i are real. By taking real and imaginary parts, we see (2) holds for the α_i separately; $\epsilon_i = \mu(\lambda) - \lambda$ will be the same for α_1 and α_2. Thus we can work in the real setting.

Let σ_i be orthogonal projection on the line $c_{\mathfrak{g}}(\mathfrak{g}^i)\alpha\mathbb{R}$ in $\mathcal{C}\mathfrak{g}$. Note that $\sigma_i(\alpha) = 0$ and $\sigma_i(c_{\mathfrak{g}}(\mathfrak{g}^j)\alpha) = 0$ for $i \neq j$. Let $\Phi \in E(\lambda, D_Y^{\mathcal{C}})$. Then

$$(1 \otimes \sigma_i)D_P^{\mathcal{C}}\pi_\alpha^*\Phi = \mu(\lambda)\Phi \otimes \sigma_i(\alpha) = 0. \tag{3.7.c}$$

Since the eigenspaces $E(\lambda, D_Y^{\mathcal{C}})$ are dense in $C^\infty(\mathcal{C}Y)$, Equation (3.7.c) holds for all $\Phi \in C^\infty(\mathcal{C}Y)$. Fix $y_0 \in Y$ and $\xi \in T_{y_0}^*Y$. Choose $f \in C^\infty(Y)$ so that $f(y_0) = 0$ and so that $df(y_0) = \xi$. Fix the index i. Then

$$\begin{aligned} 0 =&(1 \otimes \sigma_i)\{D_Y^{\mathcal{C}} \otimes 1 + 1 \otimes D_G^{\mathcal{C}} + \mathcal{E}^2 + (\chi \otimes A_G^{\mathcal{C}})\mathcal{E} \\ &+ \mathcal{E}(\chi \otimes A_G^{\mathcal{C}})\}(f\Phi \otimes \alpha)(y_0). \end{aligned} \tag{3.7.d}$$

We use Equation (3.7.c), Equation (3.7.d) and Lemma 2.7.3 to see that

$$\begin{aligned} 0 =&(1 \otimes \sigma_i)\{(A_Y^{\mathcal{C}} \otimes 1)\mathcal{E}_{spin} + \mathcal{E}_{spin}(A_Y^{\mathcal{C}} \otimes 1)\}(f\Phi \otimes \alpha)(y_0) \\ =&(1 \otimes \sigma_i)\{(c_Y(\xi) \otimes 1)\mathcal{E}_{spin} + \mathcal{E}_{spin}(c_Y(\xi) \otimes 1)\}(\Phi \otimes \alpha)(y_0) \\ =&\omega_{abi}\{c_Y(\xi)\chi c_Y(F^a)c_Y(F^b) + \frac{1}{4}\chi c_Y(F^a)c_Y(F^b)c_Y(\xi)\}\Phi(y_0). \end{aligned} \tag{3.7.e}$$

Fix the index c and let $\xi = F^c$. We compute

$$\begin{aligned} &\chi c_Y(F^a)c_Y(F^b)c_Y(F^c) = -\chi c_Y(F^a)c_Y(F^c)c_Y(F^b) - 2\delta_{bc}\chi c_Y(F^a) \\ =&\chi c_Y(F^c)c_Y(F^a)c_Y(F^b) - 2\delta_{bc}\chi c_Y(F^a) + 2\delta_{ac}\chi c_Y(F^b) \\ =&-c_Y(F^c)\chi c_Y(F^a)c_Y(F^b) - 2\delta_{bc}\chi c_Y(F^a) + 2\delta_{ac}\chi c_Y(F^b). \end{aligned}$$

Consequently Equation (3.7.e) shows that $0 = \omega_{cbi}\chi c_Y(F^b)\Phi(y_0)$ for any index c, for any index i, and for any $\Phi(y_0)$; it now follows that ω vanishes identically. Consequently

$$(\mu(\lambda) - \lambda)\Phi \otimes \alpha = \Phi \otimes D_G^{\mathcal{C}}\alpha.$$

Thus $\epsilon = \mu(\lambda) - \lambda$ is independent of λ and $\alpha \in E(\epsilon, D_G^{\mathcal{C}})$. This shows that assertion (3) holds. $\square$

3.8 Manifolds with boundary

Theorems 3.2.1 and 3.2.2 extend to this setting without much additional work. Throughout this section, we work in the category of compact manifolds with smooth boundaries.

Let M be a smooth manifold with smooth boundary. Let $\Phi \in C^\infty(M)$. The Dirichlet boundary operator sends Φ to $\Phi|_{\partial M}$; the Neumann boundary operator sends Φ to $\partial_N \Phi|_{\partial M}$. Let $\mathcal{B}$ denote Dirichlet or Neumann boundary conditions. We let

$$E(\lambda, \Delta_{M,\mathcal{B}}) = \{\Phi \in C^\infty(M) : \mathcal{B}\Phi = 0 \text{ and } \Delta_M^0 \Phi = \lambda\Phi\}$$

be the eigenspaces of the Laplacian with Dirichlet or Neumann boundary conditions. We have a decomposition

$$L^2(M) = \oplus_\lambda E(\lambda, \Delta_{M,\mathcal{B}}).$$

For example as noted in §2.8.5, if $M = [0, \pi]$ then the discrete spectral resolution of the Dirichlet and Neumann Laplacians take the form

$$\{\cos(nx), n^2\}_{n\geq 0} \text{ and } \{\sin(nx), n^2\}_{n>0}.$$

3.8.1 Theorem. *Let $\pi : Z \to Y$ be a Riemannian submersion. Let $\mathcal{B}$ denote Dirichlet or Neumann boundary conditions.*

(1) *If $\Phi \in E(\lambda, \Delta^0_{Y,\mathcal{B}})$ and if $\pi^*\Phi \in E(\mu, \Delta^0_{Z,\mathcal{B}})$, then $\lambda = \mu$.*

(2) *The following conditions are equivalent:*
  2a) *For every λ, we have $\pi^* E(\lambda, \Delta^0_{Y,\mathcal{B}}) \subset E(\lambda, \Delta^0_{Z,\mathcal{B}})$.*
  2b) *We have $\theta = 0$.*

Proof. The proof is essentially the same as that given in §3.2 with appropriate modifications to take into account the presence of the boundary. Suppose that $0 \neq \Phi \in E(\lambda, \Delta^0_{Y,\mathcal{B}})$ and that $\pi^*\Phi \in E(\lambda + \epsilon, \Delta^0_{Z,\mathcal{B}})$. We use Lemma 1.5.1 to see that

$$\epsilon\pi^*\Phi = \Delta^0_Z \pi^*\Phi - \pi^*\Delta^0_Y = \text{int}_Z(\theta) d_Z \pi^*\Phi. \tag{3.8.a}$$

By replacing Φ by $-\Phi$ if necessary, we can assume the maximal value of Φ is positive; let this maximal value be attained at $y_0 \in Y$. If y_0 is in the interior of Y, then $d_Y\Phi(y_0) = 0$. Choose z_0 so $\pi(z_0) = y_0$. Then

$$d_Z\pi^*\Phi(z_0) = \pi^* d_Y \Phi(y_0) = 0$$

so equation (3.8.a) implies $\epsilon\pi^*\Phi(z_0) = 0$ and hence $\epsilon = 0$. If $\mathcal{B} = \mathcal{B}_D$, then Φ can not attain its maximum on the boundary ∂Y and the first assertion follows. Let

$\mathcal{B} = \mathcal{B}_N$. Since $y_0 \in \partial Y$, $d_{\partial Y}\Phi(y_0) = 0$. Since $N_Y(\Phi)(y_0) = 0$, $d_Y\Phi(y_0) = 0$ and $\epsilon = 0$; this completes the proof of the first assertion.

The boundary conditions are always preserved on functions by pull back. We use Lemma 1.5.1 to see assertion (2b) implies assertion (2a). Conversely assume that assertion (2a) holds. Let $0 \neq \Phi \in E(\lambda, \Delta^0_{Y,\mathcal{B}})$ and let $\phi := \pi^*\Phi$. We use Lemma 1.5.b to see that

$$(\mu - \lambda)\phi = \operatorname{int}_Z(\theta) d_Z \pi^* \Phi.$$

By assertion (1), $\mu = \lambda$ and thus $\operatorname{int}_Z(\theta)\pi^* d_Y \Phi = 0$. Let $\Psi \in C_0^\infty(Y)$ be a smooth function on Y with compact support. By Theorem 2.3.1, we can uniformly approximate Ψ in the C^∞ topology by finite sums of eigenfunctions. Thus we have

$$\operatorname{int}_Z(\theta)\pi^* d_Y \Psi = 0$$

on $C_0^\infty(Y)$. Since θ is a horizontal co-vector, this implies $\theta = 0$ on the interior of M. Continuity then yields $\theta = 0$ on the boundary as well. $\square$

Theorems 3.4.1 and 3.4.2 extend to the category of manifolds with boundary if we impose Dirichlet, Absolute, or Relative boundary conditions on the space of smooth p forms; we refer to §2.9 for the definition of these boundary conditions. We have already noted in §2.9.4 that Theorem 3.4.1 fails with Neumann boundary conditions; we do not know if Theorem 3.4.2 fails with Neumann boundary conditions.

3.8.2 Theorem. *Let $\pi : Z \to Y$ be a Riemannian submersion. Let $\mathcal{B}$ denote Dirichlet, Absolute, or Relative boundary conditions. Let $p > 0$.*

(1) *If $\Phi \in E(\lambda, \Delta^p_{Y,\mathcal{B}})$ is non-trivial and if $\pi^*\Phi \in E(\mu, \Delta^p_{Z,\mathcal{B}})$, then $\lambda \leq \mu$.*

(2) *The following conditions are equivalent:*
 2a) *For all λ, we have $\pi^* E(\lambda, \Delta^p_{Y,\mathcal{B}}) \subset E(\lambda, \Delta^p_{Z,\mathcal{B}})$.*
 2b) *For all λ, there exists $\mu = \mu(\lambda)$ so $\pi^* E(\lambda, \Delta^p_{Y,\mathcal{B}}) \subset E(\mu, \Delta^p_{Z,\mathcal{B}})$*
 2c) *We have $\theta = 0$ and $\omega = 0$.*

To prove the first assertion, we use the same argument employing fiber products as used in the proof of Theorem 3.4.1. If $\mathcal{B} = \mathcal{B}_D$ or $\mathcal{B} = \mathcal{B}_A$, the boundary conditions are preserved automatically and there is nothing new. If $\mathcal{B} = \mathcal{B}_R$, then by assumption $\mathcal{B}_R\Phi = 0$ and $\mathcal{B}_R\pi^*\Phi = 0$. Thus

$$i_Z^*(\delta_Z \pi^* - \pi^* \delta_Y)\Phi = 0$$

so we have $i_Z^* \operatorname{int}_Z(\theta)\pi^*\Phi = 0$ and $i_Z^*\mathcal{E}\pi^*\Phi = 0$. Lemma 1.8.3 then shows inductively that

$$i_{Z(n)}^* \operatorname{int}_{Z(n)}(\theta)\pi_n^*\Phi = 0 \text{ and } i_{Z(n)}^*\mathcal{E}\pi_n^*\Phi = 0.$$

It follows that $\mathcal{B}_R\pi_n^*\Phi = 0$ so $\pi_n^*\Phi$ satisfies the given boundary conditions and the argument goes through. This proves assertion (1).

It is immediate that assertion (2a) implies assertion (2b). Suppose assertion (2c) holds. Dirichlet and Absolute boundary conditions are preserved automatically; Relative boundary conditions are preserved since $\theta = 0$ and $\omega = 0$. Thus assertion (2a) holds. Suppose that assertion (2b) holds. The proof that $\mathcal{E} = 0$ is exactly the same as before and uses the fact that the eigenspaces of the Laplacian are dense $C_0^\infty\Lambda^p(M)$. In the proof that $\theta = 0$, we used a variation in which

$$\theta(t) = (1 + t\dim(X))\theta.$$

Dirichlet and absolute boundary conditions are preserved automatically; relative boundary conditions are also preserved owing to the particular form of this variation. We suppose $\mathcal{B}_{\mathcal{R}}\Phi = 0$. Since $i_Z^*\pi^* = \pi^* i_Y^*$, we have $i_Z^*\pi^*\Phi = 0$. Since $\mathcal{E} = 0$, see that:

$$\mathcal{B}_{\mathcal{R}}\pi^*\Phi = 0 \iff i_Z^*\delta_Z\pi^*\Phi = 0 \iff i_Z^*\operatorname{int}_Z(\theta)\pi^*\Phi = 0.$$

Since $\theta(t) = (1 + t\dim(X))\theta$, this condition is preserved. The remainder of the argument is unchanged. Note that this line of argument fails for Neumann boundary conditions since the Laplacian can be negative as was shown in Lemma 2.9.4.

3.9 The Laplacian with coefficients in a flat bundle

3.9.1 Definition. Let $\tilde M$ be the universal cover of M. The fundamental group $\pi_1(M)$ acts on $\tilde M$ by deck transformations $\tilde x \to \tilde x\cdot g$ for $\tilde x \in \tilde M$ and $g \in \pi_1(M)$. Let ρ be a unitary representation of the fundamental group $\pi_1(M)$ of a Riemannian manifold M; for some k we have $\rho : \pi_1(M) \to U(k)$ or $\rho : \pi_1(M) \to O(k)$. The associated flat vector bundle V_ρ is given by:

$$V_\rho := \tilde M \times_\rho \mathbb{R}^k \text{ or } V_\rho := \tilde M \times_\rho \mathbb{C}^k$$

where we identify

$$(\tilde x, \vec v) \sim (\tilde x \cdot g, \rho(g)\vec v) \text{ for all } g \in \pi_1(M).$$

Let 1^k be the trivial real or complex vector bundle of fiber dimension k over $\tilde M$. Let $\tilde f \in C^\infty(1^k)$; $\tilde f$ is a smooth map from $\tilde M$ to $\mathbb{C}^k$. We say $\tilde f$ is *equivariant* if

$$\tilde f(\tilde x, \vec v) = \tilde f(\tilde x \cdot g, \rho(g)\vec v) \text{ for all } g \in \pi_1(M).$$

Clearly such a function $\tilde{f}$ preserves the glueings given above and descends to a section $f \in C^\infty V_\rho$. Conversely, every smooth section f to $C^\infty V_\rho$ defines an equivariant function. Define the flat connection on 1^k by:

$$\tilde{\nabla}\tilde{f} = d\tilde{f}.$$

It is then immediate that $\tilde{\nabla}$ descends to a connection ${}^\rho\nabla$ on V_ρ with zero curvature tensor. Since ρ is a unitary representation, the natural innerproduct on $\mathbb{C}^k$ descends to a fiber metric on V_ρ and we have ${}^\rho\nabla$ is a Riemannian connection as discussed in §1.9.3.

There is an alternate viewpoint which is sometimes useful. Instead of starting with a representation ρ of the fundamental group, we could start with a vector bundle V and a Riemannian connection ∇ which has zero curvature. Let V_0 be the fiber of V over the basepoint x_0. Parallel translation around a closed path γ defines the *holonomy representation*

$$\rho_\nabla(\gamma) : V_0 \to V_0.$$

The map $\gamma \to \rho_\nabla(\gamma)$ extends to a unitary representation ρ of the fundamental group $\pi_1(M)$. The associated vector bundle V_ρ is naturally isomorphic to V and the associated connection ∇_ρ is equal to ∇ under this isomorphism.

3.9.2 Example. Let $M = \mathbb{RP}^m$ be real projective space. Suppose $m \geq 2$ for the moment so $\pi_1(M) = \mathbb{Z}_2 = \{\pm 1\}$. Then $\tilde{M} = S^m$ and the deck group action is the usual action of $\{\pm 1\}$ on S^m given by multiplication; -1 acts as the antipodal map and $+1$ acts as the identity. Let $\rho(\pm 1) = \pm 1$ define the canonical identification of $\mathbb{Z}_2$ with the orthogonal group $O(1)$. The associated vector bundle is given by

$$L := S^m \times \mathbb{R}/\sim \;\text{ where } (\xi, x) \sim (-\xi, -x)$$

for $\xi \in S^m$ and $x \in \mathbb{R}$. The bundle L is a real line bundle which is isomorphic to the classifying line bundle over $\mathbb{RP}^m$. The line bundle L plays the same role in defining the first Stiefel-Whitney class for real line bundles that the classifying line bundle over $\mathbb{CP}^m$ plays for the first Chern class of a complex line bundle as discussed in §1.11. We refer to §A.2.1 for further details. We also note that if $m = 1$, the corresponding line bundle is the *Möbius line bundle.*

3.9.3 Example. Let the cyclic group of ℓ^{th} roots of unity $\mathbb{Z}_\ell$ act on the unit sphere S^{2k-1} in $\mathbb{C}^k$ by complex multiplication. Let

$$L(k;\ell) := S^{2k-1}/\mathbb{Z}_\ell$$

be the resulting quotient manifold; $L(k;\ell)$ is called a *lens space.* Let $\rho_s(\lambda) = \lambda^s$; the $\{\rho_s\}$ for $0 \leq s < \ell$ parametrize the irreducible representations of $\mathbb{Z}_\ell$. Let L_s be the complex line bundle defined by the representation ρ_s. Then

$$L_s = S^{2k-1} \times \mathbb{C}/\sim \;\text{ where } (\xi, z) \sim (\lambda\xi, \lambda^s z)$$

for $\lambda \in \mathbb{Z}_\ell$, $\xi \in S^{2k-1}$, and $z \in \mathbb{C}$.

Let ρ be a unitary representation of the fundamental group $\pi_1(M)$. We use the Levi-Civita connection on M and the connection ${}^\rho\nabla$ to covariantly differentiate tensors of all types. Motivated by Lemma 1.4.4, we define

$$d_\rho := (\text{ext} \otimes id) \circ \nabla : C^\infty(\Lambda^p M \otimes V_\rho) \to C^\infty(\Lambda^{p+1} M \otimes V_\rho)$$
$$\delta_\rho := -(\text{int} \otimes id) \circ \nabla : C^\infty(\Lambda^p M \otimes V_\rho) \to C^\infty(\Lambda^{p-1} M \otimes V_\rho).$$

Since the curvature of ${}^\rho\nabla$ vanishes, we have $d_\rho^2 = 0$ and $\delta_\rho^2 = 0$. Furthermore $d_\rho^* = \delta_\rho$ and we define the associated Laplacian

$$\Delta^p_{M,\rho} := \delta_\rho d_\rho + d_\rho \delta_\rho \text{ on } C^\infty(\Lambda^p M \otimes V_\rho).$$

The transition functions of V_ρ are locally constant; on the universal cover they are given by $\rho(g)$ for $g \in \pi_1(M)$. Thus the Laplacians and exterior differentiation and interior differentiation patch together naturally. Let $\sigma : \tilde{M} \to M$ be the natural covering projection. We then have the intertwining relation:

$$\sigma^* \Delta^p_{M,\rho} = \Delta^p_{\tilde{M}} \sigma^*. \tag{3.9.5}$$

3.9.6 Remark. The operators d_ρ and δ_ρ can be defined even if ρ is not unitary. However, δ_ρ is not the adjoint of d_ρ if ρ is not unitary. If ρ is the trivial linear representation, we recover the ordinary Laplacian.

Theorem 1.4.6 extends to this setting. We define

$$H^p(M;\rho) := \frac{\ker(d_\rho : C^\infty \Lambda^p M \to C^\infty \Lambda^{p+1} M)}{\text{image}(d_\rho : C^\infty \Lambda^{p-1} M \to C^\infty \Lambda^p M)}.$$

3.9.4 Theorem (de Rham-Hodge). *Let M be a compact Riemannian manifold without boundary. Let ρ be a unitary representation of the fundamental group of M.*

(1) *There is a natural isomorphism between the de Rham cohomology groups $H^p(M;\rho)$ defined above and the topological cohomology groups of M which are defined using sheaf cohomology with coefficients in the local system corresponding to ρ.*

(2) *The map $\phi \to [\phi] \in H^p(M;\rho)$ defines an isomorphism from* $\ker(\Delta^p_{M,\rho})$ *to $H^p(M;\rho)$.*

The case in which $\pi_1(M)$ is finite deserves particular attention. We suppose for the sake of simplicity that $\pi_1(M)$ is cyclic of order ℓ. We identify $\pi_1(M)$ with the group of ℓ^{th} roots of unity and define $\rho_s(\lambda) = \lambda^s$. The ρ_s for $0 \leq s < \ell$ are representatives for the irreducible representations of $\mathbb{Z}_\ell$. Let $\sigma : \tilde{M} \to M$ be the natural projection. We use equation (3.9.5) to see

$$\sigma^* : E(\lambda, \Delta^p_{M,\rho_s}) \to E(\lambda, \Delta_{\tilde{M}}).$$

This gives rise to the decomposition as a $\pi_1(M)$ representation space:

$$E(\lambda, \Delta^p_{\tilde{M}}) = \oplus_s E(\lambda, \Delta^p_{M,\rho_s}).$$

In other words, the spectral resolution of the operators Δ^p_{M,ρ_s} gives the equivariant spectral resolution of the operator $\Delta^p_{\tilde{M}}$ as a $\pi_1(M)$ module. We refer to [63]

3.9.7 Example. Let $M = \mathbb{RP}^m$ and let $\tilde{M} = S^m$. We can decompose any function f on S^m in the form $f = f_e + f_o$ where f_e is an even function and f_o is an odd function; f_e descends to a section of the trivial line bundle over $\mathbb{RP}^m$ and f_o descends to a section of the line bundle L over $\mathbb{RP}^m$. In §2.4, we discussed the spectral resolution of S^m if $p = 0$ in terms of spherical harmonics;

$$L^2(S^m) = \oplus_j \tilde{H}(m, j).$$

If j is even, the functions in $\tilde{H}(M, j)$ are even functions and descend to eigenfunctions on $\mathbb{RP}^m$; if j is odd, the functions in $\tilde{H}(M, j)$ are odd functions and descend to eigenfunctions for $\Delta_{\mathbb{RP}^m,\rho}$. Thus

$$L^2(\mathbb{RP}^m) = \oplus_k \tilde{H}(m, 2k) \text{ and } L^2(\mathbb{RP}^m; L) = \oplus_k \tilde{H}(m, 2k+1)$$

gives the spectral resolution of $\Delta^0_{\mathbb{RP}^m}$ and $\Delta^0_{\mathbb{RP}^m,\rho}$ where ρ is the canonical identification of $\pi_1(\mathbb{RP}^m) = \mathbb{Z}_2$ with the orthogonal group $O(2)$.

3.9.8 Example. Let $M = L(k;\ell) = S^{2k-1}/\mathbb{Z}_\ell$ be the lens space discussed in example 3.9.3. Since pull back defines an injective map

$$E(\lambda, \Delta^0_{M,\rho_s}) \to E(\lambda, \Delta^0_{S^{2k-1}}),$$

the eigenvalues of Δ^0_{M,ρ_s} are contained in the set $\{j(j+2k-2)\}$. Consider the generating function

$$F_{M,s}(t) := \sum_j \dim E(j(j+2k-2), \Delta^0_{M,\rho_s}) t^j$$

which encodes the multiplicities with which the eigenvalues occur. One can show that:

$$F_{M,s}(t) = \ell^{-1} \sum_{\lambda \in \mathbb{Z}_\ell} \lambda^{-s}(1-t^2)(1-t\lambda)^{-2k-2}.$$

We refer to [64, Theorem 4.2.13] for the proof in the case that $s = 0$; the general case follows similarly. The multiplicities can then be determined by expanding this analytic function in a Taylor series.. We refer to [63] for a further discussion of the equivariant spectrum. We also refer to Ikeda [101] who first derived this result.

If ρ_Y is a unitary representation of $\pi_1(Y)$ and if $\pi : Z \to Y$ is a Riemannian submersion, then $\pi_* \circ \rho_Y$ is a unitary representation of $\pi_1(Z)$ we shall denote by ρ_Z. Theorems 3.2.1, 3.2.2, 3.4.1, 3.4.2 extend without difficulty to this setting; we omit the proofs in the interests of brevity.

3.9.9 Theorem. *Let $\pi : Z \to Y$ be a Riemannian submersion. Let ρ_Y be a unitary representation of $\pi_1(Y)$. Let $\rho_Z := \rho_Y \circ \pi_*$ be the associated representation of $\pi_1(Z)$.*

(1) *Let $\Phi \in E(\lambda, \Delta^p_{Y,\rho_Y})$ be non-trivial. Assume that $\pi^*\Phi \in E(\mu, \Delta^p_{Z,\rho_Z})$. We then have $\lambda \le \mu$. Furthermore, if $p = 0$, then $\lambda = \mu$.*

(2) *Fix p with $0 \le p \le \dim(Y)$. The following conditions are equivalent:*

  2a) *We have $\Delta^p_{Z,\rho_Z}\pi^* = \pi^*\Delta^p_{Y,\rho_Y}$.*

  2b) *For all λ, we have $\pi^*E(\lambda, \Delta^p_{Y,\rho_Y}) \subset E(\lambda, \Delta^p_{Z,\rho_Z})$.*

  3c) *For all λ, there exists $\mu = \mu(\lambda)$ so $\pi^*E(\lambda, \Delta^p_{Y,\rho_Y}) \subset E(\mu, \Delta^p_{Z,\rho_Z})$.*

  3d) *We have $\theta = 0$. If $p > 0$, then we also have $\omega = 0$.*

3.9.10 Remark. There are similar results in the complex setting and for manifolds with boundary.

3.10 Heat Content Asymptotics

We refer to van den Berg and Gilkey [20, 21] for further information concerning the heat content asymptotics which are described in this section; see also [18, 19, 127, 128] for related work. Let M be a smooth compact Riemannian manifold of dimension m with smooth boundary ∂M. We take initial temperature Φ and pump heat into the manifold across the boundary at a constant rate determined by the flux function Ψ. Let ∂_N be the inward unit normal derivative and let Δ be the scalar Laplacian. The resulting temperature function $H^N_{\Phi,\Psi}(y;t)$ for the *Neumann heat pump* is the solution to the equations:

(1) $(\partial_t + \Delta)H^N_{\Phi,\Psi} = 0$ (3.10.a)

(2) $\lim_{t\downarrow 0} H^N_{\Phi,\Psi}(x;t) = \Phi$ (3.10.b)

(3) $\partial_N H^N_{\Phi,\Psi}(\bar{x};t) = \Psi(\bar{x})$ for $\bar{x} \in \partial M$ (3.10.c)

Equation (3.10.a) is the evolution equation, equation (3.10.b) is the initial condition, and equation (3.10.c) is the boundary condition.

Let F be an auxiliary function which is used to measure the temperature profile. For example, F could represent the *specific heat* at a point $x \in M$. We consider the *weighted heat content* energy function

$$\beta^N(\Phi, \Psi, F)(t) := \int_M H^N_{\Phi,\Psi}(t; x)F(x).$$

One can show that there is an asymptotic expansion as $t \downarrow 0$ of the form

$$\beta^N(\Phi, \Psi, F)(t) \sim \sum_{n=0}^{\infty} \beta^N_n(\Phi, \Psi, F)t^{n/2}.$$

This problem has considerable physical relevance. Carslaw and Jaeger [35, p75] noted that:

> "*The boundary condition of constant flux is of considerable practical importance. It appears if heat is generated by a flat heating element carrying electric current, if heat is generated by friction, and as an approximation in the early stages of heating a furnace or room. It also has important applications to problems on diffusion. The cooling of the Earth's surface after sunset on a clear windless night is very nearly that due to removal of heat at a constant rate per unit area per unit time, thus [this] gives the way in which the surface temperature falls after sunset.*"

We also refer to Brunt [31] for other physical applications.

There is an analogous problem for the *Dirichlet heat pump*. Let $H^D_{\Phi,\Psi}(x;t)$ be the solution to the equations:

(1) $(\partial_t + \Delta)H^D_{\Phi,\Psi} = 0$ (3.10.d)

(2) $\lim_{t\downarrow 0} H^D_{\Phi,\Psi}(x;t) = \Phi$ (3.10.e)

(3) $H^D_{\Phi,\Psi}(\bar{x};t) = \Psi(\bar{x})$ for $\bar{x} \in \partial M$ (3.10.f)

We define the weighted heat content energy function:

$$\beta^D(\Psi, \Phi, F)(t) := \int_M H^D_{\Phi,\Psi}(x;t)F(x);$$

there is an asymptotic expansion as $t \downarrow 0$ of the form:

$$\beta^D(\Psi,\Phi,F)(t) \sim \sum_{n\geq 0} \beta_n^D(\Psi,\Phi,F)t^{n/2}.$$

The heat content asymptotics β_n^N and β_n^D are *locally computable.* Let L be the second fundamental form on the boundary, let R be the Riemann curvature tensor of M and let $\rho_{ij} := R_{ikkj}$ be the Ricci tensor. Let ';' and ':' denote covariant differentiation with respect to the Levi-Civita connections of M and of ∂M respectively. We choose a local orthonormal frame $\{e_1, ..., e_m\}$ for the tangent bundle of M restricted to the boundary so that e_m is the inward unit normal. Thus $\partial_N F = F_{;m}$ for example. Let indices a, b, c etc. range from 1 through $m-1$. We adopt the Einstein convention and sum over repeated indices. We refer to [20, 21] for the proof of the following result:

3.10.1 Theorem. *We have:*

0) $\beta_0^N(\Phi,\Psi,F) = \int_M \Phi F$.

1) $\beta_1^N(\Phi,\Psi,F) = 0$.

2) $\beta_2^N(\Phi,\Psi,F) = -\int_M F\Delta\Phi + \int_{\partial M}(\Phi_{;m} - \Psi)F$.

3) $\beta_3^N(\Phi,\Psi,F) = \frac{4}{3\sqrt{\pi}}\int_{\partial M}\{(\Phi_{;m}-\Psi)F_{;m}\}$.

4) $\beta_4^N(\Phi,\Psi,F) = \frac{1}{2}\int_M(\Delta\Phi)(\Delta F) - \frac{1}{4}\int_{\partial M}\{2(\Phi_{;m}-\Psi)\Delta F + 2F_{;m}\Delta\Phi$
$-L_{aa}(\Phi_{;m}-\Psi)F_{;m}\}$.

3.10.2 Theorem. *We have:*

0) $\beta_0^D(\Phi,\Psi,F) = \int_M \Phi F$.

1) $\beta_1^D(\Phi,\Psi,F) = -\frac{2}{\sqrt{\pi}}\int_{\partial M}(\Phi-\Psi)F$.

2) $\beta_2^D(\Phi,\Psi,F) = -\int_M F\Delta\Phi + \frac{1}{2}\int_{\partial M}\{(FL_{aa} - 2F_{;m})(\Phi-\Psi)\}$.

3) $\beta_3^D(\Phi,\Psi,F) = -\frac{1}{6\sqrt{\pi}}\int_{\partial M}\big\{(\Phi-\Psi)F(L_{aa}L_{bb} - 2L_{ab}L_{ab} - 2\rho_{mm})$
$-8F\Delta\Phi + 4F(\Phi-\Psi)_{:aa} + 8(F_{;mm} - L_{aa}F_{;m})(\Phi-\Psi)\big\}$.

We remark that the formulas for β_5^N, β_6^N, β_4^D, and β_5^D are known; we omit these formulas in the interests of brevity and refer to [20, 21] for further details.

We can now use the results of §3.8 to study the heat content asymptotics for a Riemannian submersion. Recall by Lemma 1.17.2 that if $\theta = 0$, then the volume V of the fibers is constant. The main result of this section is the following:

3.10.3 Theorem. *Let $\pi : Z \to Y$ be a Riemannian submersion of compact manifolds with smooth boundary. Let $f := \pi^* F$, $\psi := \pi^*\Psi$, and let $\phi := \pi^*\Phi$. Assume that the mean curvature vector $\theta = 0$. Let $\mathcal{B} = D$ or let $\mathcal{B} = N$. For any n, we have $\beta_n^{\mathcal{B}}(\phi,\psi,f) = \beta_n^{\mathcal{B}}(\Phi,\Psi,F)\cdot V$.*

3.10.4 Remark. The heat equation asymptotics a_n discussed in §2.6 generalize to the class of manifolds with boundary. However, they behave very differently than the heat content asymptotics. In particular, there is no simple relationship between the invariants a_n of the base, the invariants a_n of the fiber, and the invariants a_n of the total space.

Proof. The heat content asymptotics reflect the heat flow across the boundary. Since

$$\beta_0(0, \Psi, F) = \int_Y \Psi F,$$

we assume henceforth $n > 0$ in proving Theorem 3.10.3.

Equations (3.10.a)-(3.10.c) and equations (3.10.d)-(3.10.f) can be decoupled; we can consider the temperature function defined by the initial condition Φ with homogeneous boundary conditions and the temperature function defined by 0 initial condition an inhomogeneous boundary conditions separately. Thus for any point y of the manifold Y we have:

$$\begin{aligned} &H^{\mathcal{B}}_{\Phi,\Psi}(y;t) = H^{\mathcal{B}}_{\Phi,0}(y;t) + H^{\mathcal{B}}_{0,\Psi}(y;t) \text{ so} \\ &\beta^{\mathcal{B}}_n(\Phi, \Psi, F) = \beta^{\mathcal{B}}_n(\Phi, 0, F) + \beta^{\mathcal{B}}_n(0, \Psi, F). \end{aligned} \tag{3.10.g}$$

Let $\{U^{\mathcal{B}}_n, \lambda^{\mathcal{B}}_n\}$ be the discrete spectral resolution of Δ_Y with given boundary conditions. Let $u^{\mathcal{B}}_n := \pi^* U^{\mathcal{B}}_n$. Expand

$$\Phi = \sum_n c^{\mathcal{B}}_n(\Phi) U^{\mathcal{B}}_n \text{ and } F = \sum_n c^{\mathcal{B}}_n(F) U^{\mathcal{B}}_n;$$

these series converge in L^2. It is then immediate from the definition that

$$H^{\mathcal{B}}_{\Phi,0}(y;t) = \sum_n e^{-t\lambda^{\mathcal{B}}_n} c^{\mathcal{B}}_n(\Phi) U^{\mathcal{B}}_n(y).$$

We compute therefore:

$$\begin{aligned} \beta^{\mathcal{B}}(\Phi, 0, F)(t) &= \sum_{i,j} e^{-t\lambda^{\mathcal{B}}_i} c^{\mathcal{B}}_i(\Phi) c^{\mathcal{B}}_j(F) \int_M U^{\mathcal{B}}_i U^{\mathcal{B}}_j \\ &= \sum_n e^{-t\lambda^{\mathcal{B}}_n} c^{\mathcal{B}}_n(\Phi) c^{\mathcal{B}}_n(F). \end{aligned}$$

Recall that we defined $f := \pi^* F$ and we defined $\phi := \pi^* \Phi$. We perform a similar calculation on Z to expand

$$\phi = \sum_n c^{\mathcal{B}}_n(\Phi) u^{\mathcal{B}}_n \text{ and } f = \sum_n c^{\mathcal{B}}_n(F) u^{\mathcal{B}}_n.$$

We have $\int_Z u_i u_j = V \int_Y U_i V_i = V\delta_{ij}$. By Theorem 3.8.1, $\Delta_Z u_n^{\mathcal{B}} = \lambda_n^{\mathcal{B}} u_n^{\mathcal{B}}$. Therefore for any point z of the manifold Z the heat content function on Z is given by:

$$h_{\phi,0}^{\mathcal{B}}(z;t) = \sum_n e^{-t\lambda_n^{\mathcal{B}}} c_n^{\mathcal{B}}(\Phi) u_n^{\mathcal{B}}(z)$$

and a similar argument using the fact that $\int_Z u_i u_j = V\delta_{ij}$ then yields

$$\begin{aligned} &\beta^{\mathcal{B}}(\phi,0,f)(t) \sim V \sum_n e^{-t\lambda_n^{\mathcal{B}}} c_n^{\mathcal{B}}(\Phi) c_n^{\mathcal{B}}(F) \text{ and} \\ &\beta_n^{\mathcal{B}}(\phi,0,f) = V\beta_n^{\mathcal{B}}(\Phi,0,F). \end{aligned} \tag{3.10.h}$$

Let U be a harmonic function with boundary value Ψ. Then $U - H_{U,0}^D$ satisfies the equations defining $H_{0,\Psi}^D$ and consequently $H_{0,\Psi}^N = U - H_{U,0}^D$. This implies that

$$\begin{aligned} &\beta(0,\Psi,F) = \int_M UF - \beta(U,0,F), \\ &\beta_0^D(0,\Psi,F) = 0, \quad \text{and} \\ &\beta_n^D(0,\Psi,F) = \beta_n(U,0,F) \text{ if } n \geq 1. \end{aligned} \tag{3.10.i}$$

Let $u = \pi^* U$; this is a harmonic function with boundary values ψ. Thus a similar argument shows:

$$\begin{aligned} &\beta_0^D(0,\psi,f) = 0, \quad \text{and} \\ &\beta_n^D(0,\psi,f) = \beta_n(u,0,f) \text{ if } n \geq 1. \end{aligned} \tag{3.10.j}$$

The assertion of Theorem 3.10.3 concerning Dirichlet boundary conditions now follows from equations (3.10.g), (3.10.h), (3.10.i), and (3.10.j).

The situation is slightly more complicated with Neumann boundary conditions. Suppose that $\int_{\partial M} \Psi = 0$. We can then find a harmonic function U so that $\partial_N \Psi$ on ∂M. We can then argue that the function

$$U - H_{U,0}^N$$

satisfies the equations defining $H_{0,\Psi}^N$ and consequently

$$H_{0,\Psi}^N = U - H_{U,0}^N.$$

Similarly, since the average value of ψ is zero on ∂Z we have

$$H_{0,\psi}^N = u - H_{u,0}^N;$$

the remainder of the argument is then the same to see that the heat content asymptotics transform properly.

To complete the proof, we must remove the assumption that the average value of Ψ vanishes. Choose a small point y_0 in the interior of Y and choose $\epsilon > 0$ so that the disk of radius ϵ about y_0 is disjoint from ∂Y. Choose a smooth function $\tilde{F}$ so that $\tilde{F}$ agrees with F near ∂Y and so that $\tilde{F}$ vanishes identically near the boundary of this disk. We let $\tilde{Y} = Y - B_\epsilon(y_0)$ and $\tilde{Z} = \pi^{-1}\tilde{Y}$. Extend Ψ so that the average value of Ψ on the boundary of $\tilde{Y}$ vanishes. The argument given in the previous paragraph shows that

$$\beta_n^N(0, \tilde{\psi}, \tilde{f})_{\tilde{Z}} = V\beta_n^N(0, \tilde{\Psi}, \tilde{F})_{\tilde{Y}}. \tag{3.10.k}$$

The heat content asymptotics reflect the flow across the boundary. The initial value of the functions $H^N(0, \psi)_Y$ and $H^N(0, \psi)_{\tilde{Y}}$ is zero. The principal of not feeling the boundary shows that

$$\beta(0, \Psi, F)_Y(t) - \beta(0, \tilde{\Psi}, \tilde{F})_{\tilde{Y}}(t)$$

vanishes to infinite order as $t \downarrow 0$ since the heat which flows across the additional boundary component is weighted with zero near the additional boundary component. Thus

$$\beta_n(0, \Psi, F)_Y = \beta_n(0, \tilde{\Psi}, \tilde{F})_{\tilde{Y}}.$$

We may therefore use equation (3.10.k) to see that

$$\beta_n^N(0, \psi, f)_Z = V\beta_n^N(0, \Psi, F)_Y.$$

The desired result now follows. $\square$

In [21], van den Berg and Gilkey computed $\beta_5^D(1, 0, 1)$. This corresponds to a heat conduction problem in a solid with constant specific heat and constant initial temperature is suddenly immersed in ice water. Van den Berg and Gilkey used H. Weyl's [199] theory of invariants to show that there exist universal constants so that

$$\begin{aligned}
\beta_5^D(1, 0, 1) = \tfrac{1}{240\sqrt{\pi}} \textstyle\int_{\partial M} \{ & b_1 \rho_{mm;mm} + b_2 L_{aa} \rho_{mm;m} + b_3 L_{ab} R_{ammb;m} \\
& + b_4 \rho_{mm}^2 + b_5 R_{ammb} R_{ammb} + b_6 L_{aa} L_{bb} \rho_{mm} + b_7 L_{ab} L_{ab} \rho_{mm} \\
& + b_8 L_{ab} L_{ac} R_{mbcm} + b_9 L_{aa} L_{bc} R_{mbcm} + b_{10} R_{ammb} R_{accb} + b_{11} L_{aa} L_{bc} R_{bddc} \\
& + b_{12} L_{ab} L_{ac} R_{bddc} + b_{13} L_{ab} L_{cd} R_{acbd} + b_{14} R_{abcm} R_{abcm} + b_{15} R_{abbm} R_{accm} \\
& + b_{16} L_{aa:b} R_{bccm} + e_1 L_{ab:c} L_{ab:c} + e_2 L_{aa} L_{bb} L_{cc} L_{dd} + e_3 L_{aa} L_{bb} L_{cd} L_{cd} \\
& + e_4 L_{ab} L_{ab} L_{cd} L_{cd} + e_5 L_{aa} L_{bc} L_{cd} L_{db} + e_6 L_{ab} L_{bc} L_{cd} L_{da} \}.
\end{aligned}$$

They computed at some length to determine these universal constants and show:

$$b_1 = 8,\ b_2 = -8,\ b_3 = 16,\ b_4 = -4,\ b_5 = 16$$
$$b_6 = -4,\ b_7 = -8,\ b_8 = 64,\ b_9 = -16,\ b_{10} = 0,$$
$$b_{11} = 0,\ b_{12} = -8,\ b_{13} = -8,\ b_{14} = 4,\ b_{15} = 8,$$
$$b_{16} = -16,\ e_1 = -8,\ e_2 = 1,\ e_3 = -4,\ e_4 = -4 - b_{12},$$
$$e_5 = -16 + b_{12},\ \text{and}\ e_6 = 48.$$

We believe the vanishing of b_{10} and b_{11} is a consequence of Theorem 3.10.3; thus this provides a new functorial property that should be useful in the computation of these invariants.

Chapter Four:

When Eigenvalues Change

4.1 Introduction

In Chapter Three, we established rigidity results and gave conditions to ensure that eigenvalues did not change. In this chapter, we study when eigenvalues can change. In §4.2, we study the geometry of circle bundles. We first study the spectral geometry of the Hopf fibration $S^3 \to S^2$. We use this example to show that Theorem 3.4.1 is sharp if $p \geq 2$; given $0 \leq \lambda \leq \mu$, we can find $\pi : Z \to Y$ and $0 \neq \Phi \in E(\lambda, \Delta_Y^p)$ so that $\pi^*\Phi \in E(\mu, \Delta_Z^p)$.

We then generalize the Hopf fibration and give examples due to Muto [143, 144] of principal circle bundles where eigenvalues change. We use these examples to build examples of holomorphic Hodge manifolds where harmonic forms of type (p, p) pull back to give eigenforms which are no longer harmonic.

In §3.6, we showed that if $P \to Y$ was a principal bundle with structure group G so that $H^1(G; \mathbb{R}) = 0$, then the eigenvalue structure was rigid; if the pull back of an eigenform was again an eigenform, then the eigenvalue did not change. In §4.3, we show this hypothesis was essential. Suppose G is a compact connected Lie group with $H^1(G; \mathbb{R}) \neq 0$. Let $\epsilon > 0$ be given. In §4.3, we show that there exists a principal bundle $P \to S^3$ with structure group G and $\Phi \in E(4, \Delta_{S^3}^2)$ so that $\Phi \in E(4 + \epsilon^2, \Delta_P^2)$; this gives more examples where eigenvalues can change.

In §3.5, we turn our attention to complex geometry and establish that eigenvalues can change. We have shown previously in Theorem 3.5.1 that eigenvalues can not change if $q = 0$. Furthermore, we have shown that eigenvalues can not decrease. We will show that these results are sharp except possibly if $(p, q) = (0, 1)$; this case is left unsettled. Let $0 \leq \lambda \leq \mu$ be given, let $q \geq 1$ be given, and let p be given with $p+q \geq 2$. We will show there exists a Hermitian submersion $\pi : V \to U$ and an eigenform $0 \neq \Phi \in E(\lambda, \Delta_U^{p,q})$ so that $\pi^*\Phi \in E(\mu, \Delta_V^{p,q})$. The case $(0,2)$ requires special argument as this is a bit exceptional.

The statement of Theorem 3.5.2 giving necessary and sufficient conditions that all the eigenvalues were preserved involved studying the conditions $J^*\omega = \pm\omega$. In §4.5, we construct examples of Hermitian submersions where $J^*\omega = \omega$ and $J^*\omega = -\omega$ for non-trivial ω.

In §4.6 we discuss the analogous theorems in spin geometry. If the structure group is Abelian, given any $\lambda \geq 0$ and $\mu \geq 0$, we can find a principal bundle $\pi : P \to \mathbb{T}^2$ with structure group G over the two dimensional torus $\mathbb{T}^2$ so that there exists $\Phi \in E(\lambda, D_{\mathbb{T}^2}^{\mathcal{C}^c})$ with $\pi_\alpha^* \in E(\mu, D_P^{\mathcal{C}^c})$ where $D_M^{\mathcal{C}^c}$ denotes the spinor operator on M. Thus eigenvalues can decrease in this setting. If the structure group G is non-Abelian, the situation is a bit more complicated and we can show that eigenvalues can increase if $\mu >> \lambda$.

4.2 Circle bundles

Muto [143, 144] gave examples of Riemannian principal S^1 bundles where eigenvalues change. We begin by considering the Hopf fibration $\pi : S^3 \to S^2$ considered in §1.13; note that S^2 is identified with the sphere of radius $\frac{1}{2}$ and has volume π while S^3 is identified with the sphere of radius 1 and has volume $2\pi^2$.

4.2.1 Theorem. *Let ν_2 be the volume element on S^2. Then $\nu_2 \in E(0, \Delta_{S^2}^2)$. Let $\pi : S^3 \to S^2$ be the Hopf fibration. Then $\pi^*\nu_2 \in E(4, \Delta_{S^3}^2)$.*

Proof. Since the volume element of any Riemannian manifold is harmonic, it is immediate that $\nu_2 \in E(0, \Delta_{S^2}^2)$. We use Lemma 1.10.1 to see that

$$\mathcal{E} = -\operatorname{ext}_S(e^1)\pi^* \operatorname{int}_Y(\mathcal{F})$$

where $\mathcal{F}$ is the curvature of the Hopf fibration and e^1 is the dual 1 form. We use equation (1.13.a) to see $\mathcal{F} = 2\nu_2$. Thus $\mathcal{E}\pi^*\nu_2 = -2e^1$. Since $de^1 = 2\pi^*\nu_2$, the desired result follows from the equations of structure given in §2.7; $\theta = 0$ since the fibers are totally geodesic. □

We can now show that Theorem 3.4.1 is sharp if $p \geq 2$. Note that we do not know if eigenvalues can change if $p = 1$.

4.2.2 Theorem. *Let $0 \leq \lambda \leq \mu$ and let $p \geq 2$ be given. There exists a principal circle bundle $\pi : P \to Y$ over some manifold Y and there exists $0 \neq \Phi \in E(\lambda, \Delta_Y^p)$ so that $\pi^*\Phi \in E(\mu, \Delta_Z^p)$.*

Proof. If $\lambda = \mu$, then the Theorem is immediate. We assume therefore $\lambda < \mu$. Let $p = 2 + r$ for $r \geq 0$. If $\lambda = 0$, by rescaling the metrics involved, we may assume without loss of generality that $\mu = 4$. Let $\mathbb{T}^r$ be the torus of dimension

r. We let $Y := S^2 \times \mathbb{T}^r$, $Z := S^3 \times \mathbb{T}^r$, and let $\pi(\xi,t) = (\pi_H(\xi),t)$ where π_H is the Hopf fibration . Let μ_r be the volume element on the torus; this is a harmonic form. Since the roles of the spheres and the torus decouple, we then have $\nu_2 \wedge \mu_r \in E(0,\Delta_Y^p)$ and $\pi^*(\nu_2 \wedge \mu_r) = \pi_H^*\nu_2 \wedge \mu_r \in E(4,\Delta_Z^p)$; this proves the theorem if $\lambda = 0$. If $\lambda \neq 0$, again, by rescaling the metrics involved, we may assume $\mu = \lambda + 4$. By rescaling the metric on the torus, we may find $0 \neq \Phi_{\mathbb{T}} \in E(\lambda, \Delta_{\mathbb{T}}^r)$. The theorem now follows from the observations:

$$\begin{aligned}&\nu_2 \wedge \Phi_{\mathbb{T}} \in E(\lambda, \Delta_Y^p) \text{ and}\\ &\pi^*(\nu_2 \wedge \Phi_{\mathbb{T}}) = \pi_H^*\nu_2 \wedge \Phi_{\mathbb{T}} \in E(\lambda+4, \Delta_Z^p). \qquad \square\end{aligned}$$

We remark that we do not know if Theorem 3.4.1 is sharp if $p = 1$; we know of no examples where eigenvalues change if $p = 1$ nor can we show that they do not.

We can use Theorem 4.2.2 to construct Riemannian submersions where several eigenvalues change:

4.2.3 Theorem. *Fix $p \geq 2$. For $1 \leq \nu \leq r$, let $0 \leq \lambda_\nu \leq \mu_\nu$ be given. There exists a Riemannian submersion $\pi : Z \to Y$ and $0 \neq \Phi_\nu \in E(\lambda_\nu, \Delta_Y^p)$ so that $\pi^*\Phi_\nu \in E(\mu_\nu, \Delta_Z^p)$ for $1 \leq \nu \leq r$.*

Proof. We use Theorem 4.2.2 to construct Riemannian submersions $\pi_\nu : Z_\nu \to Y_\nu$ and differential forms $0 \neq \Phi_\nu \in E(\lambda_\nu, \Delta_{Y_\nu}^p)$ so that $\pi_\nu^*\Phi_\nu \in E(\mu_\nu, \Delta_{Z_\nu}^p)$ for the indices $1 \leq \nu \leq r$. Let $Y := Y_1 \times ... \times Y_r$, let $Z := Z_1 \times ... \times Z_r$, and let $\pi := \pi_1 \times ... \times \pi_r$. We use pullback to regard the Φ_ν as differential forms on Y; the conclusions of the Theorem then follow since we have $\Delta_{Y_\nu}^p \Phi_\nu = \Delta_Y^p \Phi_\nu$ and since we have $\Delta_{Z_\nu}^p \pi_\nu^* \Phi_\nu = \Delta_Z^p \pi^* \Phi_\nu$. $\square$

The Hopf fibration generalizes easily. The following essentially follows from calculations of Muto:

4.2.4 Theorem. *Let ${}^L\nabla$ be a unitary connection on a complex line bundle over Y with associated curvature 2-form $\mathcal{F} = \mathcal{F}({}^L\nabla)$ and associated principal circle bundle $S = S(L)$. Let $\Phi \in E(\lambda, \Delta_Y^p)$. Assume that $d_Y\Phi = 0$, that $d_Y \operatorname{int}_Y(\mathcal{F})\Phi = 0$, and that $-\operatorname{ext}_Y(\mathcal{F})\operatorname{int}_Y(\mathcal{F})\Phi = \epsilon\Phi$ for ϵ constant. Then $\pi^*\Phi \in E(\lambda+\epsilon, \Delta_S^p)$.*

Proof. We use Lemma 1.10.1. Since $\theta = 0$ and since $d_Y\Phi = 0$,

$$\begin{aligned}\Delta_S^p\pi^*\Phi - \pi^*\Delta_Y^p\Phi &= d_S\mathcal{E}\pi^*\Phi = -d_S\{e^1 \wedge \pi^*(\operatorname{int}_Y(\mathcal{F})\Phi)\}\\ &= -\,\pi^*\{\operatorname{ext}_Y(\mathcal{F})\operatorname{int}_Y(\mathcal{F})\Phi\} = \epsilon\pi^*\Phi. \quad \square\end{aligned}$$

We can use this Theorem to construct more examples where eigenvalues change.

4.2.5 Theorem. *Let Y be a homogeneous manifold with $H^2(Y;\mathbb{R}) \neq 0$. There exists a complex line bundle L over Y and a unitary connection ${}^L\nabla$ on L with normalized curvature $\mathcal{F}$ such that:*

(1) *We have $\epsilon := |\mathcal{F}|^2$ is constant.*
(2) *We have $0 \neq \mathcal{F} \in E(0, \Delta_Y^2)$.*
(3) *We have $\pi_S^*\mathcal{F} \in E(\epsilon, \Delta_S^2)$.*

Proof. We use Lemma 1.11.4 to find a unitary connection ${}^L\nabla$ on a complex line bundle L over Y such that

$$0 \neq \Phi := \mathcal{F}({}^L\nabla) \in E(0, \Delta_Y^2).$$

Let

$$0 \neq \epsilon = |\Phi|^2 = -\operatorname{int}_Y(\mathcal{F})\Phi;$$

we have

$$-\operatorname{ext}_Y(\mathcal{F})\operatorname{int}_Y(\mathcal{F})\Phi = \epsilon\Phi.$$

Since Φ is harmonic, $d_Y\Phi = 0$. By Theorem 4.2.4, it suffices to show $d_Y\epsilon = 0$ or equivalently that ϵ is constant to show that $\pi^*\Phi \in E(\epsilon, \Delta_S^2)$. Let $\mathcal{G}$ be the connected component of the identity in the isometry group of Y. Pull back induces a natural action of $\mathcal{G}$ on the de Rham cohomology groups $H^2(Y;\mathbb{R})$. Since $\mathcal{G}$ is connected, the *homotopy axiom* for cohomology implies this action is trivial. We use the de Rham-Hodge Theorem 1.4.6 to identify $H^2(Y;\mathbb{R})$ with $E(0, \Delta_Y^2)$. We may now conclude that $E(0, \Delta_Y^2)$ is fixed by this action of $\mathcal{G}$. Thus $\Phi \circ \psi = \Phi$ for any $\psi \in \mathcal{G}$ and thus $|\Phi|^2$ is fixed by $\mathcal{G}$. Since Y is a homogeneous space, $\mathcal{G}$ acts transitively on Y. This shows $|\Phi|^2$ is constant. $\square$

4.2.6 Remark. the *generalized Hopf fibration* $S^1 \to S^{2n+1} \to \mathbb{CP}^n$ provides an example of this phenomena. We have $H^2(\mathbb{CP}^n;\mathbb{R}) = \mathbb{R} \neq 0$. The associated line bundle is the Hopf line bundle and the normalized curvature generates $H^2(\mathbb{CP}^n;\mathbb{R})$.

This phenomena of changing eigenvalues also appears in complex geometry. Let L be a holomorphic line bundle over a complex manifold Y. We say that L is **positive** if the curvature $\mathcal{F}$ of L is the Kaehler form of a Hermitian metric on Y; there is a possible sign convention which plays no role in our development. For example, the hyperplane bundle H is a positive line bundle over complex projective space $\mathbb{CP}^n$ and the associated metric is the *Fubini-Study metric.* More generally, if Y is a holomorphic submanifold of $\mathbb{CP}^\nu$ for some ν, then the restriction of H to Y is a positive line bundle over Y and the metric on Y is the restriction of the Fubini-Study metric to Y. Conversely, if Y admits a positive line bundle L, then there exists a holomorphic embedding $\alpha : Y \to \mathbb{CP}^\nu$ for some ν and a positive integer k so that $L^{\otimes k} = \alpha^*H$. Such a manifold is said to be a *Hodge manifold.* Thus we may identify the set of Hodge manifolds with the set of smooth algebraic varieties.

4.2.7 Theorem. *Let $\mathcal{F}$ be the curvature of a positive line bundle L over a complex manifold Y of real dimension $2n$. Let $\pi : S \to Y$ be the associated S^1 principal bundle. Then $0 \neq \mathcal{F}^p \in E(0, \Delta_Y^{2p})$ and $\pi^*\mathcal{F}^p \in E(p(n+1-p), \Delta_S^{2p})$ for $1 \leq p \leq n$.*

Proof. Let L be a positive line bundle over a holomorphic manifold Y. Let $\mathcal{F}$ be the curvature of L. Then $\mathcal{F}$ is the *Kaehler form* of the metric on Y so $\mathcal{F}^p$ is harmonic for $1 \leq p \leq n := \dim_{\mathbb{C}} Y$. We have

$$\operatorname{int}_Y(\mathcal{F})\mathcal{F}^p = -p(n+1-p)\mathcal{F}^{p-1}$$

is harmonic and

$$-\operatorname{ext}_Y(\mathcal{F})\operatorname{int}_Y(\mathcal{F})\mathcal{F}^p = p(n+1-p)\mathcal{F}^p.$$

The Theorem now follows from Theorem 4.2.4. $\square$

4.3 Principal Bundles

The following result shows that the hypothesis $H^1(G;\mathbb{R}) = 0$ in assertion (2) of Theorem 3.6.1 was essential.

4.3.1 Theorem. *Let Y be a compact Riemannian manifold. Suppose that there exists $\Phi \in E(\lambda, \Delta_Y^2)$ so that $\lambda \neq 0$, so that $|\Phi|^2 = a$ is constant, and so that $d_Y\Phi = 0$. Let G be a compact connected Lie group with $H^1(G;\mathbb{R}) \neq 0$, and let π be projection on the second factor of $P := G \times Y$. For any $\epsilon > 0$, there exists a metric $g_P(\epsilon)$ on P so that $\pi : P \to Y$ is a Riemannian principal G bundle and so that $\pi^*(\Phi) \in E(\lambda + \epsilon^2 a, \Delta_P^2)$.*

4.3.2 Remark. Let $\pi : S^3 \to S^2$ be the Hopf fibration. Let ν_2 be the volume element on S^2. We showed $\pi^*\nu_2$ belongs to $E(\lambda, \Delta_{S^3}^2)$. Since ν_2 has constant length, $\pi^*\nu_2$ has constant length. Since $d\nu_2 = 0$, we have $d\pi^*\nu_2 = 0$. Thus we may take $Y = S^3$ and $\Phi := \pi^*\nu_2$ in applying this result. We may apply Theorem 4.2.5 to construct other examples where Y is a circle bundle over a homogeneous manifold with non-vanishing second cohomology group and where the 2 form Φ in question will be the pullback of the curvature of the associated line bundle.

Proof. Assume that $\Phi \in E(\lambda, \Delta_Y^2)$ with $\lambda \neq 0$ and that $d_Y\Phi = 0$. Let

$$\chi := \lambda^{-1}\delta_Y\Phi \text{ then } d_Y\chi = \Phi.$$

We assume $|\Phi|^2 = a$ is constant. Then we have that

$$\operatorname{int}_Y(\Phi)\Phi = -|\Phi|^2 = -a.$$

Let G be a compact connected Lie group. Let $\{e_i\}$ and $\{e^i\}$ be orthonormal bases for the Lie algebra and the Lie co-algebra of G. Since $H^1(G;\mathbb{R}) \neq 0$, we may assume $e^1 \in E(0, \Delta^1_G)$. Let π be projection on the second factor of $P := G \times Y$. We define a metric $ds^2_P(\epsilon)$ by requiring that $\mathfrak{H}\Psi := \Psi - \epsilon\chi(\Psi)e_1$ is the horizontal lift of Ψ and by requiring that $\{e_i\}$ is an orthonormal frame for $\mathcal{V}$. The splitting is G equivariant and with this metric, π defines a Riemannian principal G bundle. Note that $\{e^1 + \epsilon\chi, e^2, ...\}$ is the corresponding dual orthonormal frame for $\mathcal{V}^*$. We compute

$$\begin{aligned}[F_a, F_b] =&[f_a, f_b] - \epsilon f_a(\chi(f_b)) + \epsilon f_b(\chi(f_a))\\ =&([f_a, f_b] - \epsilon\chi([f_a, f_b])e_1) - \epsilon d_P\chi(f_a, f_b)e_1.\end{aligned}$$

Consequently $\mathcal{E}\pi^*\Phi = -\operatorname{ext}_P(e^1 + \epsilon\chi)\pi^* \operatorname{int}_P(\epsilon\Phi)$. Since $d_Y\Phi = 0$ and since the fibers of π are minimal, we see

$$(\Delta^2_P\pi^* - \pi^*\Delta^2_Y)(\Phi) = \epsilon^2 d_P(\chi|\Phi|^2) = a\epsilon^2\Phi. \quad \square$$

4.3.3 Remark:. The technique of Riemannian products used in the proof of Lemma 2.9.4 can now be used to construct examples for any $p \geq 2$ where eigenvalues change.

4.4 Complex geometry

There are analogous examples for the complex Laplacian.

4.4.1 Theorem. *Let $\pi : Z \to Y$ be a Hermitian submersion. Let $0 \leq \lambda < \mu < \infty$, let $q \geq 1$ and let $p + q \geq 2$. There exists a Hermitian submersion $\pi : V \to U$ and $0 \neq \Phi \in E(\lambda, \Delta^{p,q}_U)$ so that $\pi^*\Phi \in E(\mu, \Delta^{p,q}_V)$.*

We begin by reducing the proof of Theorem 4.4.1 to the special case $\lambda = 0$ and $(p, q) = (1, 1)$ or $(p, q) = (0, 2)$:

4.4.2 Lemma. *Suppose there is a Hermitian submersion $\pi_1 : Z_1 \to Y_1$ and non-zero $\Phi_1 \in E(0, \Delta^{r,s}_{Y_1})$ so $\pi_1^*\Phi_1 \in E(\sigma, \Delta^{r,s}_{Z_1})$ for some $\sigma > 0$. Let $0 \leq \lambda < \mu$, let $r \leq p$, and let $s \leq q$ be given. Then there is a Hermitian submersion $\pi : Z \to Y$ and $0 \neq \Phi \in E(\lambda, \Delta^{p,q}_Y)$ so $\pi^*\Phi \in E(\mu, \Delta^{p,q}_Z)$.*

Proof. If M is a complex manifold with a Hermitian metric, let $M(c)$ denote M with the scaled metric $c^{-2}ds^2_M$. Since $\Delta^{p,q}_{M(c)} = c^2\Delta^{p,q}_M$,

$$E(\lambda, \Delta^{p,q}_M) = E(c^2\lambda, \Delta^{p,q}_{M(c)}).$$

Give $M := M_1 \times M_2$ the product metric and product holomorphic structure. Then

$$E(\lambda_1, \Delta_{M_1}^{p_1,q_1}) \wedge E(\lambda_2, \Delta_{M_2}^{p_2,q_2}) \subset E(\lambda_1 + \lambda_2, \Delta_M^{(p_1+p_2,q_1+q_2)}).$$

Assume the conditions of the Lemma hold. Choose $c > 0$ so that $\mu = c^2\sigma + \lambda$. Let $\mathbb{T}$ be a holomorphic flat torus of complex dimension at least $\max(p-r, q-s)$. By rescaling the metric on $\mathbb{T}$, we may choose $0 \neq \Phi_2 \in E(\lambda, \Delta_{\mathbb{T}}^{p-r,q-s})$. Let

$$Y := Y_1(c) \times \mathbb{T},\ Z := Z_1(c) \times \mathbb{T},\ \text{and}\ \pi(z_1, w) = (\pi_1(z_1), w).$$

Then π is a Hermitian submersion. Let $\Phi := \Phi_1 \wedge \Phi_2$. Then $0 \neq \Phi \in E(\lambda, \Delta_Y^{p,q})$ and

$$\pi^*\Phi = (\pi_1^*\Phi_1) \wedge \Phi_2 \in E(c^2\sigma + \lambda = \mu, \Delta_Z^{p,q}). \quad \square$$

4.4.3 Forms of type (1,1). Over $Y = Y_2$, let $S_0 = Y \times S^1$ be the trivial circle bundle and with $\mathcal{F}_0 = 0$. Let S_1 be a circle bundle with

$$\mathcal{F}_1 = \frac{1}{2\pi}(dx \wedge dy).$$

Let $Z = W(S_0, S_1)$ be the fiber product discussed in §1.8. Let e^0 and e^1 be the corresponding dual vertical covectors. We then have

$$d_Z e^0 = 0 \text{ and } d_Z e^1 = \frac{1}{2\pi}\pi^*(dx \wedge dy).$$

Define an almost complex structure J on Z by requiring that g_Z is Hermitian, that π^* preserves J, that $J(e^0) = -e^1$, and that $J(e^1) = e^0$. Let

$$\xi^1 := e^0 - \sqrt{-1}e^1 \text{ and } \xi^2 := \pi^*(dx - \sqrt{-1}dy)$$

be a frame for $\Lambda^{0,1}(Z)$. Then

$$d_Z\xi^1 = -\frac{\sqrt{-1}}{2\pi}\pi^*(dx \wedge dy) \in C^\infty\Lambda^{1,1}(Z) \text{ and } d_Z\xi^2 = 0$$

so the *Nirenberg-Neulander theorem* A.5.1 shows J is an *integrable almost complex structure*. As the metric g_Y is flat,

$$dx \wedge dy \in E(0, \Delta_Y^{1,1}) \cap E(0, \Delta_Y^2) \cap E(0, \bar\Delta_Y^{1,1}).$$

Thus

$$\begin{aligned} 0 =& d_Y(dx \wedge dy) = \delta_Y(dx \wedge dy) = \partial_Y(dx \wedge dy) \\ =& \partial_Y^*(dx \wedge dy) = \bar\partial_Y(dx \wedge dy) = \bar\partial_Y^*(dx \wedge dy). \end{aligned}$$

We use Lemma 1.10.1 (7) and Lemma 2.7.2 to see that

$$\begin{aligned}
&\bar\partial_Z^*\pi^*(dx\wedge dy) = -\pi_Z^{1,0}\operatorname{ext}_Z(e^1)\pi^*\operatorname{int}_Y(\mathcal{F}_1)dx\wedge dy\\
&\quad = \pi_Z^{1,0}e^1/2\pi = -\frac{\sqrt{-1}}{4\pi}(e^0+\sqrt{-1}e^1)\\
&\Delta_Z^{1,1}\pi^*(dx\wedge dy) = \bar\partial_Z\bar\partial_Z^*\pi^*(dx\wedge dy) = \frac{1}{4\pi}\pi_Z^{1,1}d_Ze^1\\
&\quad = \frac{1}{8\pi^2}\pi^*(dx\wedge dy).
\end{aligned}$$

This shows $dx\wedge dy\in E(0,\Delta_Y^{1,1})$ and $\pi^*(dx\wedge dy)\in E(\frac{1}{8\pi^2},\Delta_Z^{1,1})$. This provides an example where a harmonic form of type (1,1) pulls back to an eigenform corresponding to a non-zero eigenvalue.

4.4.4 Forms of type (0,2). Let $z=(z^1,z^2)$ for $z^i=x^i+\sqrt{-1}y^i$ be complex coordinates on $Y=Y_4$. Let

$$\Phi := (dx^1-\sqrt{-1}dy^1)\wedge(dx^2-\sqrt{-1}dy^2).$$

Then $\Phi\in E(0,\Delta_Y^{0,2})$ and Φ generates the line bundle $\Lambda^{0,2}(Y)$. Use Lemma 1.12.1 to construct line bundles L_i over Y so

$$\mathcal{F}_1=\frac{1}{2\pi}(dx^1\wedge dx^2) \text{ and } \mathcal{F}_2=\frac{1}{2\pi}(dx^1\wedge dy^2).$$

These are not holomorphic line bundles since the curvatures are not (1,1) forms. Let $Z := W(S(L_1),S(L_2))$ be the fiber product of the associated unit circle bundles. Let e^i be the associated vertical covectors. We define an almost complex structure J on Z by requiring g_Z is Hermitian, that π^* preserves J, and that $J(e^1)=-e^2$ and $J(e^2)=e^1$. Let ξ^i span $\Lambda^{0,1}(Z)$ for

$$\xi^1 := \pi^*d\bar z^1,\ \xi^2 := \pi^*d\bar z^2, \text{ and } \xi^3 := e^1-\sqrt{-1}e^2.$$

Then $d\xi^1=0$, $d\xi^2=0$, and

$$\begin{aligned}
d_Z\xi^3 =& \frac{1}{2\pi}\pi^*(dx^1\wedge(dx^2-\sqrt{-1}dy^2))\\
=&\frac{1}{4\pi}\pi^*\{(dx^1+\sqrt{-1}dy^1)\wedge(dx^2-\sqrt{-1}dy^2)\\
&+(dx^1-\sqrt{-1}dy^1)\wedge(dx^2-\sqrt{-1}dy^2)\}
\end{aligned}\tag{4.4.a}$$

This decomposes $d_Z\xi^3$ as the sum of forms of type (1,1) and (0,2) so $d_Z\xi^3$ has no (2,0) component. Thus the almost complex structure J is integrable and by the Nirenberg Neulander theorem defines a complex structure on Z. We compute $\mathcal{E}\pi^*\Phi = \frac{1}{2\pi}(e^1 - \sqrt{-1}e^2)$. This is of type $(0,1)$ and

$$d_Z\mathcal{E}\pi^*\Phi = \frac{1}{4\pi^2}\pi^*(dx^1 \wedge dx^2 - \sqrt{-1}dx^1 \wedge dy^2).$$

We use equation (4.4.a) to see $\pi_Z^{0,2}d_Z\mathcal{E}\pi^*\Phi = \frac{1}{8\pi^2}\pi^*\Phi$ so $\pi^*\Phi \in E(\frac{1}{8\pi^2}, \Delta_Z^{0,2})$. This provides an example where a harmonic form of type (0,2) pulls back to an eigenform corresponding to a non-zero eigenvalue.

Proof of Theorem 4.4.1. By Lemma 4.4.2, it suffices to prove Theorem 4.4.1 in the special cases $(p,q) = (1,1)$ and $(p,q) = (0,2)$ with $\lambda = 0$. The first case is handled in §4.4.3 and the second case is handled in §4.4.4. $\square$

4.4.5 Holomorphic line bundles. The examples of §4.4.3 and §4.4.4 involved manifolds Y with flat metrics. We conclude this section by constructing other families of examples where eigenvalues change where the metric on the base is not flat. We restrict to the case $p = q \geq 1$ for the sake of simplicity. Let L be a holomorphic line bundle over Y. Let $\langle,\rangle$ be a fiber metric on L. If s_h is a local non-vanishing holomorphic section to L, let

$$\nabla_L s_h := \partial_Y \log\langle s_h, s_h\rangle \cdot s_h.$$

Let $\tilde{s}_h = e^F s_h$ be another local non-vanishing holomorphic section to L where F is a locally defined holomorphic function on Y. Since $\bar\partial_Y F = 0$, we have

$$\begin{aligned}\nabla_L\tilde{s}_h =&(dF + \partial_Y \log\langle s_h, s_h\rangle)e^F s_h = (\partial_Y F + \partial_Y \log\langle s_h, s_h\rangle)\tilde{s}_h\\ =&\partial_Y \log\langle \tilde{s}_h, \tilde{s}_h\rangle \cdot \tilde{s}_h.\end{aligned}$$

Thus ∇_L is invariantly defined and

$$\mathcal{F} = -\sqrt{-1}\bar\partial_Y\partial_Y \log\langle s_h, s_h\rangle.$$

We see that ∇_L is Riemannian since

$$\langle\nabla_L s_h, s_h\rangle + \langle s_h, \nabla_L s_h\rangle = d\langle s_h, s_h\rangle.$$

4.4.6 Hodge manifolds. Let L be a positive line bundle over a Hodge manifold Y and let

$$Z := Z(j,k) := W(S(L^{\otimes j}), S(L^{\otimes k}))$$

be the fiber product of the circle bundles defined by the circle bundles of the tensor powers of L. Let e^j be the corresponding vertical covectors. We extend the almost complex structure from Y to Z by defining

$$J(e^j) = -e^k \text{ and } J(e^k) := e^j.$$

We use the Nirenberg-Neulander theorem to see that J is integrable; the integrability condition on horizontal covectors is immediate so we must only check the vertical component;

$$d(e^j - \sqrt{-1}e^k) = (j - \sqrt{-1}k)\pi^*\mathcal{F}.$$

4.4.7 Theorem. *Let $1 \leq p \leq \bar{m}$ and let $\mu := (j^2+k^2)p(\bar{m}+1-p)$. Then*

$$\mathcal{F}^p \in E(0, \Delta_Y^{2p}) \cap E(0, \Delta_Y^{p,p}), \text{ and}$$
$$\pi^*(\mathcal{F}^P) \in E(\mu, \Delta_Z^{2p}) \cap E(\frac{1}{2}\mu, \Delta_Z^{p,p}).$$

Proof. We have $\mathcal{F}^p$ is a harmonic form of type (p,p). Since Y is Kaehler, we have that

$$E(0, \Delta_Y^{2p}) \cap C^\infty \Lambda^{p,p}(Y) = E(0, \Delta_Y^{p,p})$$

so $\mathcal{F}^p \in E(0, \Delta_Y^{p,p})$. We compute

$$\pi_Z^{p,p-1}\mathcal{E}\pi^*\mathcal{F}^p = \frac{1}{2}(j - \sqrt{-1}k)p(\bar{m}+1-p)(e^j + \sqrt{-1}e^k) \wedge \pi^*\mathcal{F}^{p-1}$$

so $d\pi_Z^{p,p-1}\mathcal{E}\pi^*\mathcal{F}^p = \frac{1}{2}\mu\pi^*\mathcal{F}^p$. Thus $\pi^*\mathcal{F}^p \in E(\frac{1}{2}\mu, \Delta_Z^{p,p})$; the proof of the corresponding assertion in the real case is similar. $\square$

Note that the manifold Z constructed above is in general not Kaehler. For example, if L is the Hopf line bundle over the Riemann sphere S^2 and if we have $(j,k) = (0,1)$, then $Z = S^1 \times S^3$ is the Hopf manifold and $\pi : S^1 \times S^3 \to S^2$ is essentially just the Hopf fibration where we normalize the metrics suitably.

4.5 Hermitian submersions

Let $\pi : Z \to Y$ be a Hermitian submersion with minimal fibers. Let $p > 0$ and let $q > 0$. In Theorem 3.5.2 we saw that π^* preserves the eigenspaces on forms of type $(p,0)$ if and only if $J^*\omega = -\omega$ and that π^* preserves the eigenspaces of forms of type $(0,q)$ if and only if $J^*\omega = \omega$. We now give examples to illustrate these two cases. The case $J^*\omega = \omega$ is relatively easy; the case $J^*\omega = -\omega$ requires more work.

4.5.1 Hermitian submersions with $J^*\omega = \omega$. Let Y be a Riemann surface so $\dim_{\mathbb{C}} Y = 1$. Then the distribution $\mathcal{H}_{1,0}$ is a 1 dimensional complex foliation and hence $\mathcal{H}_{1,0}$ is necessarily integrable. Thus $J^*\omega = \omega$. The submersion constructed in §4.4.3 dealing with examples of forms of type $(1,1)$ gives an example

$$\pi : W(S_0, S_1) \to S^1 \times S^1$$

with non-trivial curvature tensor ω satisfying $J^*\omega = \omega$.

4.5.2 Hermitian submersions with $J^*\omega=-\omega$. We have $J=-1$ on $\Lambda^{2,0}\bigoplus\Lambda^{0,2}$ and $J=+1$ on $\Lambda^{1,1}$. Let S_i be circle bundles over a torus $\mathbb{T}^k$ with curvatures $\mathcal{F}^i$ and corresponding dual vertical covectors e^i. We assume that $J^*\mathcal{F}^i=-\mathcal{F}^i$ or equivalently that we may decompose $\mathcal{F}^i=\xi^i+\bar{\xi}^i$ for $\xi^i\in\Lambda^{2,0}$. We define an almost complex structure on $Z=W(S_0,S_1)$ by requiring that g_Z is Hermitian, that π^* preserves J, that $J(e^0)=-e^1$, and that $J(e^1)=e^0$. We compute

$$\begin{aligned}d(e^0-\sqrt{-1}e^1)&=\pi^*(\mathcal{F}^0-\sqrt{-1}\mathcal{F}^1)\\&=\pi^*(\xi^0-\sqrt{-1}\xi^1)+\pi^*(\bar{\xi}^0-\sqrt{-1}\bar{\xi}^1).\end{aligned}$$

The Nirenberg-Neulander integrability condition is satisfied if and only if

$$\xi^0=\sqrt{-1}\xi^1. \tag{4.5.a}$$

Define horizontal 2-forms ω^i by the evaluation:

$$\omega^i(f_a,f_b)=\frac{1}{2}g_Z(e_i,[f_a,f_b]).$$

We then have

$$\omega^i(f_a,f_b)=\frac{1}{2}e^i([f_a,f_b])=-\frac{1}{2}de^i(f_a,f_b)=-\frac{1}{2}\pi^*\mathcal{F}^i(f_a,f_b).$$

Thus $\omega^i=-\frac{1}{2}\pi^*\mathcal{F}^i$ and $J^*\omega^i=-\omega^i$. Thus it suffices to give an example where equation (4.5.a) is satisfied. Let

$$\xi^0:=\frac{1}{4\pi}(dx^1+\sqrt{-1}dy^1)\wedge(dx^2+\sqrt{-1}dy^2).$$

Then we have

$$\begin{aligned}\mathcal{F}^0&=\frac{1}{2\pi}(dx^1\wedge dx^2-dy^1\wedge dy^2),\ \text{and}\\\mathcal{F}^1&=\frac{1}{2\pi}(-dx^1\wedge dy^2+dx^2\wedge dy^1).\end{aligned}$$

We use Lemma 1.12.1 to construct bundles L_i over the torus with

$$\begin{aligned}\mathcal{F}^2&=\frac{1}{2\pi}(dx^1\wedge dx^2),\ \mathcal{F}^3=\frac{1}{2\pi}(dy^1\wedge dy^2),\\\mathcal{F}^4&=\frac{1}{2\pi}(dx^1\wedge dy^2),\ \mathcal{F}^5=\frac{1}{2\pi}(dx^2\wedge dy^1).\end{aligned}$$

Since $\mathcal{F}(L_i^*)=-\mathcal{F}(L_i)$ and

$$\mathcal{F}(L_i\otimes L_j)=\mathcal{F}(L_i)+\mathcal{F}(L_j),$$

$L_0:=L_2\otimes L_3^*$ and $L_1:=L_4^*\otimes L_5$ define circle bundles over the torus with the desired curvatures.

4.6 Spin geometry

We now generalize results concerning differential forms to the spinor setting. In Theorem 4.6.1, we deal with Abelian structure groups; in Theorem 4.6.2, we deal with non-Abelian structure groups. By taking $0 \le \mu < \lambda$ in Theorem 4.6.1, we see that eigenvalues can decrease so Theorem 3.4.1 can fail for spinors. Theorem 4.6.2 shows Theorem 3.6.1 also can fail for spinors.

4.6.1 Theorem. *Let G be a compact connected Abelian Lie group. Let $Y = \mathbb{T}^2$. For any $\lambda \ge 0$ and for any $\mu \ge 0$, there exists a principal bundle $\pi : P \to Y$ with structure group G, there exists $0 \ne \Phi \in E(\lambda, D_Y^{\mathcal{C}^c})$, and there exists $\alpha \in \mathcal{C}^c\mathfrak{g}$ such that $\pi_\alpha^*\Phi \in E(\mu, D_P^{\mathcal{C}^c})$.*

Proof. We first recall some notational conventions extablished previously. If P is a principal bundle, let $\mathfrak{g}_i$ be the vertical vector fields discussed in §1.16. The curvature tensor is then given by: $\omega(F_a, F_b) = g_P([F_a, F_b], \mathfrak{g}_i)\mathfrak{g}^i = 2\omega_{abi}\mathfrak{g}^i$. The following endomorphism defined in equation (2.7.e) played an important role in the intertwining formulas of Lemma 2.7.3:

$$\mathcal{E}_{spin} = \tfrac{1}{4}\Sigma_{a<b}\chi c_Y(F^a)c_Y(F^b) \otimes c_{\mathfrak{g}}(\omega(F_a, F_b)). \tag{4.6.a}$$

Let G be a compact connected Abelian Lie group; this means that we may express $G = S^1 \times ... \times S^1$. Let $x = (x^1, ..., x^r)$ for $x^i \in [0, 2\pi]$ be the usual periodic parameters on G. Note that ∂_i^x need not be an orthonormal basis for the Lie algebra $\mathfrak{g}$ of G. Let $\vec{\varrho} = (\varrho_1, ..., \varrho_r)$ be a collection of integers. If $\lambda = \mu = 0$, let $\vec{\varrho} = 0$; otherwise choose $\vec{\varrho} \ne 0$. Let $y := (y^1, y^2)$ for $y^a \in [0, 2\pi]$ be the usual periodic parameters on $Y := S^1 \times S^1$. For $\varepsilon_a \in \mathbb{R}^+$, let

$$ds_Y^2 := \varepsilon_1^{-2}dy^1 \circ dy^1 + \varepsilon_2^{-2}dy^2 \circ dy^2.$$

Let $F_a := \varepsilon_a\partial_a^y$ be the associated orthonormal frame for TY. By Lemma 1.12.1, there exist connections ∇_i on complex line bundles L_i over Y with curvatures

$$\mathcal{F}_i = \frac{1}{2\pi}\varrho_i dy^1 \wedge dy^2.$$

Let P be the associated principal bundle with structure group G; P is *not* topologically trivial if $\vec{\varrho} \ne 0$. Then we have

$$\omega(F_1, F_2) = 4\varepsilon_1\varepsilon_2\xi \text{ where } \xi := -\frac{1}{16\pi}\varrho_i dx^i \in \mathfrak{g}.$$

Thus by Equation (4.6.a), $\mathcal{E}_{spin} = \varepsilon_1\varepsilon_2\chi c_Y(F^1)c_Y(F^2) \otimes c_{\mathfrak{g}}(\xi)$. Note that

(1) $\{\chi c_Y(F^1)c_Y(F^2)\}^2 = -1$.

(2) $c_Y(F^1)$ commutes with $\chi c_Y(F^1)\chi_Y(F^2)$.

(3) $c_Y(F^1)^2 = -1$.

Choose $0 \neq \Phi_0 \in \mathcal{C}^c Y$ so

$$c_Y(F^1)\Phi_0 = \chi c_Y(F^1)c_Y(F^2)\Phi_0 = \sqrt{-1}\Phi_0.$$

Let $\Phi_n(y) := e^{-\sqrt{-1}ny^1}\Phi_0$. Then

$$A_P^{\mathcal{C}^c}\pi_\alpha^*\Phi_n = \Phi_n \otimes (n\varepsilon_1 + \sqrt{-1}\varepsilon_1\varepsilon_2 c_{\mathfrak{g}}(\xi))\alpha$$

and $A_Y^{\mathcal{C}^c}\Phi_n = n\varepsilon_1\Phi_n$. Choose n and $\varepsilon_1 \neq 0$ so $\varepsilon_1 n = \sqrt{\lambda}$. The eigenvalues of $\sqrt{-1}c_{\mathfrak{g}}(\xi)$ are $\pm|\xi|$. Choose $\varepsilon_2 \neq 0$ and $0 \neq \alpha$ so that

$$(n\varepsilon_1 + \sqrt{-1}\varepsilon_1\varepsilon_2 c_{\mathfrak{g}}(\xi))\alpha = \sqrt{\mu}\alpha.$$

Then $\Phi_n \in E(\lambda, D_Y^{\mathcal{C}^c})$ and $\pi_\alpha^*\Phi_n \in E(\mu, D_P^{\mathcal{C}^c})$. $\square$

The situation for non-Abelian structure groups is a bit different. Let $\epsilon_G \geq 0$ be the smallest eigenvalue of $D_G^{\mathcal{C}}$.

4.6.2 Theorem. *Let $\lambda \geq 0$ and let μ be given with $\lambda + \epsilon_G \leq \mu$. If $\lambda = 0$, let $Y = \mathbb{T}^2$; if $\lambda > 0$, let $Y = S^3$. Let G be non-Abelian. There exists a principal bundle $\pi : P \to Y$ with structure group G, there exists $0 \neq \Phi \in E(\lambda, D_Y^{\mathcal{C}^c})$, and there exists $\alpha \in \mathcal{C}^c\mathfrak{g}$ such that $\pi_\alpha^*\Phi \in E(\mu, D_P^{\mathcal{C}^c})$.*

Proof. Suppose first $\lambda = 0$. We choose $e_i \in \mathfrak{g}$ so $[e_1, e_2] = \xi \neq 0$. Let $Y = \mathbb{T}^2$ with the flat product metric and let $P := G \times Y$ where π is projection on the second factor. Note that P is topologically trivial in contrast to the principal bundles given in the proof of Theorem 4.6.1. For $\varrho \in \mathbb{R}$, we define a non-trivial horizontal structure by setting $F_1 := \partial_1^y + 4\varrho e_1$ and $F_2 := \partial_2^y + e_2$. Then

$$\omega(F_1, F_2) = 4\varrho\xi.$$

Choose $0 \neq \Phi \in \mathcal{C}^c\mathbb{R}^2$ so $c_Y(dy^1)c_Y(dy^2)\Phi = \sqrt{-1}\Phi$. Let

$$\mathcal{U}(\varrho) := A_G^{\mathcal{C}^c} + \sqrt{-1}\varrho c_{\mathfrak{g}}(\xi).$$

Since $\chi^2 = 1$, $D_P^{\mathcal{C}^c}\pi_\alpha^*\Phi = \Phi \otimes \mathcal{U}(\varrho)^2\alpha$. Let $\epsilon(\varrho)$ be the smallest eigenvalue of $\mathcal{U}(\varrho)^2$. Then $\epsilon(0) = \epsilon_G$ and $\epsilon(\varrho) \to \infty$ as $\varrho \to \infty$. Thus we can find ϱ so that $\epsilon(\varrho) = \mu \geq \epsilon_G$. Choose

$$0 \neq \alpha \in \mathcal{C}^c\mathfrak{g} \text{ so } \mathcal{U}(\varrho)^2\alpha = \mu\alpha.$$

Then $0 \neq \Phi \in E(0, D_Y^{\mathcal{C}^c})$ and $\pi_\alpha^*\Phi \in E(\mu, D_P^{\mathcal{C}^c})$.

Next suppose $\lambda \neq 0$. Let g_0 be the standard metric on S^3 and let $\mathfrak{so}(3)$ be the Lie algebra of S^3. By Equation (A.4.b),

$$D^{\mathcal{C}^c}_{(S^3,g_0)} = \tfrac{9}{4} \text{ on } \mathcal{C}\mathfrak{so}(3).$$

Let $Y := (S^3, \varepsilon^{-2} g_0)$. For suitably chosen ε, $D^{\mathcal{C}^c}_Y = \lambda$ on $\mathcal{C}^c\mathfrak{so}(3)$. Let $P := G \times Y$ where π is projection on the second factor. Choose $0 \neq \xi \in \mathfrak{g}$. Let $\varrho \in \mathbb{R}$. Let

$$\begin{aligned}
&F_1(y) := \varepsilon y \cdot i,\ F_2(y) := \varepsilon y \cdot j,\ F_3(y) := \varepsilon y \cdot k\\
&[F_1, F_2] = 2\varepsilon F_3,\ [F_2, F_3] = 2\varepsilon F_1,\ [F_3, F_1] = 2\varepsilon F_1,\\
&f_1 := F_1 - 2\varrho\xi,\ f_2 := F_2,\ f_3 := F_3.
\end{aligned}$$

The F_a are an orthonormal frame for $\mathfrak{so}(3)$. Let the f_a define the horizontal space $\mathcal{H}$ of P. Then $\omega(F_2, F_3) = 4\varepsilon\varrho\xi$, $\omega(F_3, F_1) = 0$, and $\omega(F_1, F_2) = 0$. Thus

$$\begin{aligned}
A^{\mathcal{C}^c}_P =& \sqrt{\lambda} c_Y(F^1) c_Y(F^2) c_Y(F^3) \otimes 1 + \chi \otimes A^{\mathcal{C}^c}_G\\
&+ \varepsilon\varrho\chi c_Y(F^2) c_Y(F^3) \otimes c_{\mathfrak{g}}(\xi),\\
D^{\mathcal{C}^c}_P =& \lambda + (\varepsilon\varrho c_Y(F^2) c_Y(F^3) \otimes c_{\mathfrak{g}}(\xi) + 1 \otimes A^{\mathcal{C}^c}_G)^2.
\end{aligned}$$

Choose $\Phi \in \mathcal{C}^c\mathfrak{so}(3)$ so that $c_Y(F^2) c_Y(F^3)\Phi = \sqrt{-1}\Phi$. Let

$$\mathcal{U}(\varrho) := A^{\mathcal{C}^c}_G + \varepsilon\varrho\sqrt{-1} c_{\mathfrak{g}}(\xi).$$

Then $D^{\mathcal{C}^c}_P \pi^*_\alpha \Phi = \Phi \otimes (\lambda + \mathcal{U}(\varrho)^2)\alpha$. To complete the proof, we choose ϱ so

$$\mathcal{U}(\varrho)^2\alpha = \mu - \lambda \geq \epsilon_G. \quad \square$$

We note that constant ϵ_G in Theorem 4.6.2 need not be optimal. For example, suppose that $G = S^3$. Let $\{e_i\}$ for $i = 1, 2, 3$ be the usual basis for the Lie algebra of S^3 as discussed in §1.13. Then $\epsilon_G = \frac{9}{4}$. Let $\xi = e^1$. Then

$$\mathcal{U}(\varrho) = 3/2 c_{\mathfrak{g}}(\mathfrak{g}^1) c_{\mathfrak{g}}(\mathfrak{g}^2) c_{\mathfrak{g}}(\mathfrak{g}^3) + \varepsilon\varrho\sqrt{-1} c_{\mathfrak{g}}(e^1).$$

Since $c_{\mathfrak{g}}(\mathfrak{g}^1)$ commutes with $c_{\mathfrak{g}}(\mathfrak{g}^1) c_{\mathfrak{g}}(\mathfrak{g}^2) c_{\mathfrak{g}}(\mathfrak{g}^3)$, for suitably chosen values of ϱ, 0 is an eigenvalue of the self-adjoint endomorphism $\mathcal{U}(\varrho)$. Thus the conclusions of Theorem 4.6.2 hold for any $\lambda \leq \mu$ and not just $\lambda + \frac{9}{4} \leq \mu$. We refer to W. Kramer & U. Semmelmann [115] and to Moroianu [136, 137, 138, 139, 140] for other papers in this area.

Chapter Five:
Other Topics

5.1 Introduction

In this section, we take up some related topics. In §5.2, we discuss obstructions to the existence of metrics of positive scalar curvature on Riemannian manifolds in the spinor category. The eta invariant, a spectral invariant, plays a crucial role in this setting as do geometric fiber bundles with $\mathbb{HP}^2$ fibers. Thus the spectral geometry of Riemannian submersions enters into the picture. This represents joint work of the first author with B. Botvinnik and S. Stolz. The $\hat{A}$-roof genus is a spectral invariant. Let M be a compact manifold of dimension $m \geq 5$ which either is simply connected or which has finite cyclic fundamental group. Suppose first that M admits a spin structure s. Then M admits a metric of positive scalar curvature if and only if $\hat{A}(M, s) = 0$. If the dimension m is divisible by 4, then $\hat{A}(M, s)$ takes values in $\mathbb{Z}$ and is independent of the spin structure s which is chosen. If $m \equiv 1$ or if $m \equiv 2$ mod 8, then $\hat{A}(M, s)$ takes values in $\mathbb{Z}_2$ and can depend on the particular spin structure chosen. This invariant vanishes otherwise. Thus there is no obstruction to constructing a metric of positive scalar curvature on such a manifold if $m \equiv 3$, if $m \equiv 5$, if $m \equiv 6$, or if $m \equiv 7$ mod 8. The restriction that $m \geq 5$ is essential; the situation is very different if $m = 4$. There are analogous results if M does not admit a spin structure but if the universal cover of M admits a spin structure.

In §5.3, we study the Ricci curvature. If Y is a compact manifold which admits a metric of positive Ricci curvature, then Meyers theorem shows that the fundamental group $\pi_1(M)$ is finite. Let $\pi : P \to M$ be a principal bundle with compact connected structure group G. If P admits a bundle metric with positive

Ricci curvature, then $\pi_1(P)$ is finite. Conversely, we show that if $\pi_1(P)$ is finite and if the base Y admits a metric of positive scalar curvature, then P admits a bundle metric with positive scalar curvature. The geometry of the Laplacian again plays a central role in the analysis and again the spectral geometry of Riemannian submersions is crucial.

In §5.4, we discuss some unsolved problems. In previous sections, we have shown that eigenvalues do not change on functions and can change on forms if the degree is at least 2. We do not know what the situation is regarding 1 forms. We also discuss some unresolved problems in the spinor context.

5.2 Manifolds with positive scalar curvature

Let Y be a compact connected Riemannian manifold without boundary of dimension m. We assume $m \geq 5$ for the most part to ensure that certain surgery arguments work. Let $\mathcal{R} := R_{ijji}$ be the scalar curvature. Let $B(r, P)$ denote the geodesic ball of radius r about a point P of Y where r is less than the injectivity radius of M. Let $V_Y(r, P)$ be the Riemannian volume of $B(r, P)$ and let $V_{\mathbb{E}}(r, 0)$ be the volume of the corresponding ball in Euclidean space. There is an asymptotic expansions as $r \downarrow 0$:

$$V_Y(r, P) = V_{\mathbb{E}}(r, 0)\{1 - \frac{1}{6m}\mathcal{R}(x)r^2 + O(r^4)\};$$

see, for example, Gray [78, 79] where several terms in the asymptotic expansion are computed. Thus from the point of geometry, we can interpret the condition that the scalar curvature is positive as meaning that the volume of small geodesic balls in Y grows less rapidly than the corresponding volume growth in flat space.

Suppose that the dimension is 2 and that Y admits a metric of positive scalar curvature. The Gauss-Bonnet theorem:

$$\chi(M) = \frac{1}{4\pi}\int_M \mathcal{R}$$

then implies $\chi(M) > 0$ so $M = S^2$ or $M = \mathbb{RP}^2$. Conversely, of course, both these manifolds admit metrics of positive scalar curvature. Suppose that Y admits a spin structure s, see §A.2 for details. Let $A^{\mathcal{S}}_{Y,s}$ be the *Dirac operator* and let $D^{\mathcal{S}}_{Y,s} := (A^{\mathcal{S}}_{Y,s})^2$ be the *spin Laplacian.* If the scalar curvature $\mathcal{R}$ is positive, Theorem 2.6.5 shows there are no harmonic spinors; this used the Lichnerowicz formula discussed in §2.6.4.

This observation can be used to construct an obstruction to the existence of metrics of positive scalar curvature. If the dimension m is divisible by 4, we may decompose the spinors into positive and negative chirality. This decomposes $A^{\mathcal{S}}_{Y,s} = A^{\mathcal{S}}_{Y,s,+} + A^{\mathcal{S}}_{Y,s,-}$ where

$$A^{\mathcal{S}}_{Y,s,\pm} : C^\infty(\mathcal{S}_\pm) \to C^\infty(\mathcal{S}_\mp).$$

We define
$$\hat{A}(Y,s) := \dim\ker(A^{\mathcal{S}}_{Y,s,+}) - \dim\ker(A^{\mathcal{S}}_{Y,s,-});$$
this is the index of the spin complex. The Atiyah-Singer index theorem expresses $\hat{A}(Y,s)$ as a polynomial in the Pontrjagin numbers of M so $\hat{A}(M)$ is determined by the differentiable structure of the manifold and is independent of the particular spin structure s chosen or the particular metric g. Let p_i are the Pontrjagin classes of Y. In dimensions $m = 4$ and $m = 8$, we have:
$$\hat{A}(Y^4) = \int_{Y^4} -\frac{1}{24}/p_1(TY) \text{ and } \hat{A}(Y^8) = \int_{Y^8} \frac{1}{5760}/(7p_1^2 - 4p_2)(TY).$$
We refer to Hirzebruch [98] for further details. The discussion given above shows that if $\hat{A}(Y) \neq 0$, then Y does *not* admit a metric of positive scalar curvature.

The Kummer surface K^4 is a complex hypersurface of real dimension 4 in $\mathbb{CP}^3$. It is defined by the homogeneous equations
$$z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0.$$
This 4 dimensional manifold is a simply connected compact spin manifold with $\hat{A}(K^4) = 2$. Thus K^4 does *not* admit a metric of positive scalar curvature.

If $m \equiv 1 \bmod 8$ we reduce the dimension of the kernel mod 2 to define
$$\hat{A}(Y,s) := \dim\ker(D^{\mathcal{S}}_{Y,s}) \in \mathbb{Z}_2;$$
this is independent of the metric chosen but does depend on the spin structure. Similarly, if $m \equiv 2 \bmod 8$, we define
$$\hat{A}(Y,s) := \frac{1}{2}\dim\ker(D^{\mathcal{S}}_{Y,s}) \in \mathbb{Z}_2.$$
Again, this is independent of the metric but depends on the spin structure. We define $\hat{A}(Y,s) = 0$ in the remaining dimensions.

If Y admits a metric of positive scalar curvature, then there are no harmonic spinors and $\hat{A}(Y,s) = 0$. Gromov and Lawson [81, 82] conjectured the converse might hold in the simply connected setting; see also related work of Schoen and Yau [172]. Note that if $\pi_1(Y)$ is trivial, there is only one spin structure. This conjecture has been established by Stolz [179] who proved:

5.2.1 Theorem. *Let Y be a connected compact spin manifold of dimension m which is at least 5. Then $\hat{A}(Y) = 0$ if and only if Y admits a metric of positive scalar curvature.*

Note that there is no obstruction if $m \equiv 3, 5, 6, 7 \bmod 8$. If $\pi_1(Y) \neq 0$, the situation is a bit more complicated. We suppose the fundamental group is finite cyclic to simplify the discussion. We refer to [30, 116] for the proof of the following result:

5.2.2 Theorem. *Let Y be a connected compact spin manifold of dimension m which is at least 5 with $\pi_1(Y) = \mathbb{Z}_n$. Then $\hat{A}(Y, s) = 0$ for all spin structures s on Y if and only if Y admits a metric of positive scalar curvature.*

There are analogues of this theorem if Y is not spin, but if the universal cover of Y is spin. We refer to [29, 117] for the proof of the first assertion of following result and to [9, 65] for the proof of the second assertion of the following result.

5.2.3 Theorem. *Let Y be a connected compact manifold of dimension m which is at least 5 with $\pi_1(Y) = \mathbb{Z}_n$ whose universal cover $\tilde{Y}$ is spin.*

(1) *Assume Y is orientable. Then $\hat{A}(\tilde{Y}) = 0$ if and only if Y admits a metric of positive scalar curvature.*
(2) *Assume Y is not orientable but $w_2(Y) = 0$. Assume $m \equiv 2$ mod 4. Then $\hat{A}(\tilde{Y}) = 0$ if and only if Y admits a metric of positive scalar curvature.*

There are many related results. The Gromov-Lawson conjecture has been proved for spherical space form groups and for a short list of infinite groups including free groups, free Abelian groups, and fundamental groups of orientable surfaces. We refer to [9, 131, 163, 164, 165, 167, 168, 169, 179, 180] for other related papers.

We sketch the proof of Theorem 5.2.1 briefly. We consider triples (Y, s, f) where Y is a compact spin manifold of dimension m, where s is a spin structure on Y, and where $f : Y \mapsto B\mathbb{Z}_n$ gives Y a $\mathbb{Z}_n$ structure; if $\pi_1(Y) = \mathbb{Z}_n$, there is a canonical structure f. Let $MSpin_m(B\mathbb{Z}_n)$ be the bordism groups; these are defined by introducing the equivalence relation $(M_1, s_1, f_1) \sim (M_2, s_2, f_2)$ if there exists a compact manifold N with boundary M_1 disjoint union $-M_2$ so that the structures extend over N. Standard surgery theory arguments and the work of Gromov and Lawson [81] and Schoen and Yau [172] show that

5.2.4 Theorem. *Let M be a connected spin manifold of dimension m which is at least 5 with cyclic fundamental group $\mathbb{Z}_n$. Suppose there exists a manifold M_1 which admits a metric of positive scalar curvature so $[(M, s, f)] = [(M_1, s_1, f_1)]$ in $MSpin_m(B\mathbb{Z}_n)$. Then M admits a metric of positive scalar curvature.*

This result shows that the Gromov-Lawson conjecture reduces to a question in equivariant spin bordism. All the torsion in the coefficient ring $MSpin_m$ is 2 torsion; thus only the prime 2 enters. We assume hence forth that the fundamental group $\mathbb{Z}_n$ is a cyclic 2 group.

The spin bordism groups, unfortunately, are still too large to deal with. It is at this point that the geometry of Riemannian submersions enters. Let $\mathbb{HP}^2$ be quaternion projective space with the usual homogeneous metric of positive scalar curvature. Let

$$\mathbb{HP}^2 \to Z \to Y$$

be a fiber bundle where the transition functions are the group of isometries $PSp(3)$ of $\mathbb{HP}^2$. We assume given a $\mathbb{Z}_n$ structure on the base. Since $\mathbb{HP}^2$ is simply connected, this induces a canonical $\mathbb{Z}_n$ structure on the total space E and all $\mathbb{Z}_n$

structures on E arise in this fashion. Let $T_m(\mathbb{Z}_n)$ be the subgroup of $MSpin_m(\mathbb{Z}_n)$ generated by the total spaces E of such fibrations. Let $ko_m(\mathbb{Z}_n)$ be the connective K theory groups. Results of Stolz [179, 180] then show that

$$ko_m(B\mathbb{Z}_n) = \frac{MSpin_m(B\mathbb{Z}_n)}{T_m(B\mathbb{Z}_n)}.$$

The following Lemma shows that the elements of T_m are represented by manifolds that admit metrics of positive scalar curvature. They are therefore irrelevant and the question of the existence of metrics of positive scalar curvature then reduces to computations in connective K theory. It is beyond the scope of the present monograph to give these arguments; they involve the eta invariant which is another spectral invariant. We will content ourselves, therefore, with establishing that such fibrations admit metrics of positive scalar curvature as this is a crucial point in the discussion. In fact, it turns out that it is necessary to study the eta invariant of such a fibration; this is another example in which the spectral geometry of a Riemannian submersion arises. We shall omit a discussion of the eta invariant as it would take us relatively far afield.

5.2.5 Lemma. *Let $\pi : Z \to Y$ be a geometrical $\mathbb{HP}^2$ fiber bundle. Then there exists a metric g_Z on Z with positive scalar curvature.*

Proof. Let g_F be the standard metric of positive scalar curvature on the fiber $\mathbb{HP}^2$. Let g_Y be any Riemannian metric on the base. Let F_x be the fiber of Z over any point x of Y. Since the structure group is $PSp(3)$, we use Besse [22, 9.59] to see that exists a metric g_Z on the total space Z so that:

(1) The induced metric on each F_x is g_F.

(2) Each g_F is totally geodesic.

(3) The map $\pi : Z \to Y$ is a Riemannian submersion.

Let $\mathcal{V}$ and $\mathcal{H}$ be the vertical and horizontal distributions of the submersion. We define the canonical variation $g_Z(t)$ of the metric by imposing the three conditions:

$$g_Z(t)|_{\mathcal{V}} = tg_F,\ \ g_Z(t)|_{\mathcal{H}} = \pi^*(g_Y),\ \text{ and } g_Z(t)(\mathcal{V},\mathcal{H}) = 0.$$

Let $\mathcal{R}_F$ and $\mathcal{R}_Z(t)$ be the scalar curvature of the metrics on F and on Z. Then

$$\mathcal{R}_Z(t) = t^{-1}\mathcal{R}_F + O(1);$$

see Besse [22, 9.70]. In particular, $\mathcal{R}_Z(t) \to \infty$ as $t \downarrow 0$. $\square$

5.3 Manifolds with positive Ricci curvature

Let Y be a compact connected Riemannian manifold without boundary. If $\{e_i\}$ is an orthonormal basis for the tangent space T_PM, and if $\xi,\eta\in T_PM$, we define the Ricci tensor

$$\rho(\xi,\eta):=\sum_i R(\xi,e_i,e_i,\eta);$$

this is a symmetric bilinear form independent of the choice of $\{e_i\}$. If this form is positive definite, i.e. if $\rho(\xi,\xi)>0$ for $\xi>0$, then Y is said to have positive Ricci curvature. This condition imposes restrictions on the topology of Y. If Y has positive Ricci curvature, then Meyer's theorem implies the fundamental group $\pi_1(Y)$ is finite. Note that the converse implication fails; there are simply connected Riemannian manifolds which do not admit metrics of positive Ricci curvature. For example, as noted in §5.2, the Kummer surface K^4 does not admit a metric of positive scalar curvature. Since the scalar curvature is the trace of the Ricci tensor, this manifold also does not admit a metric of positive Ricci curvature.

Let Y be a compact connected Riemannian manifold without boundary which admits a metric of positive Ricci curvature. Let P be a principal bundle over Y with compact connected structure group G. If P admits a metric with positive Ricci curvature, then $\pi_1(P)$ is finite. The following converse was proved by Gilkey, Park, and Tuschmann [75]; the proof of this result forms the focus of this section involving, as it does, a significant amount of spectral geometry.

5.3.1 Theorem (Gilkey-Park-Tuschmann). *Let g_Y be a metric of positive Ricci curvature on a compact connected Riemannian manifold. Let P be a principal bundle over Y with compact connected structure group G. Assume that $\pi_1(P)$ is finite. Then there exists a G invariant metric g_P on P with positive Ricci curvature so that $\pi:(P,g_P)\mapsto(Y,g_Y)$ is a Riemannian submersion.*

If the structure group G admits a metric with positive Ricci curvature, this result follows from work of Bérard Bergery [12]; the fibers can be taken to be totally geodesic in this special case. Bérard Bergery also showed that under the assumptions of Theorem 5.3.1, that P always admits a metric of positive Ricci curvature. However, his argument does not yield a G invariant metric nor does it yield a metric so that π a Riemannian submersion.

The case of a circle bundle is the crucial one in the work of Bérard Bergery and we summarize his argument: Let L be a complex line bundle over Y with associated circle bundle S; $\pi_1(S)$ is finite if and only if $0\neq c_1(L)$ in $H^2(Y;\mathbb{R})$. By Lemma 1.11.4, we may choose a connection on L so the associated curvature 2-form $\mathcal{F}$ is harmonic. By Lemma 1.10.1, we may use the connection to define a splitting of $T(S)=\mathcal{V}\oplus\mathcal{H}$ into the vertical and horizontal subspaces of the projection $\pi:S\to Y$. Let $ds^2_{\mathcal{H}}$ be the pull back of the metric on Y to the horizontal distribution $\mathcal{H}$. Let e^1 be the 1-form which is dual to the circular flow

∂_t and consider the metric

$$ds_S^2 = e^{-2C_1}(e^1 \circ e^1) + ds_{\mathcal{H}}^2.$$

Then for $C_1 >> 0$, the Ricci tensor of g_S is positive semi-definite. Since $\pi_1(S)$ is finite, $c_1(L) \neq 0$ so there exists $y_0 \in Y$ so that

$$\mathcal{F}(y_0) \neq 0.$$

At this point, the Ricci curvature of g_S is positive definite. Aubin [8] has shown that a manifold S which admits a metric with positive semi-definite Ricci tensor which is positive definite at a point admits a metric with positive Ricci curvature everywhere; this result then shows P admits a metric of positive Ricci curvature.

The difficulty with this proof is that when the work of Aubin is applied, the G invariance of the metric is lost and π is no longer a Riemannian submersion. Consequently, we shall proceed a bit differently. The case of a principal circle bundle S is crucial. We choose a connection on S with harmonic curvature $\mathcal{F}$ to split the tangent space of S into horizontal and vertical subspaces. However, instead of considering a constant rescaling, we consider a metric on S of the form

$$ds_S^2 := e^{2(-C_1+\varrho\phi)}(e^1)^2 + ds_{\mathcal{H}}^2;$$

i.e. we rescale the volume of the circle fibers by a varying conformal factor. Here ϕ is a smooth function on Y and C_1 and ϱ are parameters to be specified later. Let $\mathcal{O}$ be a non-empty open set so that

$$|\mathcal{F}|^2 \geq \epsilon_1 > 0 \text{ on } \mathcal{O}.$$

On $\mathcal{O}$, the Ricci tensor is positive for $C_1 >> 0$. On $\mathcal{O}^c$, we will choose ϕ to give the positivity of the Ricci tensor for ϱ small. Thus a crucial feature of our construction, in contrast to that given in [12] is that the fibers have *non-constant* volumes so the fibers are not minimal and hence are not totally geodesic. This proves Theorem 5.3.1 in the case that $G = S^1$. The general case will follow by considering first G/H where H is the connected component of the center of G and then by considering a tower of circle bundles.

The remainder of this section is devoted to the proof of Theorem 5.3.1. Let Y admit a metric of positive Ricci curvature. We begin with a technical lemma from algebraic topology concerning. Let P be a principle torus bundle over Y. Let L_i be the associated line bundles. We show that $\pi_1(P)$ is finite if and only if the Chern classes $c_1(L_i)$ are linearly independent in $H^2(Y;\mathbb{R})$. Then we give several technical results regarding the geometry of circle bundles. These Lemmas follow from the O'Neill formulas [150]. We shall give a self-contained proof of the formulas which we shall need for the convenience of the reader. They are quite straightforward; in any event, it is no more difficult to derive them directly than to adapt the O'Neill formulas to the setting at hand. We also refer to the discussion in Besse [22]). We will use these technical results to prove Theorem 5.3.1 in the special case that $G = S^1$. We then generalize this argument to torus bundles of higher rank and finally to general compact connected Lie groups to complete the proof of Theorem 5.3.1. We are grateful to Professor Tuschmann for giving us permission to present the joint work with the first and third authors in this present monograph.

5.3.2 Lemma. *Let (Y, g_Y) be a compact connected Riemannian manifold with positive Ricci curvature. Let P be a principal torus bundle over Y with associated complex line bundles L_1, ..., L_r. Then $\pi_1(P)$ is finite if and only if the first Chern classes $c_1(L_i)$ are linearly independent in $H^2(Y;\mathbb{R})$.*

Proof. The fundamental group of Y is finite. By passing to a finite cover, we may assume that $\pi_1(Y) = 0$. The long exact sequence of a fibration yields:

$$...\pi_2(Y) \to \pi_1(\mathbb{T}^r) \to \pi_1(P) \to 0.$$

Thus $\pi_1(P)$ is Abelian and $\pi_1(P)$ is finite if and only if the map $\delta : \pi_2(Y) \to \pi_1(\mathbb{T}^r)$ is a rational surjection. We dualize this map and apply the Hurewicz isomorphism to construct

$$\delta^* : H^1(\mathbb{T}^r;\mathbb{R}) \to H^2(Y;\mathbb{R});$$

δ is a rational surjection if and only if δ^* is injective. Let $B\mathbb{T}^r$ be the classifying space of the r dimensional torus. We identify $H^1(\mathbb{T}^r;\mathbb{R})$ with $H^2(B\mathbb{T}^r;\mathbb{R})$ to regard

$$\delta^* : H^2(B\mathbb{T}^r;\mathbb{R}) \to H^2(Y;\mathbb{R}).$$

We use naturality to see then that δ^* is given by the first Chern class. More precisely, if $x_1, ..., x_r$ are the canonical generators of $H^2(B\mathbb{T}^r;\mathbb{R})$, then

$$\delta^*(\Sigma_i a_i x_i) = \Sigma_i a_i c_1(L_i).$$

Consequently $\pi_1(P)$ is finite if and only if $\{c_1(L_1), ..., c_1(L_r)\}$ are linearly independent in $H^2(Y;\mathbb{R})$. $\square$

We wish to study the curvature tensor of a circle bundle. We adopt the notation of Lemma 1.10.1. Let L be a complex line bundle over a manifold Y which is equipped with a unitary connection ∇. We use the construction described in Lemma 1.15.2 to define a Riemannian metric on the circle bundle $S(L)$ by requiring that

(1) Pushforward $\pi_* : \mathcal{H} \to TY$ is an isometry.
(2) We have $\mathcal{V} \perp \mathcal{H}$.
(3) We have $|\partial_t^2| = e^{2f}$ where $f \in C^\infty(Y)$ will be a suitably chosen warping function.

We shall denote the metric on Y by

$$ds_Y^2 := g_{ab}(y)dy^a \circ dy^b.$$

Let $\{F_a\}$ be a local orthonormal frame for the horizontal space. We begin by computing the Christoffel symbols in this setting. Let

$$\Gamma_{ttt} = g_S(\nabla_{\partial_t}\partial_t, \partial_t) \text{ and } \Gamma_{tta} = g_S(\nabla_{\partial_t}\partial_t, \partial_a)$$

etc. Let $\mathcal{A}_a := \mathcal{A}_s(\partial_a)$ be the connection 1 form; we suppress the role of s for notational convenience.

5.3.3 Lemma.

(1) $g_S(\partial_t,\partial_t)=e^{2f}$, $g_S(\partial_t,\partial_a)=e^{2f}\mathcal{A}_a$, *and* $g_S(\partial_a,\partial_b)=g^Y_{ab}+e^{2f}\mathcal{A}_a\mathcal{A}_b$.

(2) $\Gamma_{ttt}=0$, $\quad\Gamma_{tta}=-e^{2f}\partial_a f$, $\quad\Gamma_{att}=\Gamma_{tat}=e^{2f}\partial_a f$,
$\Gamma_{atb}=\Gamma_{tab}=\frac12\{\partial_a(e^{2f}\mathcal{A}_b)-\partial_b(e^{2f}\mathcal{A}_a)\}$, *and*
$\Gamma_{abt}=\frac12\{\partial_a(e^{2f}\mathcal{A}_b)+\partial_b(e^{2f}\mathcal{A}_a)\}$.

Proof. We prove the first assertion by computing:

$$\begin{aligned}
g_S(\partial_t,\partial_t)=&e^{2f},\\
g_S(\partial_t,\partial_a)=&g_S(\partial_t,\partial_a-\mathcal{A}_a\partial_t)+g_S(\partial_t,\mathcal{A}_a\partial_t)=e^{2f}\mathcal{A}_a,\ \text{and}\\
g_S(\partial_a,\partial_b)=&g_S(\partial_a-\mathcal{A}_a\partial_t,\partial_b-\mathcal{A}_b\partial_t)+g_S(\partial_a-\mathcal{A}_a\partial_t,\mathcal{A}_b\partial_t)\\
&+g_S(\mathcal{A}_a\partial_t,\partial_b-\mathcal{A}_b\partial_t)+\mathcal{A}_a\mathcal{A}_b g_S(\partial_t,\partial_t)\\
=&g^Y_{ab}+e^{2f}\mathcal{A}_a\mathcal{A}_b.
\end{aligned}$$

The Christoffel symbols are given by $2\Gamma_{ijk}=g_{ik/j}+g_{jk/i}-g_{ij/k}$. The second assertion now follows. $\square$

We fix $y_0\in Y$ and choose a coordinate system y centered at y_0. We further normalize the coordinate system by assuming that $g_{ab}=\delta_{ab}+O(|y|^2)$; for example, we could take geodesic normal coordinates centered at y_0. We choose s so that $({}^L\nabla s)(y_0)=0$ or equivalently so $\mathcal{A}=O(|y|)$.

5.3.4 Lemma.

(1) $g_S^{tt}=e^{-2f}+O(|y|^2)$, $g_S^{tb}=-\mathcal{A}_b+O(|y|^2)$, *and* $g_S^{ab}=\delta_{ab}+O(|y|^2)$.

(2) $\Gamma_{tt}{}^t(y_0)=0$, $\partial_t\Gamma\equiv0$, $\Gamma_{ab}{}^c(y_0)=0$, $\Gamma_{ab}{}^t(y_0)=\frac12(\partial_b\mathcal{A}_a+\partial_a\mathcal{A}_b)(y_0)$, *and* $\Gamma_{at}{}^b(y_0)=\frac12 e^{2f}(\partial_a\mathcal{A}_b-\partial_b\mathcal{A}_a)(y_0)$.

(3) $\Gamma_{ta}{}^t=\partial_a(f)-\frac12 e^{2f}\mathcal{A}_b(\partial_a\mathcal{A}_b-\partial_b\mathcal{A}_a)+O(|y|^2)$.

(4) $\partial_a\Gamma_{ta}{}^t(y_0)=\{\partial_a\partial_a(f)-\frac12 e^{2f}(\partial_a\mathcal{A}_b)\mathcal{F}_{ab}\}(y_0)$.

(5) $\partial_a\Gamma_{bb}{}^t(y_0)-\partial_b\Gamma_{ab}{}^t(y_0)=\{\frac12\mathcal{F}_{ab;b}+2\partial_b f\cdot\partial_a\mathcal{A}_b-\partial_a f\cdot\partial_b\mathcal{A}_b-\partial_b f\cdot\partial_b\mathcal{A}_a\}(y_0)$.

(6) $\partial_a(\Gamma_{bb}{}^a-\Gamma^Y_{bb}{}^a)(y_0)=-(e^{2f}\mathcal{F}_{ab}\partial_a\mathcal{A}_b)(y_0)$.

(7) $\partial_b(\Gamma_{ab}{}^a-\Gamma^Y_{ab}{}^a)(y_0)=-\frac12(e^{2f}\mathcal{F}_{ab}\partial_b\mathcal{A}_a)(y_0)$.

Proof. Let $C(y):=C_0(y)+C_1(y)$ where $C_i(y)$ are matrix valued functions with $C_1(y_0)=0$. Then

$$\begin{aligned}
C^{-1}(y)=&\{C_0(y)(I+C_0(y)^{-1}C_1(y))\}^{-1}\\
=&\{I-C_0(y)^{-1}C_1(y)+O(|y|^2)\}C_0(y)^{-1}\\
=&C_0(y)^{-1}-C_0(y)^{-1}C_1(y)C_0(y)^{-1}+O(|y|^2).
\end{aligned}$$

Applying this to $C_0 = diag(e^{2f}, 1, ..., 1)$ and to $C_1 = ds_S^2 - C_0$ yields assertion (1). The first identities of (2) are immediate. We use assertion (1) to raise indices and derive assertion (2) from Lemma 5.3.3; note that $\Gamma_{abc}(y_0) = 0$. To prove assertion (3), we compute

$$\Gamma_{ta}{}^t = \Gamma_{tat} g_S^{tt} + \Gamma_{tab} g_S^{tb} = \partial_a(f) - \tfrac{1}{2} e^{2f} \mathcal{A}_b(\partial_a \mathcal{A}_b - \partial_b \mathcal{A}_a) + O(|y|^2).$$

We differentiate this expression with respect to a and evaluate at y_0 to prove assertion (4). The proof of assertion (5) is similar:

$$\begin{aligned}\Gamma_{ab}{}^t &= \Gamma_{abt} g_S^{tt} + \Gamma_{abc} g_S^{ct} \\ &= \tfrac{1}{2} e^{-2f}\{\partial_a(e^{2f}\mathcal{A}_b) + \partial_b(e^{2f}\mathcal{A}_a)\} + O(|y|^2) \\ &= \tfrac{1}{2}(\partial_a \mathcal{A}_b + \partial_b \mathcal{A}_a + 2\partial_a(f)\mathcal{A}_b + 2\partial_b(f)\mathcal{A}_a) + O(|y|^2)\end{aligned}$$

and

$$\begin{aligned}&\partial_a \Gamma_{bb}{}^t(y_0) - \partial_b \Gamma_{ab}{}^t(y_0) \\ &\quad = \tfrac{1}{2}\{2\partial_a\partial_b \mathcal{A}_b - \partial_b\partial_a \mathcal{A}_b - \partial_b\partial_b \mathcal{A}_a \\ &\qquad + 4\partial_b f \cdot \partial_a \mathcal{A}_b - 2\partial_a f \cdot \partial_b \mathcal{A}_b - 2\partial_b f \cdot \partial_b \mathcal{A}_a)\}(y_0).\end{aligned}$$

We prove assertions the final two assertions by computing:

$$\begin{aligned}\Gamma_{bb}{}^a &= g^{ad}\Gamma_{bbd} + g^{at}\Gamma_{bbt} = \Gamma_{bba} - \mathcal{A}_a \Gamma_{bbt} + O(|y|^2) \\ &= \Gamma^Y_{bb}{}^a + \tfrac{1}{2} e^{2f}\{2\partial_b(\mathcal{A}_b \mathcal{A}_a) - \partial_a(\mathcal{A}_b^2) - 2\mathcal{A}_a \partial_b \mathcal{A}_b)\} + O(|y|^2), \text{ and} \\ \Gamma_{ab}{}^a &= g^{ad}\Gamma_{abd} + g^{at}\Gamma_{abt} = \Gamma_{aba} - \mathcal{A}_a \Gamma_{abt} + O(|y|^2) \\ &= \Gamma^Y_{ab}{}^a + \tfrac{1}{2} e^{2f}\{\partial_b(\mathcal{A}_a^2) - \mathcal{A}_a(\partial_a \mathcal{A}_b + \partial_b \mathcal{A}_a)\}. \quad \square\end{aligned}$$

We can now compute the curvature tensor.

5.3.5 Lemma.

(1) $R^S(\partial_t, F_a, F_a, \partial_t) = e^{2f}\{-f_{;aa} - f_{;a} f_{;a} + \sum_b \frac{1}{4} e^{2f} \mathcal{F}_{ab}\mathcal{F}_{ab}\}$.

(2) $R^S(F_a, F_b, F_b, \partial_t) = \frac{1}{2} e^{2f}\{\mathcal{F}_{ab;b} + 3 f_{;b}\mathcal{F}_{ab}\}$.

(3) $R^S(F_a, F_b, F_b, F_a) = R^Y(\partial_a, \partial_b, \partial_b, \partial_a) - \frac{3}{4} e^{2f}\mathcal{F}_{ab}\mathcal{F}_{ab}$.

Proof. Fix y_0. Adopt the normalizations of Lemma 5.3.4 where the coordinates on Y are chosen so that $F = \partial_a$ at y_0 for some a. We use Lemma 5.3.4 to compute at y_0:

$$\begin{aligned}
&R^S(\partial_t,F_a,F_a,\partial_t)\\
&\quad=g_S(\partial_t\Gamma_{aa}{}^b\partial_b+\partial_t\Gamma_{aa}{}^t\partial_t-\partial_a\Gamma_{ta}{}^b\partial_b-\partial_a\Gamma_{ta}{}^t\partial_t,\partial_t)\\
&\qquad+g_S(\Gamma_{tb}{}^c\Gamma_{aa}{}^b\partial_c+\Gamma_{tb}{}^t\Gamma_{aa}{}^b\partial_t+\Gamma_{tt}{}^c\Gamma_{aa}{}^t\partial_c+\Gamma_{tt}{}^t\Gamma_{aa}{}^t\partial_t,\partial_t)\\
&\qquad-g_S(\Gamma_{ab}{}^c\Gamma_{ta}{}^b\partial_c+\Gamma_{ab}{}^t\Gamma_{ta}{}^b\partial_t+\Gamma_{at}{}^c\Gamma_{ta}{}^t\partial_c+\Gamma_{at}{}^t\Gamma_{ta}{}^t\partial_t,\partial_t)\\
&\quad=\{e^{2f}(-\partial_a\Gamma_{ta}{}^t-\Gamma_{ab}{}^t\Gamma_{ta}{}^b-\Gamma_{at}{}^t\Gamma_{ta}{}^t)\}(y_0)\\
&\quad=\{e^{2f}(-\partial_a\partial_a f+\tfrac12e^{2f}(\partial_a\mathcal{A}_b)(\partial_a\mathcal{A}_b-\partial_b\mathcal{A}_a)\\
&\qquad-\tfrac14e^{2f}(\partial_a\mathcal{A}_b+\partial_b\mathcal{A}_a)(\partial_a\mathcal{A}_b-\partial_b\mathcal{A}_a)-(\partial_af)^2)\}(y_0)\\
&\quad=\{e^{2f}(-\partial_a\partial_af-(\partial_af)^2+\tfrac14e^{2f}(\partial_a\mathcal{A}_b-\partial_b\mathcal{A}_a)^2)\}(y_0).
\end{aligned}$$

This proves the first assertion. Next we compute at y_0

$$\begin{aligned}
R^S(F_a,&F_b,F_b,\partial_t)=e^{2f}\{\partial_a\Gamma_{bb}{}^t-\partial_b\Gamma_{ab}{}^t+\Gamma_{at}{}^t\Gamma_{bb}{}^t-\Gamma_{bt}{}^t\Gamma_{ab}{}^t\}\\
&=e^{2f}\{\tfrac12\mathcal{F}_{ab;b}+2\partial_bf\cdot\partial_a\mathcal{A}_b-\partial_af\cdot\partial_b\mathcal{A}_b-\partial_bf\cdot\partial_b\mathcal{A}_a)\\
&\quad+\partial_af\cdot\partial_b\mathcal{A}_b-\tfrac12\partial_bf(\partial_a\mathcal{A}_b+\partial_b\mathcal{A}_a)\}\\
&=\tfrac12e^{2f}\{\mathcal{F}_{ab;b}+3f_{;b}\mathcal{F}_{ab}\}.
\end{aligned}$$

This proves the second assertion. We prove assertion (3) by computing at y_0:

$$\begin{aligned}
&R^S(F_a,F_b,F_b,F_a)\\
&=\partial_a\Gamma_{bb}{}^a-\partial_b\Gamma_{ab}{}^a+\Gamma_{at}{}^a\Gamma_{bb}{}^t-\Gamma_{bt}{}^a\Gamma_{ab}{}^t\\
&=R^Y(\partial_a,\partial_b,\partial_b,\partial_a)+e^{2f}\{-\partial_a\mathcal{A}_b\mathcal{F}_{ab}+\tfrac12\partial_b\mathcal{A}_a\mathcal{F}_{ab}\\
&\quad+\tfrac14(\partial_b\mathcal{A}_a+\partial_a\mathcal{A}_b)\mathcal{F}_{ab}\}.
\end{aligned}$$

The final assertion of Lemma 5.3.5 follows from the first three and from the quadratic formula. □

Let $\Delta=\delta d$ be the scalar Laplacian. Let ρ_S and ρ_Y be the Ricci tensors of S and of Y. The next Lemma is an immediate consequence of Lemma 5.3.5:

5.3.6 Lemma. *Let F be a unit horizontal tangent vector. Let $\mathcal{T}:=e^{-f}\partial_t$.*

(1) $\rho_S(\mathcal{T},\mathcal{T})=\Delta f-|df|^2+\frac14e^{2f}|\mathcal{F}|^2$.

(2) $\rho_S(\mathcal{T},F)=\frac12e^f\{-\delta\mathcal{F}(F)+3\mathcal{F}(F,df)\}$.

(3) $\rho_S(F,F)=\rho_Y(F,F)-\frac12e^{2f}|\operatorname{int}(F)\mathcal{F}|^2-f_{;FF}-F(f)^2$.

(4) *The metric g_S has positive Ricci curvature if and only if*
$\forall F,\ 4\rho_S(\mathcal{T},\mathcal{T})\rho_S(F,F)>\rho_S(\mathcal{T},F)^2$.

We first prove Theorem 5.3.1 in the special case that $G = S^1$. Choose a unitary connection on the complex line bundle L so the curvature $\mathcal{F}$ is harmonic. Since $c_1(L)$ does not vanish in $H^2(Y;\mathbb{R})$, $\mathcal{F}$ does not vanish identically. Since $\mathcal{F}$ is harmonic, $\delta\mathcal{F} = 0$.

(1) Choose $\epsilon > 0$ so $\rho_S(F,F) \geq \epsilon\ \forall F$.
(2) Choose a non-empty open set $\mathcal{O}$ and $\epsilon_1 > 0$ so that $|\mathcal{F}(y)|^2 \geq \epsilon_1\ \forall y \in \mathcal{O}$.
(3) Choose a smooth function ϕ so that $\Delta\phi(y) \geq 1$ for $y \notin \mathcal{O}$.
(4) Let $f := -C_1 + \varrho\phi$. Choose C_1 so $e^{2f}|\operatorname{int}(F)\mathcal{F}|^2 < \frac{\epsilon}{3}\ \forall\ \varrho \in (0,1)$.
(5) Choose $\epsilon_2 > 0$ so $e^{2f}/4 \geq \epsilon_2\ \forall\ \varrho \in (0,1)$.
(6) Choose $\varrho_0 > 0$ so $|\varrho\phi_{;FF}| + |\varrho^2 F(\phi)^2| \leq \frac{\epsilon}{3}\ \forall F$ if $\varrho \in (0,\varrho_0)$. Then $\rho_S(F,F) \geq \frac{\epsilon}{3}$.
(7) Choose K so that $\rho_S(\mathcal{T},\mathcal{T}) \geq \epsilon_1\epsilon_2 - K\varrho$ on $\mathcal{O}$, $\rho_S(\mathcal{T},\mathcal{T}) \geq \varrho - K\varrho^2$ on $\mathcal{O}^c$, and so that $\rho_S(\mathcal{T},F)^2 \leq K\varrho^2$.

It now follows that $\rho_S(\mathcal{T},\mathcal{T})\rho_S(F,F)$ dominates $\rho_S(\mathcal{T},F)^2$ for small values of ϱ and thus ρ_S is positive definite.

Next we prove Theorem 5.3.1 if G is a torus of rank r; we use induction on r. Let L_1, ..., L_r be the complex line bundles defined by the torus bundle P. Let $\mathbb{T}^r := S^1 \times ... \times S^1$ be the r dimensional torus. Assume $\pi_1(P)$ is finite. Let P_{r-1} be the torus bundle of rank $r-1$ defined by the complex line bundles L_1, ..., L_{r-1}. We use the long exact sequence of the fibration

$$S^1 \to P \to P_{r-1}$$

to see $\pi_1(P_{r-1})$ is finite. By induction, we may find a metric g_{r-1} on P_{r-1} so $\pi_{r-1} : P_{r-1} \to Y$ is a Riemannian submersion, so g_{r-1} has positive Ricci tensor, and so that g_{r-1} is invariant under the action of $\mathbb{T}^{r-1}$.

We work equivariantly. Let $\tilde{L} := \pi_{r-1}^* L_r$; this complex line bundle admits a natural $\mathbb{T}^{r-1}$ action. Let $\tilde{\nabla}_0$ be an arbitrary unitary connection on $\tilde{L}$. We average $\tilde{\nabla}_0$ over the action of $\mathbb{T}^{r-1}$ to construct a $\mathbb{T}^{r-1}$ invariant connection $\tilde{\nabla}_1$. Let $\mathcal{F}_1$ be the curvature of $\tilde{\nabla}_1$; this is invariant under the $\mathbb{T}^{r-1}$ action. Let $\mathcal{F}$ be the harmonic representative of $c_1(\tilde{L})$. Since $\mathbb{T}^{r-1}$ is a connected Lie group, it acts trivially on the cohomology of P_{r-1}. Since $\mathbb{T}^{r-1}$ acts by isometries, it preserves the harmonic forms and thus $\mathcal{F}$ is $\mathbb{T}^{r-1}$ invariant. We decompose $\mathcal{F} = \mathcal{F}_1 + d\omega$; by averaging ω over $\mathbb{T}^{r-1}$, we may assume ω is $\mathbb{T}^{r-1}$ invariant. Let $\tilde{\nabla} := \tilde{\nabla}_1 + \omega$. Then $\tilde{\nabla}$ is a $\mathbb{T}^{r-1}$ invariant connection with harmonic curvature $\mathcal{F}$. Choose $\mathcal{O}$ so that $|\mathcal{F}|^2 \geq \epsilon_1 > 0$ on $\mathcal{O}$. Since $\mathcal{F}$ is $\mathbb{T}^{r-1}$ invariant, we can choose $\mathcal{O}$ to be $\mathbb{T}^{r-1}$ invariant and hence $\mathcal{O}^c$ is $\mathbb{T}^{r-1}$ invariant. Choose ϕ so $\Delta\phi \geq 1$ on $\mathcal{O}^c$; by averaging ϕ over the action of $\mathbb{T}^{r-1}$, we can assume ϕ is $\mathbb{T}^{r-1}$ invariant. Thus the metric constructed in the case $r = 1$ yields a metric on P which is both S^1

and $\mathbb{T}^{r-1}$ invariant and hence $\mathbb{T}^r$ invariant. Since the composition of Riemannian submersions is a Riemannian submersion, the remaining assertions of Theorem 5.3.1 now follow.

We can now complete the proof of Theorem 5.3.1 for general groups. Let C be the connected component of the center of a compact connected Lie group G. Since C is a compact connected Abelian Lie group, $C = \mathbb{T}^r$ is a torus of rank r for some r. Let $P \to Y$ be a principal G bundle with $\pi_1(P)$ finite. Let $\tilde{G} := G/C$ and let $\tilde{Y} := P/C$. Then the natural projection from $\tilde{Y}$ to Y defines a principal $\tilde{G}$ bundle. Since G is reductive, $\tilde{G}$ is semisimple so $\tilde{G}$ admits a metric of positive Ricci curvature. The argument in Bérard Bergery [12] shows that $\tilde{Y}$ admits a $\tilde{G}$ invariant metric with positive Ricci curvature so that $\pi : (\tilde{Y}, g_{\tilde{Y}}) \to (Y, g_Y)$ is a Riemannian submersion with totally geodesic fibers. The natural projection from P to $\tilde{Y}$ defines a principal $\mathbb{T}^r$ torus bundle. The structure group G acts on this principal bundle since C is central. We average over the group G at each stage in applying the construction used to prove Theorem 5.3.1 in the case $r > 1$ and the remainder of the argument is the same. $\square$

The proof of Theorem 5.3.1 uses what might be called a *Kaluza-Klein* ansatz. We conclude this section with a brief historical summary of some related work.

Nash [149], Poor [158], and Bérard-Bergery [12] used Riemannian submersions with totally geodesic fibers to construct metrics of positive Ricci curvature on the total spaces of certain compact fiber bundles and on certain vector bundles of rank at least 2. The same construction was used by D'Atri & Ziller [41], Jensen [104], and by Wang & Ziller [191] to construct left invariant and bi invariant Einstein metrics with positive scalar curvature on certain compact Lie groups G where G was viewed as a principal H bundle $H \to G \to G/H$.

Several other authors [36, 42, 113, 189] constructed Einstein metrics with positive scalar curvature on the total spaces of principal torus bundles over products of Kaehler Einstein manifolds with positive first Chern class. Cheeger [38], Derdzinski & Rigas [44], and Strake & Walschap [183] used the same method to construct metrics of non negative sectional curvature on certain manifolds. Weinstein [198] studied symplectic structures on fat bundles. A principal ingredient in the constructions used by all these authors involved rescaling the fibers by a constant factor.

Non constant warping has also been used. Nabonnand [145] gave examples of (non-compact) complete manifolds with positive Ricci curvature and infinite fundamental group. Bérard Bergery [15] showed that if M is complete and admits a metric with non-negative Ricci curvature, then $M \times \mathbb{R}^k$ carries a metric with positive Ricci curvature for $k \geq 3$. For example, if M is the K^3 surface, then $M \times \mathbb{R}^3$ admits a metric with positive Ricci curvature but not positive sectional curvature. Gromoll & Meyer [80] used warped products to construct metrics of positive Ricci curvature on certain non compact manifolds. Anderson [2, 4] and Sha & Yang [172] used warped products to construct metrics of positive Ricci curvature on the connected sum of certain simply connected manifolds with positive Ricci curvature. Wei [197] showed that if L is a simply connected nilpotent Lie group, then $L \times \mathbb{R}^p$ admits a complete Riemannian metric with positive Ricci curvature for p sufficiently large.

5.4 Unsolved problems

We do not know of any examples of manifolds without boundary where eigenvalues change if $p = 1$; if eigenvalues change for $p = 1$, the fibers can not be minimal by Theorem 3.6.1. By Theorem 3.2.1, eigenvalues do not change if $p = 0$. Lemma 2.9.4 provides an example where the eigenvalues of the Neumann Laplacian change.

In the complex setting, let $\pi : Z \to Y$ be a Hermitian submersion. If we suppose that $\theta = 0$, that $\Phi \in E(\lambda, \Delta_Y^{0,1})$, and that $\pi^*\Phi \in E(\lambda + \epsilon, \Delta_Z^{0,1})$, we see that $\epsilon\pi^*\Phi = \pi^{0,1}\mathcal{E}\pi^*\bar{\partial}_Y\Phi$. The left hand side is a horizontal $(0,1)$ form; the right hand side is a vertical $(0,1)$ form. Consequently $\epsilon = 0$. Thus to construct an example where an eigenvalue changes for a $(0,1)$ form, we must consider Hermitian submersions where the fibers are not minimal. We know of no examples where eigenvalues can change but are unable to prove that they can not.

We have studied, briefly, the spectral geometry of the form valued Laplacian for manifolds with boundary. It would be interesting to have a similar examination performed for the spinor Laplacian using spectral boundary conditions.

Appendix

A.1 Introduction

In the appendix, we provide additional material concerning complex and spin geometry. It is included as an appendix to avoid complicating the flow of our exposition in the main part of the monograph. In §A.2, we provide a brief introduction to spin geometry. We review the Clifford algebra, spinor and pinor groups, and construction of a spin structure on a manifold. In §A.3, we define the Dirac operator and derive its basic properties. In §A.4, we discuss some examples on the torus. In §2.7, we discuss projectable spinors and define pull back in the spinor category.

In §A.5, we turn our attention to complex geometry. We define the complex analogue of the exterior and interior derivatives, define the complex Laplacian, and state the Nirenberg-Neulander theorem. We conclude our discussion of complex geometry in §A.6 with a brief expository account of Hodge geometry.

A.2 Spin geometry

There is a third natural operator of Laplace type which will is crucial in many applications. We refer to [6, 64, 120] for details concerning spin geometry. In this appendix, we shall only sketch the constructions briefly here. The *Clifford algebra* $\mathrm{Clif}(m)$ is the universal unital real algebra generated by $\mathbb{R}^m$ subject to the relations

$$v * w + w * v = -2(v, w)1$$

where (v, w) is the standard inner product on $\mathbb{R}^m$. If c is a linear map from $\mathbb{R}^m$ to a unital algebra $\mathcal{A}$ so that $c(\xi)^2 = -|\xi|^2$, we polarize to see that c satisfies the

Clifford commutation relations

$$c(v)c(w) + c(w)c(v) = -2(v, w)1_{\mathcal{A}}$$

and thus extends to a representation of the Clifford algebra. The symbol of the de Rham operator $d + \delta$ is $c := \text{ext} - \text{int}$. This gives a representation of the Clifford algebra $\text{Clif}(T^*M)$ on $\Lambda^*(M)$ for any Riemannian manifold M; $d+\delta$ is an operator of Dirac type. The map $\phi \mapsto c(\phi) \cdot 1$ extends to a linear isomorphism between $\text{Clif}(T^*M)$ and $\Lambda^*(M)$; this is not, of course, an algebra morphism.

Since the generating relation is homogeneous of even degree, the Clifford algebra inherits a natural bigrading

$$\text{Clif}(m) = \text{Clif}^e(m) \oplus \text{Clif}^o(m).$$

Let χ be the *parity operator* ;

$$\chi(\xi) := \begin{cases} +1 & \text{if } \xi \in \text{Clif}^e(m), \\ -1 & \text{if } \xi \in \text{Clif}^o(m). \end{cases}$$

If $m \geq 3$, then the fundamental group of the special orthogonal group $\text{SO}(m)$ is $\mathbb{Z}_2$; we use the Clifford algebra to describe the universal cover. Let the *pinor group* $\text{Pin}(m)$ be the set of all elements $\omega \in \text{Clif}(\mathbb{R}^m)$ which can be written in the form $\omega = v_1 * ... * v_k$ for some k where the v_i are unit vectors in $\mathbb{R}^m$; $\text{Pin}(m)$ is a smooth connected manifold which is given the structure of a Lie group by Clifford multiplication. If v is a unit vector in $\mathbb{R}^m$ and if $x \in \mathbb{R}^m$, let

$$\rho(v) : x \mapsto v * x * v^{-1}$$

be conjugation by v; $\rho(v)$ is reflection in the hyperplane $v^\perp$ so $\rho(v) \in O(m)$. We set $\rho(w) : x \to w * x * w^{-1}$ to extend ρ to a representation from $\text{Pin}(m)$ to $\text{O}(m)$; ρ defines a short exact sequence

$$\mathbb{Z}_2 \to \text{Pin}(m) \to \text{O}(m).$$

This shows that if $m \geq 3$, then $\text{Pin}(m)$ is a universal cover of $O(m)$. Since $O(m)$ has two arc components, there are two universal covers of $O(m)$; the other universal cover is defined by taking the $+$ sign when defining the Clifford algebra. We let the *spinor group* be the subgroup of $\text{Spin}(m)$ generated by taking even products;

$$\text{Spin}(m) := \text{Pin}(m) \cap \text{Clif}^e(m) = \text{Pin}(m) \cap \ker \chi.$$

Let $\text{U}(n)$ be the unitary group. We complexify to define the *complex spinor group*

$$\text{Clif}^c(\mathbb{R}^m) := \text{Clif}(\mathbb{R}^m) \otimes_{\mathbb{R}} \mathbb{C}, \text{ and}$$
$$\text{Spin}^c(m) := \text{Spin}(m) \otimes_{\mathbb{Z}_2} U(1) \subset \text{Clif}_c(\mathbb{R}^m).$$

Let S^n be the unit sphere in $\mathbb{R}^{n+1}$, let $\mathbb{RP}^n := S^n/\mathbb{Z}_2$ be real projective space, and let $\text{SU}(n)$ be the special unitary group. We have that

$$\text{SO}(3) = \mathbb{RP}^3, \ \text{Spin}(3) = S^3 = \text{SU}(2),$$
$$\text{Spin}^c(3) = \text{U}(2), \text{ and } \text{Spin}(4) = S^3 \times S^3.$$

A.2.1 Definition. The *Stiefel-Whitney classes* are $\mathbb{Z}_2$ characteristic classes of a real vector bundle. They are characterized by the properties:

a) If $\dim(V) = r$, then $w(V) = 1 + w_1(V) + ... + w_r(V)$ for $w_i \in H^i(M; \mathbb{Z}_2)$.
b) If $f : M_1 \to M_2$, then $f^*(w(V)) = w(f^*V)$.
c) We have $w(V \oplus W) = w(V)w(W)$.
d) If L is the Möbius line bundle over S^1, then $w_1(L) \neq 0$.

A.2.2 Remark. The first Stiefel Whitney class measures the failure of a vector bundle to be orientable; $w_1(V) \neq 0$ in $H^1(M; \mathbb{Z}_2)$ if and only if V is not orientable. Let $L_{\mathbb{RP}}$ be the *classifying line bundle* over the infinite dimensional real projective space $\mathbb{RP}^\infty$; this is the limit as $k \to \infty$ of the real line bundles over $\mathbb{RP}^k$ discussed in §3.9.2. Let $\mathrm{Vect}^1_{\mathbb{R}}(Y)$ be the set of isomorphism classes of real line bundles over Y and let $[Y, \mathbb{RP}^\infty]$ denote the set of homotopy classes of maps from Y to $\mathbb{RP}^\infty$. The map $f \to f^*L_{\mathbb{RP}}$ defines a natural isomorphism of functors:

$$[Y, \mathbb{RP}^\infty] = \mathrm{Vect}^1_{\mathbb{R}}(Y).$$

Note that $\mathbb{RP}^\infty$ can be replaced by $\mathbb{RP}^k$ for any k so that $2k > \dim(Y)$. This construction is complete analogous with the construction of the first Chern class given in §1.11. We use tensor product to give $\mathrm{Vect}^1_{\mathbb{R}}(Y)$ the structure of an Abelian group. Let $< x >$ be inhomogeneous coordinates. The map which sends the product $< x_i > \times < y_j >$ to $< x_i y_j >$ makes $\mathbb{RP}^\infty$ and $\mathbb{CP}^\infty$ into H groups; these are "groups" in the homotopy category, see Spanier [178] for details. The H-group structure on $\mathbb{RP}^\infty$ gives the set of homotopy classes $[X, \mathbb{RP}^\infty]$ a group structure and the isomorphism given above between $\mathrm{Vect}^1_{\mathbb{R}}(Y)$ and $H^1(Y; \mathbb{Z}_2)$ is a group isomorphism.

A.2.3 Definition. The second Stiefel Whitney class is the obstruction to the existence of a spin structure if the vector bundle V in question is orientable. Let $\mathcal{O}_\alpha$ be a simple cover of a Riemannian manifold M. Over each $\mathcal{O}_\alpha$, let s_α be an orthonormal frame for the tangent bundle TM. On the overlap, we can express $s_\alpha = \phi_{\alpha\beta} s_\beta$ where $\phi_{\alpha\beta}$ maps the overlap $\mathcal{O}_\alpha \cap \mathcal{O}_\beta$ to the orthogonal group $\mathrm{O}(r)$. The transition functions $\phi_{\alpha\beta}$ satisfy the *cocycle condition*:

$$\phi_{\alpha\beta}\phi_{\beta\gamma} = \phi_{\alpha\gamma}.$$

We say that M is *orientable* if we can reduce the structure group to $\mathrm{SO}(r)$; this means that we can choose the local frames so that $\det(\phi_{\alpha\beta}) = 1$; M is orientable if and only if $w_1(TM) = 0$. We say that V admits a *spin structure* if M is orientable and if we can define lifts $\tilde{\phi}_{\alpha\beta}$ to $\mathrm{Spin}(r)$ preserving the cocycle condition. Let L be a real line bundle over M. By chosing a fiber metric for L, we can reduce the structure group to $\mathrm{O}(1) = \pm 1$. Let $\varepsilon_{\alpha\beta}$ be the transition functions of L. If s is a spin structure on V , we twist the structure s by L to define a new structure

s_L with lifts $\tilde\phi_{\alpha\beta}\varepsilon_{\alpha\beta}$. Let $\mathrm{Vect}^1_{\mathbb{R}}(M)$ be the set of isomorphism classes of real and complex line bundles over M. We use the map $s \mapsto s_L$ to parametrize inequivalent spin structures on M by $\mathrm{Vect}^1_{\mathbb{R}}(M)$. We use tensor product to make $\mathrm{Vect}^1_{\mathbb{R}}(M)$ into an Abelian group. The first Stiefel-Whitney class is a group isomorphism from $\mathrm{Vect}^1_{\mathbb{R}}(M)$ to $H^1(M;\mathbb{Z}_2)$ which provides a natural equivalence between these two functors. Thus inequivalent spin and spin^c structures on V are parametrized by $H^1(M;\mathbb{Z}_2)$ and $H^2(M;\mathbb{Z})$. There exist inequivalent spin structures if and only if $H^1(M;\mathbb{Z}_2) \neq 0$.

A.2.4 Definition. We say that a Riemannian manifold M is a *spin manifold* or is *spin* if M admits a spin structure; this condition does not depend on the particular metric which is chosen.

A.2.5 Examples.

(1) The sphere S^m is spin for any m. Note that $\mathbb{RP}^1 = S^1$.

(2) Let $m > 1$. Real projective space $\mathbb{RP}^m$ is orientable if and only if m is odd and spin if and only if $m \equiv 3 \bmod 4$. We have $H^1(\mathbb{RP}^m;\mathbb{Z}_2) = \mathbb{Z}_2$ so there are 2 inequivalent spin structures on $\mathbb{RP}^{4j-1}$.

(3) Complex projective space $\mathbb{CP}^m$ admits a spin structure if and only if m is odd. The spin structure is unique since $\mathbb{CP}^m$ is simply connected.

(4) The group of n^{th} roots of unity acts by complex multiplication on the unit sphere S^{2k-1} in $\mathbb{C}^k$. For $k \geq 2$, the *lens space* $L(k;n)$ is the quotient $S^{2k-1}/\mathbb{Z}_n$; these manifolds were discussed in §3.9.8. If k is odd and if n is even, $L(k;n)$ does not admit a spin structure; $L(k,n)$ admits a spin structure if n is odd or if k and n are both even. The spin structure is unique if n is odd; there are two spin structures if n is even.

(5) The product of spin manifolds is spin; the product of spin^c manifolds is spin^c. The connected sum of spin manifolds is spin; the connected sum of spin^c manifolds is spin^c.

A.3 The Dirac operator

Left Clifford multiplication defines a representation of the spinor group $\mathrm{Spin}(m)$ on the Clifford algebra $\mathrm{Clif}(m)$. If M has a spin structure, let $\mathcal{C}M$ be the associated vector bundle; this is *not* the Clifford algebra bundle defined by T^*M. Right Clifford multiplication commutes with left Clifford multiplication so $\mathcal{C}M$ inherits a right $\mathrm{Clif}(m)$ module structure. There is a natural connection $\nabla^{\mathcal{C}}$ on $\mathcal{C}M$ called the *spin connection* which is preserved by the right $\mathrm{Clif}(m)$ module structure. Left Clifford multiplication induces a natural map c_M from T^*M to the bundle of endomorphisms of $\mathcal{C}M$ so that $c_M(\xi)^2 = -|\xi|^2 I$. Relative to an orthonormal frame $\{F_\mu\}$ for TM and relative to the induced frame for $\mathcal{C}M$, the connection 1 form $A^{\mathcal{C}}$ of $\nabla^{\mathcal{C}}$, and the operators $A_M^{\mathcal{C}}$ and $D_M^{\mathcal{C}}$ are defined by

$$\begin{aligned} A^{\mathcal{C}} :=& \tfrac{1}{4}F^\mu \otimes c_M(F^\nu)c_M(F^\sigma)^M\Gamma_{\mu\nu\sigma} \in T^*M \otimes \operatorname{End}(\mathcal{C}M), \\ A_M^{\mathcal{C}} :=& c_M \circ \nabla^{\mathcal{C}} : C^\infty(\mathcal{C}M) \to C^\infty(\mathcal{C}M), \text{ and} \\ D_M^{\mathcal{C}} :=& (A_M^{\mathcal{C}})^2 : C^\infty(\mathcal{C}M) \to C^\infty(\mathcal{C}M); \end{aligned} \tag{A.3.a}$$

$A_M^{\mathcal{C}}$ and $D_M^{\mathcal{C}}$ are elliptic self-adjoint partial differential operators which commute with the right $\operatorname{Clif}(m)$ module structure on $\mathcal{C}M$.

The representation of $\operatorname{Spin}(m)$ on the complexification of the Clifford algebra $\operatorname{Clif}^c(m)$ by left multiplication is not irreducible. If $m = 2k$ is even, this representation decomposes as a direct sum of 2^k copies of a fundamental representation $\mathcal{S}$ called the spin representation. Let $\mathcal{S}M$ be the associated vector bundle and let $A_M^{\mathcal{S}}$ be the associated first order operator. Then $A_M^{\mathcal{S}}$ is the Dirac operator and $D_M^{\mathcal{S}} := (A_M^{\mathcal{S}})^2$ is the spin Laplacian. We may decompose

$$\mathcal{C}^cM := \mathcal{C}M \otimes_{\mathbb{R}} \mathbb{C} = 2^k \cdot \mathcal{S}M$$

as 2^k copies of the fundamental representation; this decomposition induces decompositions

$$A_M^{\mathcal{C}^c} := A_M^{\mathcal{C}} \otimes 1 \text{ and } D_M^{\mathcal{C}^c} := D_M^{\mathcal{C}} \otimes 1$$

as the direct sum of 2^k copies of the operators $A_M^{\mathcal{S}}$ and $D_M^{\mathcal{S}}$ respectively. Thus the spectral theory of the operators $A_M^{\mathcal{C}}$ and $D_M^{\mathcal{C}}$ is essentially the same as the spectral theory of the operators $A_M^{\mathcal{S}}$ and $D_M^{\mathcal{S}}$. There is a similar decomposition if m is odd; we omit details in the interests of brevity.

There are significant advantages in some instances to working with the bundle $\mathcal{C}M$ and the operators $A_M^{\mathcal{C}}$ and $D_M^{\mathcal{C}}$ rather than with the spinor bundles and the operators $A_M^{\mathcal{S}}$ and $D_M^{\mathcal{S}}$. The $\operatorname{Clif}(m)$ module structure enables one to define a refined index of the Dirac operator which takes values in the real K theory groups, see Hitchin [99]; this plays crucial role in the statement of the Gromov-Lawson-Rosenberg conjecture, see Gromov-Lawson [81, 82] and Rosenberg [163] for details; see also related work of Schoen and Yau [172].

When working with projectable spinors in §2.7.3, §2.8, and §4.6, it is more convenient to work with the full Clifford bundle; the crucial distinction is that the parity of the dimension m does not enter in defining the bundle $\mathcal{C}M$ whereas the case m even is quite different from the case m odd when defining the spinor representations. The use of $\mathcal{C}M$ rather than $\mathcal{S}M$ will simplify this discussion. On the other hand, in studying the Gromov-Lawson conjecture, we use the bundle $\mathcal{S}(M)$ as it was the one we are more familar with in this context; the other viewpoint is equally prevalant in the literature on this subject.

A.4 Spinors on the torus and sphere

Let $x = (x^1, ..., x^r)$ be the usual periodic parameters on the r dimensional torus $\mathbb{T} := \mathbb{T}^r$. Give $\mathbb{T}$ the flat metric. Since $\Gamma = 0$, relative to the natural trivialization of $\mathcal{C}\mathbb{T}^r$ induced by the framing $\partial_i := \partial/\partial x^i$ of $T(\mathbb{T}^r)$ we have

$$A_{\mathbb{T}}^{\mathcal{C}} = \Sigma_i c_{\mathbb{T}}(dx^i)\partial_i \text{ and } D_{\mathbb{T}}^{\mathcal{C}} = -\Sigma_i \partial_i^2. \tag{A.4.a}$$

We may identify the unit sphere S^3 with the unit quaternions. Let e_1, e_2, and e_3 be the usual basis for the Lie algebra $\mathfrak{g}$ of left invariant vector fields on S^3. Let $\nabla_{e_i} e_j = \Gamma_{ijk} e_k$. If σ is a permutation of the indices,

$$\Gamma_{\sigma(1)\sigma(2)\sigma(3)} = sign(\sigma);$$

the remaining Christoffel symbols vanish. This framing of TS^3 defines a natural trivialization of $\mathcal{C}(S^3)$. We summarize the equations of structure below:

$$\begin{aligned}
&e_1(x) = x \cdot i,\ e_2(x) = x \cdot j,\ e_3(x) = x \cdot k,\\
&[e_1, e_2] = 2e_3,\ [e_2, e_3] = 2e_1,\ [e_3, e_1] = 2e_2,\\
&A_{S^3}^{\mathcal{C}} = c_S(e^1)e_1 + c_S(e^2)e_2 + c_S(e^3)e_3 + 3/2c_S(e^1)c_S(e^2)c_S(e^3),\\
&D_{S^3}^{\mathcal{C}} = -(e_1^2 + e_2^2 + e_3^2) \otimes 1 - 3c_S(e_1)c_S(e_2)e_3\\
&\qquad - 3c_S(e_2)c_S(e_3)e_1 - 3c_S(e_3)c_S(e_2)e_1 + \tfrac{9}{4}.
\end{aligned} \tag{A.4.b}$$

A.5 Complex geometry

There is a complex analogue of the real Laplacian. Let $z := (z^1, ..., z^m)$ be a system of local holomorphic coordinates on a holomorphic manifold M of complex dimension m where $z^j := x^j + \sqrt{-1}y^j$. We complexify the real tangent and cotangent bundles and define

$$\begin{aligned}
&dz^j := dx^j + \sqrt{-1}dy^j,\ d\bar{z}^j := dx^j - \sqrt{-1}dy^j,\\
&dz^I := dz^{i_1} \wedge ... \wedge dz^{i_p},\ d\bar{z}^J = d\bar{z}^{j_1} \wedge ... \wedge d\bar{z}^{j_q},\\
&\Lambda^{p,q}(M) := \mathrm{Span}_{|I|=p,|J|=q}\{dz^I \wedge d\bar{z}^J\},\\
&\partial_j^z := \tfrac{1}{2}(\partial_j^x - \sqrt{-1}\partial_j^y),\ \partial_j^{\bar{z}} := \tfrac{1}{2}(\partial_j^x + \sqrt{-1}\partial_j^y),\\
&J(\partial_j^x) = \partial_j^y,\ J(\partial_j^y) = -\partial_j^x,\ J^*(dx^j) = -dy^j,\ J^*(dy^j) = dx^j\\
&\partial \textstyle\sum_{I,J} f_{I,J} dz^I \wedge d\bar{z}^J := \sum_{j,I,J} \partial_j^z(f_{I,J}) dz^j \wedge dz^I \wedge d\bar{z}^J,\\
&\bar{\partial} \textstyle\sum_{I,J} f_{I,J} dz^I \wedge d\bar{z}^J := \sum_{j,I,J} \partial_j^{\bar{z}}(f_{I,J}) d\bar{z}^j \wedge dz^I \wedge d\bar{z}^J.
\end{aligned} \tag{A.5.a}$$

A complex function f is holomorphic if and only if $\bar\partial f = 0$. Since $d = \partial + \bar\partial$,

$$\partial\partial = 0,\ \ \bar\partial\partial + \partial\bar\partial = 0,\ \text{ and } \bar\partial\bar\partial = 0.$$

A Riemannian metric g_M is *Hermitian* if

$$g_M(X, Y) = g_M(JX, JY)$$

for all real tangent vector fields. We extend such a metric to the complexified tangent bundle to be complex linear in the first factor and conjugate linear in the second factor. We define the *Dolbeault Laplacian* by

$$\Delta^{p,q} := (\bar\partial\bar\partial^* + \bar\partial^*\bar\partial) \text{ on } C^\infty\Lambda^{p,q}(M).$$

Let ξ be a real cotangent vector. We decompose

$$\xi = \xi^{(1,0)} + \xi^{(0,1)}$$

into forms of bi-degrees $(0,1)$ and $(1,0)$; these two forms are complex conjugates of each other. Since ξ is real,

$$\bar\xi^{(0,1)} = \xi^{(1,0)}.$$

We use equation (A.5.a) to see that

$$\begin{aligned}&\sigma_L(\bar\partial)(\xi) = \operatorname{ext}(\xi^{(0,1)}) \text{ and}\\ &\sigma_L(\bar\partial^*)(\xi) = -\sigma_L(\bar\partial)(\xi) = -\operatorname{int}(\xi^{(1,0)}).\end{aligned}$$

We define

$$c(\xi) = \operatorname{ext}(\xi^{(0,1)}) - \operatorname{int}(\xi^{(1,0)}).$$

This yields a Clifford module structure since

$$c(\xi)^2 = -\tfrac{1}{2}|\xi|^2.$$

Thus modulo a suitable normalizing constant, $\bar\partial + \bar\partial^*$ is an operator of Dirac type and $\Delta^{(p,q)}$ is an operator of Laplace type.

Let $\pi : Z \to Y$ be a Riemannian submersion. In the complex setting, we assume that Z and Y are holomorphic, that π is holomorphic, and that the metrics on Z and on Y are Hermitian. The complexification of pull back defines

$$\pi^* : C^\infty\Lambda^{p,q}(Y) \to C^\infty\Lambda^{p,q}(Z).$$

The linear map J defined in equation (A.5.a) satisfies $J^2 = -1$ and is an *almost complex structure* on M. Not every almost complex structure arises from a complex structure; there is an integrability condition described in the the *Nirenberg-Neulander theorem*. Let M be an almost complex manifold. This means that we assume given a linear map

$$J : TM \to TM \text{ so that } J^2 = -1.$$

We extend J to the cotangent bundle by duality; thus

$$(JX, \theta) := (X, J^*\theta).$$

We mimic the construction given in equation (A.5.a). We decompose

$$T^*M \otimes \mathbb{C} = T^{(1,0)}(M) \oplus T^{(0,1)}(M)$$

where

$$T^{(1,0)}M := \mathrm{Span}\{\xi - \sqrt{-1}J\xi\} \text{ and } T^{(0,1)}M := \mathrm{Span}\{\xi + \sqrt{-1}J\xi\}.$$

We then define

$$\Lambda^{(p,q)}(M) := \Lambda^p(T^{(1,0)}M) \otimes \Lambda^q(T^{(0,1)}M).$$

This gives a decomposition

$$\Lambda^*(M) \otimes \mathbb{C} = \oplus_{p,q}\Lambda^{(p,q)}(M).$$

Let $\pi^{(p,q)}$ denote the natural projection of $\Lambda^*(M) \otimes \mathbb{C}$ on $\Lambda^{(p,q)}(M)$. We define

$$\partial^{(p,q)} := \pi^{(p+1,q)}d : C^\infty\Lambda^{(p,q)}(M) \to C^\infty\Lambda^{(p+1,q)}(M)$$

and

$$\bar\partial^{(p,q)} := \pi^{(p,q+1)}d : C^\infty\Lambda^{(p,q)}(M) \to C^\infty\Lambda^{(p,q+1)}(M).$$

Let

$$T_cM = \mathrm{Span}\{X - \sqrt{-1}JX\} \subset TM \otimes \mathbb{C}$$

be the complex tangent bundle. This is the span of the holomorphic tangent vectors. Similarly let

$$\bar T_cM = \mathrm{Span}\{X + \sqrt{-1}JX\} \subset TM \otimes \mathbb{C}$$

be the span of the anti-holomorphic tangent vectors. We have

$$TM \otimes_{\mathbb{R}} \mathbb{C} = T_cM \oplus \bar T_cM.$$

We say T_cM is integrable if the *complex Frobenius condition* is satisfied. This means that

$$[X, Y] \in C^\infty T_cM \text{ for all } X, Y \in C^\infty T_cM.$$

The following integrability result was used in §1.14 to study the Hopf manifold and in §4.4.3 to study forms of type $(1, 1)$.

A.5.1 Theorem (Nirenberg-Neulander). *The following conditions are equivalent and define the notion of an integrable almost complex structure:*

(1) *The operator J arises from a holomorphic structure on M.*

(2) *We have $d = \partial + \bar{\partial}$.*

(3) *We have $\bar{\partial}^{(0,1)}\bar{\partial}^{(0,0)} = 0$.*

(4) *The complex tangent bundle is integrable.*

A.6 Hodge geometry

Let L be a holomorphic line bundle over holomorphic manifold Y. Let $\langle , \rangle$ be a fiber metric on L. If s_h is a local non-vanishing holomorphic section to L, let

$$\nabla_L s_h := \partial_Y \log\langle s_h, s_h\rangle \cdot s_h.$$

Let $\tilde{s}_h = e^F s_h$ be another local non-vanishing holomorphic section to L where F is a locally defined holomorphic function on Y. Since $\bar{\partial}_Y F = 0$, we have

$$\begin{aligned}\nabla_L \tilde{s}_h =&(dF + \partial_Y \log\langle s_h, s_h\rangle)e^F s_h = (\partial_Y F + \partial_Y \log\langle s_h, s_h\rangle)\tilde{s}_h\\ =&\partial_Y \log\langle \tilde{s}_h, \tilde{s}_h\rangle \cdot \tilde{s}_h.\end{aligned}$$

Thus ∇_L is invariantly defined and

$$\mathcal{F} = -\sqrt{-1}\bar{\partial}_Y \partial_Y \log\langle s_h, s_h\rangle.$$

We see that ∇_L is Riemannian since

$$\langle \nabla_L s_h, s_h\rangle + \langle s_h, \nabla_L s_h\rangle = d\langle s_h, s_h\rangle.$$

If M is a Riemann surface, then the Chern form $\frac{1}{2\pi}\mathcal{F}$ has a geometrical interepretation. The line bundle L admits a meromorphic section s and we have

$$\int_M \frac{1}{2\pi}\mathcal{F} = \#\text{zeros}(s) - \#\text{poles}(s).$$

A.6.1 Definition. We say that L is a *positive* line bundle over Y if the curvature $\mathcal{F}(L)$ is the Kaehler form of a Hermitian metric on Y; there is a possible sign convention which plays no role in our development. If M admits a positive line bundle, M is said to be Hodge.

A.6.2 Example. The hyperplane bundle H is a positive line bundle over complex projective space $\mathbb{CP}^\nu$ and the associated metric is the Fubini-Study metric. More generally, if Y is any holomorphic submanifold of $\mathbb{CP}^\nu$, then the restriction of the hyperplane bundle to Y is a positive line bundle over Y and the metric on Y is the restriction of the Fubini-Study metric to Y.

The following converse to Example A.6.2 is well known.

A.6.3 Theorem. *Suppose M is a compact holomorphic manifold which admits a positive line bundle L. Then there exists a holomorphic embedding $\alpha : Y \to \mathbb{CP}^\nu$ for some ν and a positive integer k so that $L^{\otimes k} = \alpha^*(H)$. The image of α is a smooth algebraic subvariety of $\mathbb{CP}^\nu$.*

References

[1] K Abe, *On a generalization of the Hopf fibration: I, II, III*, I: Tohoku Math. J.**29**(1977) 335–374; II: Tohoku Math. J.**30**(1978) 177–210; III: Monatsh. Math. **99** (1985), 169–198.

[2] M T Anderson, *Short geodesics and gravitational instantons*, J. Diff. Geo. **31** (1990), 265–275.

[3] N Anghel, *On the index of Callias-type operators*, Geom. Funct. Anal. **3** (1993), 431–438.

[4] ———, *Metrics of positive Ricci curvature with large diameters*, Manuscripta Math. **68** (1990), 405–415.

[5] M Asorey , J Carinena, and L Ibort, *Generalized canonical transformations for time dependent systems*, J Math Phys **24** (1983), 2745–2750.

[6] M F Atiyah, R Bott, & A Shaprio, *Clifford Modules*, Topology 3 Suppl. 1 (1964), 3–38.

[7] M F Atiyah, V K Patodi, and I M Singer, *Spectral asymmetry and Riemannian geometry I, II, III:*, Math. Proc. Cambr. Phil. Soc. **77** (1975) 43–69; **78** (1975) 405–432; **9 (1976) 71–99**.

[8] T Aubin, *Métriques riemanniennes et courbure*, J. Diff. Geo. **4** (1970), 383–424.

[9] E Barrera-Yanez, Ph. D. thesis, University of Oregon (1997).

[10] H Baum, *Une borne supérieure pour la première valeur propre de l'opérateur de Dirac sur une variétériemannienne spinorielle compacte à courbure scalaire positive*, C. R. Acad. Sci. Paris Sér. I Math. **311** (1990), 389–392.

[11] E Bedford and T Suwa, *Eigenvalues of Hopf manifolds*, Proc. AMS **60** (1976), 259–264.

[12] L Bérard Bergery, *Certains fibrés à courbure de Ricci positive*, C R Acad. Sc. Paris **286** (1978), 929–931.

[13] ———, *La courbure scalaire des variétés riemanniennes*, Bourbaki Seminar (1979/80), 225–245.

[14] ———, *Sur la courbure des métriques riemanniennes invariantes des groupes de Lie et des espaces homogènes*, Ann. Sci. École Norm. Sup. (4) 11 (1978), 543–576.

[15] ———, *Quelques exemples des variétés riemanniennes complètes non compactes à courbure de Ricci positive*, C R Acad. Sc. Paris **302** (1986), 159–161.

[16] L Berard Bergery and J P Bourguignon, *Laplacians and Riemannian submersions with totally geodesic fibers*, Illinois J Math **26** (1982), 181–200.

[17] ———, *Laplacian and Riemannian submersions with totally geodesic fibres*, in **Global differential geometry and global analysis (1979) Lecture Notes in Math** 838; Springer, Berlin-New York (1981), 30–35.

[18] M van den Berg, S Desjardins, and P Gilkey, *Functoriality and heat content asymptotics for operators of Laplace type*, Topological Methods in Nonlinear Analysis **2** (1993), 147–162.

[19] M van den Berg and P Gilkey, *Heat Content Asymptotics of a Riemannian Manifold with Boundary*, Journal of Functional Analysis **120** (1994), 48–71.

[20] ———, *Heat equation with inhomogeneous Dirichlet conditions*, to appear Communications in Analysis and Geometry.

[21] ———, *A remark on the heat equation with inhomogeneous Neumann boundary conditions*, preprint.

[22] A L Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete; 3. Folge, Bd. 10., Springer-Verlag, Berlin, New York, 1987.

[23] G Besson, *A Kato type inequality for Riemannian submersions with totally geodesic fibers*, Ann. Global Anal. Geom. **4** (1986), 273–289.

[24] G Besson and M Bordoni, *On the spectrum of Riemannian submersions with totally geodesic fibers*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. (9) Mat. Appl. **1** (1990), 335–340.

[25] D E Blair, *On a generalization of the Hopf fibration*, An. sti. Univ. Al. I. Cuza Iasi, n. Ser., Sect. Ia **17** (1971), 171–177.

[26] M Boiteux, *Theory of nonbijective canonical transformations in mechanics*, J Math Phys **23** (1982), 1311–1314.

[27] M Bordoni, *On the spectrum of fiber spaces*, Polytech. Inst. Bucharest Sci. Bull. Electr. Engrg., Differential geometry and topology applications in physics and technics **53** (1991), 71–78.

[28] B Botvinnik and P Gilkey, *The eta invariant and metrics of positive scalar curvature*, Math. Ann. **302** (1995), 507–517.

[29] ———, *The Gromov-Lawson-Rosenberg conjecture: the twisted case*, Houston Math. J. **23** (1997), 143–160.

[30] B Botvinnik, P Gilkey, and S. Stolz, *The-Gromov-Lawson-Rosenberg conjecture for groups with periodic cohomology*, J. Diff. Geo. **46** (1997), 374–405.

[31] Brunt, Quart J. R. Met. Soc. **58** (1932), 389.

[32] U Bunke, *Relative index theory*, J. Funct. Anal. **105** (199), 63–76.

[33] F Burstall, *Non-linear functional analysis and harmonic maps*, Ph D Thesis.

[34] R Carr, *Construction of manifolds of positive scalar curvature*, Trans. AMS **307** (1988), 63–74.

[35] H S Carslaw and J C Jaeger, *Conduction of Heat in Solids*, Oxford Science Publications, Oxford University Press, 1946.

[36] L Castellani, R D'Auria, and P Fré, $SU(3) \times SU(2) \times U(1)$ *from* $D = 11$ *supergravity*, Nuclear Phys. B **239** (1984), 610–652.

[37] H Cerdeira, *Hydrogen atom in the four dimensional oscillator representation: matrix elements of* z^n, J Phys A **18** (1985), 2719–2727.

[38] J Cheeger, *Some examples of manifolds of non-negative curvature*, J. Diff. Geo. **8** (1973), 623–628.

[39] A Connes and G Skandalis, *The longitudinal index theorem for foliations*, Publ. Res. Inst. Math. Sci. **20** (1984), 1139–1183.

[40] X Dai, Xianzhe and W Zhang, *Circle bundles and the Kreck-Stolz invariant*, Trans. AMS **347** (1995), 3587–3593.

[41] J D'Atri and W Ziller, *Naturally reductive metrics and Einstein metrics on compact Lie groups*, Mem. AMS **215** (1979).

[42] R D'Auria, P Fré, and P van Nieuwenhuizen, $n = 2$ *matter coupled supergravity from compactification on a coset* G/H *possessing an additional killing vector*, Phys. Lett. B. **136** (1984), 347–353.

[43] M H Dehgahni and Y Sobouti, *Liouville's equation: the full symmetries of* r^{-1} *potentials*, Astron Astrophys **275** (1993), 91–95.

[44] A Derdzinski and A Rigas, *Unflat connections in 3 sphere bundles over* S^4, Trans. AMS **265** (1981), 485–493.

[45] S Desjardins and P Gilkey, *Heat content asymptotics for operators of Laplace type with Neumann boundary conditions*, Math Z **215** (1994), 251–268.

[46] S Donaldson, *The Seiberg-Witten equations and* 4*-manifold topology*, Bull. AMS **33** (1996), 45–70.

[47] T Eguchi, P Gilkey, and A J Hanson, *Gravitation, gauge theory, and differential geometry*, Phys Rep **66** (1980), 213–393.

[48] N Ejiri, *Some compact hypersurfaces of constant scalar curvature in a sphere*, J. Geom. **19** (1982), 197–199.

[49] R H Escobales, *Sufficient conditions for a bundle-like foliation to admit a Riemannian submersion onto its leaf space*, Proc. AMS **84** (1982), 280–284.

[50] ______, *Riemannian submersions from complex projective space*, J. Diff. Geo. **13** (1978), 93–107.

[51] ______, *Riemannian submersions with totally geodesic fibers*, J. Diff. Geo. (1975), 253–276.

[52] S. Ferry, A. Ranicki, and J. Rosenberg, *A history and survey of the Novikov conjecture. Novikov conjectures, index theorems and rigidity*, Vol. 1 (Oberwolfach, 1993), 7–66, London Math. Soc. Lecture Note Ser., 226, Cambridge Univ. Press, Cambridge, 1995.

[53] R. Fintushel and R. Stern, *Immersed spheres in 4-manifolds and the immersed Thom conjecture*, Turkish J. Math. **19** (1995), 145–157.

[54] T. Friedrich and I. Kath, *Variétés riemanniennes compactes de dimension 7 admettant des spineurs de Killing*, C. R. Acad. Sci. Paris Sér. I Math. **307** (1988), 967–969.

[55] T. Friedrich and H. Kurke, *Compact four-dimensional self-dual Einstein manifolds with positive scalar curvature*, Math. Nachr. **106** (1982), 271–299.

[56] B. Fuglede, *Harmonic morphisms between Riemannian manifolds*, Ann. Inst. Fourier **28** (1978), 107–144.

[57] ______, *Harmonic morphism between semi-Riemannian manifolds*, Ann. Acad. Sci. Fennicae **21** (1996), 31-50.

[58] K Furutani, *On the group of diffeomorphism commuting with an elliptic operator*, J Math Soc Japan **35** (1983), 153–162.

[59] ______, *On a differentiable map commuting with an elliptic pseudo differential operator*, J Math Kyoto Univ **24** (1984), 197–203.

[60] A. Futaki, *Scalar-flat closed manifolds not admitting positive scalar curvature metrics*, Invent. Math. **112** (1993), 23–29.

[61] P. Gajer, *Riemannian metrics of positive scalar curvature on compact manifolds with boundary*, Ann. Global Anal. Geom. **5** (1987), 179–191.

[62] K. Galicki and H. B. Lawson Jr., *Quaternionic reduction and quaternionic orbifolds*, Math. Ann. **282** (1988), 1–21.

[63] P B Gilkey, *The geometry of spherical space form groups*, ISBN 9971-50-927-X, World Scientific, Singapore, 1989.

[64] ______, *Invariance Theory, the Heat Equation, and the Atiyah-Singer Index theorem (2^{nd} edition)*, ISBN 0-8493-7874-4, CRC Press, Boca Raton, Florida, 1994.

[65] ______, *The eta invariant of pin manifolds with cyclic fundamental groups*, preprint.

[66] P B Gilkey, J V Leahy, and JH Park, *The spectral geometry of the Hopf fibration*, Journal Physics A **29** (1996), 5645–5656.

[67] ______, *Eigenforms of the spin Laplacian and projectable spinors for principal bundles*, to appear Nuclear Physics B.

[68] ______, *Eigenvalues of the form valued Laplacian for Riemannian submersions*, to appear Proc. AMS.

[69] ______, *The eigenforms of the complex Laplacian for a Hermitian submersion*, preprint.

[70] P B Gilkey and JH Park, *Riemannian submersions which preserve the eigenforms of the Laplacian*, Illinois J Math **40** (1996), 194–201.

[71] ______, *Eigenvalues of the Laplacian and Riemannian submersions*, Yokohama Math J **43** (1995), 7–11.

[72] ______, *Eigenvalues of the Laplacian for sphere bundles*, Topological Methods in Non-linear Analysis **5** (1995), 341–344.

[73] ______, *The Bochner Laplacian, Riemannian submersions, Heat content asymptotics and heat equation asymptotics*, to appear Czech Math J.

[74] ______, *A note on the eigenfunctions of the Laplacian for a twisted holomorphic product.*, Comm. Korean Math. Soc. **12** (1997), 325–332.

[75] P Gilkey, JH Park, and W Tuschmann, *Invariant metrics of positive Ricci curvature on principal bundles*, to appear Math. Zeit.

[76] S I Goldberg and T Ishihara, *Riemannian submersions commuting with the Laplacian*, J Diff. Geo. **13** (1978), 139–144.

[77] J M Gracia-Bondía, *Hydrogen atom in the phase-space formulation of quantum mechanics*, Phys. Rev. A **30** (1984), 691–697.

[78] A Gray, *The volume of a small geodesic ball in a Riemannian manifold*, Mich. Math. J **20** (1973), 327–344.

[79] ______, *Tubes*, Addison-Wesley Publishing Company ISBN 0-201-15676-8, 1990.

[80] D Gromoll and W Meyer, *Examples of complete manifolds with positive Ricci curvature*, J Diff. Geo. **21** (1985), 195–211.

[81] M Gromov and H B Lawson, *Spin and scalar curvature in the presence of a fundamental group I*, Ann. of Math. **111** (1980), 209–230.

[82] ______, *The classification of simply connected manifolds of positive scalar curvature*, Ann. of Math. **111** (1980), 423–434.

[83] ______, *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*, Publ. Math. IH.E.S. **58** (1983), 83–196.

[84] G Grubb, *Functional calculus of pseudodifferential boundary problems. 2nd ed.*, Progress in Mathematics, Birkhaeuser, 1996.

[85] ______, *Pseudodifferential boundary problems and applications*, Jahresber. Dtsch. Math.-Ver. **99** (1997), 110–121.

[86] G Grubb and R Seeley, *Developpements asymptotiques pour l'operateur d'Atiyah-Patodi-Singer*, C. R. Acad. Sci., Paris, Ser. I **317** (1993), 1123–1126.

[87] ______, *Weakly parametric pseudodifferential operators and Atiyah-Patodi-Singer boundary problems*, Invent. Math. **121** (1995), 481–529.

[88] ______, *Zeta and eta functions for Atiyah-Patodi-Singer operators*, J Geom. Anal. **6** (1996), 31–77.

[89] S Gudmundsson, *Harmonic morphisms as sphere bundles over compact Riemann surfaces*, preprint Lund University (1996).

[90] ______, *The Harmonic Morphisms Bibliography*, (available on the internet: http://www.maths.lth.se/matematiklu/personal/sigma/harmonic/bibliography.html.

[91] M Gursky, *Locally conformally flat four- and six-manifolds of positive scalar curvature and positive Euler characteristic*, Indiana Univ. Math. J **43** (1994), 747–774.

[92] B Hajduk, *Metrics of positive scalar curvature on spheres and the Gromov-Lawson conjecture*, Math. Ann. **280** (1988), 409–415.

[93] ———, *On the obstruction group to existence of Riemannian metrics of positive scalar curvature*, Global differential geometry and global analysis. Lecture Notes in Math., 1481, Springer, Berlin (1991), 62–72.

[94] ———, *The index of the Dirac operator and scalar curvature: geometric topology or topological geometry?*, Wiadom. Mat. **32** (1996), 1–14.

[95] O Hijazi, *Première valeur propre de l'opérateur de Dirac et nombre de Yamabe*, C. R. Acad. Sci. Paris Sér. I Math. **313** (1991), 865–868.

[96] ———, *A conformal lower bound for the smallest eigenvalue of the Dirac operator and Killing spinors*, Comm. Math. Phys. **104** (1986), 151–162.

[97] O Hijazi and J Milhorat, *Décomposition du fibrédes spineurs d'une variété spin Kähler-quaternionienne sous l'action de la 4-forme fondamentale*, J Geom. Phys. **15** (1995), 320–332.

[98] F Hirzebruch, *Neue Topologische Methoden in der Algebraischen Geometrie*, Springer-Verlag, Berlin, 1956.

[99] N Hitchin, *Harmonic Spinors*, Adv in Math **14** (1974), 1–55.

[100] P A Hogan, *Kaluza-Klein theory derived from a Riemannian submersion*, J Math. Phys. **25** (1984), 2301–2305.

[101] A Ikeda, *On the spectrum of a Riemannian manifold of positive constant curvature*, Osaka J Math. **17** (1980), 75–93.

[102] Ikuta and A Kazuko, *Riemannian submersion with* 2 *dimensional base space*, TRU Math. **18** (1982), 89–105.

[103] T Ishihara, *A mapping of Riemannian manifolds which preserves harmonic functions*, J Math Kyoto Univ **19** (1979), 215–229.

[104] G Jensen, *Einstein metrics on principal fiber bundles*, J Diff. Geo. **8** (1973), 599–614.

[105] D L Johnson, *Kaehler submersions and holomorphic connections*, J Diff. Geo. **15** (1980), 71–79.

[106] J Kazdan, *Deformation to positive scalar curvature on complete manifolds*, Math. Ann. **261** (1982), 227–234.

[107] N Kibler and T Négadi, *On the use of nonbijective canonical transformations in Chemical Physics*, Lettere al Nuovo Cimento **37** (1983), 225–228.

[108] ———, *On the connection between the hydrogen atom and the harmonic oscillator*, Croatia Chemica Acta **57** (1984), 1509–1523.

[109] N Kibler, A Ronveaux, and T Négadi, *On the hydrogen oscillator connection: passage formulas between wave functions*, J Math Phys **27** (1986), 1541–1548.

[110] N Kibler and P Winternitz P, *Lie algebras under constraints and nonbijective canonical transformations*, J Phys A **21** (1988), 1787–1803.

[111] K-D. Kirchberg, *An estimation for the first eigenvalue of the Dirac operator on closed Kähler manifolds of positive scalar curvature*, Ann. Global Anal. Geom. **4** (1986), 291–325.

[112] ———, *The first eigenvalue of the Dirac operator on Kähler manifolds. J Geom. Phys.* **7** (1990), 449–468.

[113] S Kobyashi, *Topology of positively pinched Kaehler manifolds*, Tohôku Math. J **15** (1963), 121–139.

[114] W Kramer, *Lifts of spinors on principal-G-bundles*, preprint.

[115] W Kramer and U Semmelmann, *Private communication re Lifts of spinors on principal-G-bundles.*

[116] S Kwasik and R Schultz, *Positive scalar curvature and periodic fundamental groups*, Comment. Math. Helv. **65** (1990), 271–286.

[117] ———, *Fake spherical space forms of constant positive scalar curvature*, Comment. Math. Helv. **71** (1996), 1–40.

[118] J Kwon and Y. Suh, *On sectional and Ricci curvatures of semi-Riemannian submersions*, Kodai Math. J **20** (1997), 53–66.

[119] H B Lawson, *Complete manifolds of positive scalar curvature*, Proceedings of the 1981 Shanghai symposium on differential geometry and differential equations (Shanghai/Hefei, 1981), 147–181.

[120] H B Lawson and M L Michelsohn, *Spin Geometry*, Princeton Univ. Press, 1989.

[121] C LeBrun, *On the scalar curvature of complex surfaces*, Geom. Funct. Anal. **5** (1995), 619–628.

[122] M Lewkowicz, *Positive scalar curvature and local actions of nonabelian Lie groups*, J Diff. Geo. **31** (1990), 29–45.

[123] A Lichnerowicz, *Spineurs harmoniques*, C R. Acad. Sci. Paris **257** (1963), 7–9.

[124] ———, *Sur les zéros des spineurs-twisteurs*, C. R. Acad. Sci. Paris Sér. I Math. **310** (1990), 19–22.

[125] A Lizak, *Some classes of harmonic mappings of Riemannian submersion*, Pr. Nauk. Politech. Szczecin. 380, Inst. Mat. 11 (1988), 173–180.

[126] R Mazzeo, D Pollack, and K Uhlenbeck, *Connected sum constructions for constant scalar curvature metrics*, Topological Methods in Non-linear Analysis **6** (1995), 207–233.

[127] D M McAvity, *Heat kernel asymptotics for mixed boundary conditions*, Class. Quant. Grav **9** (1992), 1983–1998.

[128] D M McAvity, *Surface energy from heat content asymptotics*, J Phys. A : Math. Gen **26** (1993), 823–830.

[129] H P McKean and I M Singer, *Curvature and eigenvalues of the Laplacian*, J Diff. Geo **1** (1967), 43–69.

[130] P A Mello and M Moshinsky, *Linear canonical transformations*, J Math Phys **16** (1975), 2017–2028.

[131] T Miyazaki, *On the existence of positive curvature metrics on non simply connected manifolds*, J Fac. Sci. Univ. Tokyo Sect IA **30** (1984), 549–561.

[132] ———, *Simply connected spin manifolds with positive scalar curvature*, Proc. AMS **93** (1985), 730–734.

[133] S Montaldo, *p-harmonic maps and stability of Riemannian submersions*, Boll. Un. Mat. Ital. A **10** (1996), 537–550.

[134] H Moriyoshi, *Positive scalar curvature and KO characteristic numbers*, Proc. AMS 100 (1987), 585–588.

[135] ———, *Positive scalar curvature and higher $\hat{A}$-genus*, J Fac. Sci. Univ. Tokyo Sect. IA Math. **35** (1988), 199–224.

[136] A Moroianu, *Opérateur de Dirac et Submersions Riemanniennes*, Thesis École Polytechnique Palaiseau (1996).

[137] ———, *Spineurs et variétés de Hodge*, to appear in Rev. Roum. Math. Pures Appl..

[138] ———, *La première valeur propre de l'opérateur de Dirac sur les variétés kählérinnes compactes*, C. R. Acad. Sci. Paris t 319 Série I (1994), 1057-1062.

[139] ———, *La première valeur propre de l'opérateur de Dirac sur les variétés kählérinnes compactes*, Comm. Math. Phys. **169** (1995), 373–384.

[140] ———, *Spin^c Manifolds and Complex Contact Structures*, SFB Preprint 243 Berlin (1996).

[141] J Moore, John D, *Lectures on Seiberg-Witten invariants*, Lecture Notes in Mathematics, 1629. Springer-Verlag, Berlin (1996).

[142] Y. Muto, *Riemannian submersions and critical Riemannian metrics*, J Math Soc. Japan **29** (1977), 493–511.

[143] ———, *Some eigenforms of the Laplace-Beltrami operators in a Riemannian submersion*, J Korean Math. Soc. **15** (1978), 39–57.

[144] ———, *Riemannian submersion and the Laplace-Beltrami operator*, Kodai Math J **1** (1978), 329–338.

[145] P Nabonnand, *Sur les variétés riemanniennes complètes à courbure de Ricci positive*, C. R. Acad. Sci. **291A** (1980), 591–593.

[146] P T Nagy, *On bundle-like conformal deformation of a Riemannian submersion*, Acta Math. Acad. Sci. Hungar. **39** (1982), 155–161.

[147] ———, *Nonhorizontal geodesics of a Riemannian submersion*, Acta Sci. Math. (Szeged) **45** (1983), 347–355.

[148] F Narita, *Riemannian submersion with isometric reflections with respect to the fibers*, Kodai Math. J **16** (1993), 416–427.

[149] J C Nash, *Positive Ricci curvature on fiber bundles*, J Diff. Geo. **14** (1979), 241–265.

[150] B O'Neil, *Semi-Riemannian geometry with applications to relativity*, Acad. Press, New York, 1983.

[151] JH Park, *On the Chern forms and conformally invariant tensor B_0 of Kaehler manifolds*, Kyungpook Math. J **29** (1989), 37–40.

[152] ———, *The Laplace Beltrami operator and Riemannian submersion with minimal and not totally geodesic fibers*, Bull. Korean Math. Soc. **27** (1990), 39–47.

[153] ———, *A note on eigenfunctions of the Laplacian of warped products* , Honam Math. **16** (1994), 23–25.

[154] ______, *Eigen 1-forms and Riemannian submersions*, Bull. Korean Math. Soc. **33** (1996), 477–480.

[155] ______, *The spectral geometry of Riemannian submersions for manifolds with boundary*, preprint.

[156] JH Park and S Yorozu, *Transversal conformal fields of foliations*, Nihonkai Math. J **4** (1993), 73–85.

[157] ______, *Transverse fields preserving the transverse Ricci field of a foliation*, J Korean Math. Soc. **27** (1990), 167–175.

[158] W A Poor, *Some exotic spheres with positive Ricci curvature*, Math. Annalen **216** (1975), 245–252.

[159] A Ranjan, *Riemannian submersions of compact simple Lie groups with connected totally geodesic fibres*, Math. Z. **191** (1986), 239–246.

[160] ______, *Riemannian submersions of spheres with totally geodesic fibres*, Osaka J Math. **22** (1985), 43–260.

[161] H Reckziegel, *Horizontal lifts of isometric immersions into the bundle space of a pseudo-Riemannian submersion*, **Global differential geometry and global analysis 1984**Lecture Notes in Math. 1156, Springer, Berlin-New York (1985), 264–279.

[162] J Roe, *Exotic cohomology and index theory*, Bull. AMS **23** (1990), 447–453.

[163] J Rosenberg, *C*-algebras, positive scalar curvature, and the Novikov Conjecture*, Publ. Math. IHES (1983), no. 58, 197–212.

[164] ______, *Geometric Methods in Operator Algebras*, Pitman Research Notes, vol. 123, pp. 341–374.

[165] ______, *C*-algebras, positive scalar curvature, and the Novikov Conjecture III*, Topology **25** (1986), 319–336.

[166] ______, *The KO-assembly map and positive scalar curvature*, Algebraic topology Poznań 1989 Lecture Notes in Math., 1474, Springer, Berlin, 1991, 170–182.

[167] J Rosenberg and S Stolz, *Algebraic Topology and its applications*, G Carlsson, R Cohen, W C Hsiang, and J D S Jones eds, MSRI Publications V27 Springer, vol. 27, 1994, pp. 241–267.

[168] ______, *A stable version of the Gromov-Lawson conjecture*, The Čech centennial (Boston, MA, 1993), 405–418, Contemp. Math., 181, AMS, Providence, RI, 1995.

[169] ______, *A general approach to the Gromov-Lawson-Rosenberg conjecture*, (preprint).

[170] J Rosenberg and S Weinberger, *G-indices and applications*, Ann. Sci. École Norm. Sup. **21** (1988), 479–495.

[171] R Schoen, *A remark on minimal hypercones*, Proc. Nat. Acad. Sci. USA **79** (1982), 4523–4524.

[172] R Schoen and S T Yau, *The structure of manifolds with positive scalar curvature*, Manuscripta Math. **28** (1979), 159–183.

[173] ______, *Complete three-dimensional manifolds with positive Ricci curvature and scalar curvature*, Ann. of Math. Stud **102** (1982), 209–228.

[174] ______, *The structure of manifolds with positive scalar curvature*, **Directions in partial differential equations** Publ. Math. Res. Center Univ. Wisconsin **54** (1987), 235–242.

[175] R Schultz, *Positive scalar curvature and odd order Abelian fundamental groups*, (preprint).

[176] JP Sha and D-G Yang, *Positive Ricci curvature on compact simply connected 4-manifolds*, Differential geometry: Riemannian geometry 529–538, Proc. Sympos. Pure Math., 54, Part 3, AMS, Providence, RI, 1993.

[177] ______, *Positive Ricci curvature on the connected sums of* $S^n \times S^m$, J Diff. Geo. **33** (1991), 127–137.

[178] E Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1968.

[179] S Stolz, *Simply connected manifolds of positive scalar curvature*, Ann. of Math. **196** (1992), 511–540.

[180] ______, *Splitting certain MSpin module spectra*, Topology **33** (1994), 159–180.

[181] ______, *A conjecture concerning positive Ricci curvature and the Witten genus*, Math. Ann. **304** (1996), 785–800.

[182] ______, *Positive scalar curvature metrics—existence and classification questions*, Proceedings of the International Congress of Mathematicians **2** (1994), 625–636.

[183] M Strake and G Walschap, *Connection metrics of nonnegative curvature on vector bundles*, Manuscripta Math. **66** (1990), 309–318.

[184] R. Takagi and S. Yorozu, *Notes on the Laplace-Beltrami operator on a foliated Riemannian manifold with a bundle-like metric*, Nihonkai Math. J. **1** (1990), 89–106.

[185] S Tanno, *Killing vectors on contact Riemannian manifolds and fiberings related to the Hopf fibration*, Tohoku Math. J, II. **23** (1971), 313–333.

[186] Gr. Tsagas, *The spectrum of the Hopf fibration*, Tensor New Ser. **50** (1991), 1–5.

[187] ______, *The spectrum of the Hopf fibration*, Rend. Ist. Mat. Univ. Trieste **22** (1990), 28–35.

[188] C Taubes, *SW* $\Rightarrow$ *Gr: from the Seiberg-Witten equations to pseudo holomorphic curves*, J AMS **9** (1996), 845–918.

[189] M Y. Wang and W Ziller, *On normal homogeneous Einstein manifolds*, Ann. Sci. École Norm. Sup. **18** (1985), 563–633.

[190] ______, *Existence and non existence of homogeneous Einstein metrics*, Invent Math **84** (1986), 177–194.

[191] ______, *Einstein metrics on principal torus bundles*, J Diff. Geo. **31** (1990), 215–248.

[192] B Watson, *Manifold maps commuting with the Laplacian*, J Diff. Geo. **8** (1973), 85–94.

[193] ______, *δ commuting mappings and Betti numbers*, Tohoku Math J **27** (1975), 135–152.
[194] ______, *Almost Hermitian submersions*, J Diff. Geo. **11** (1976), 147–165.
[195] ______, *G, G'-Riemannian submersions and nonlinear gauge field equations of general relativity,* **Global analysis–analysis on manifolds** Teubner-Texte Math. **57** (1983), 324–349.
[196] ______, *Riemannian submersions and instantons*, Math. Modelling **1** (1980), 381–393.
[197] G Wei, *Examples of complete manifolds of positive Ricci curvature with nilpotent isometry groups*, Bull. A M S. **19** (1988), 311–313.
[198] A Weinstein, *Fat bundles and symplectic manifolds*, Adv. in Math. **37** (1980), 239–250.
[199] H Weyl, *The Classical Groups*, Princeton University Press, 1946.
[200] Y. L Xin, *Riemannian submersion and equivariant harmonic maps*, **Differential geometry (Shanghai, 1991)** World Sci. Publishing, 272–287.
[201] H Yanamoto, *Conformally flat spaces of positive scalar curvature*, Res. Rep. Nagaoka Tech. College **8** (1972), 119–121.
[202] S T Yau, *On the structure of complete manifolds with positive scalar curvature*, Differential geometry and complex analysis (1985), 219–222.
[203] G Yu, *Zero in the spectrum conjecture, positive scalar curvature and asymptotic dimension*, Invent. Math. **127** (199), 99–126.

Index