Nova Série
BOLETIM
DA SOCIEDADE BRASILEIRA DE MATEMÁTICA

Bol. Soc. Bras. Mat., Vol. 28, N. 2, 141-153

Lagrangian flows: The dynamics of globally minimizing orbits

Ricardo Mañé

Abstract. The objective of this note is to present some results, to be proved in a forthcoming paper, about certain special solutions of the Euler-Lagrange equations on closed manifolds. Our main results extend to time dependent periodic Lagrangians with minor modifications.
We have chosen the autonomous case because this formally simpler framework allows to reach more easily the core of our concepts and results. Moreover the autonomous case exhibits certain special features involving the energy as a first integral that deserve special attention. They are closely related to the link found by Carneiro [**C**] between the energy and Mather's action function [**Ma**].

1. The Autonomous Case

Let L be a Lagrangian on a closed manifold M, i.e. $L: TM \to \mathbb{R}$ is a C^∞ function and has positive definite Hessian on the fibers. The Euler-Lagrange equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}(x,\dot{x})\right) - \frac{\partial L}{\partial v}(x,\dot{x}) = 0 \tag{E-L}$$

generates a smooth flow $f_t: TM \to TM$ defined as follows. Given $w = (p, v) \in TM$, denote $x_w: \mathbb{R} \to M$ the solution of (E-L) with initial condition:

$$x_w(0) = p$$
$$\dot{x}_w(0) = v.$$

Now define $f_t: TM \to TM$ by $f_t(w) = (x_w(t), \dot{x}_w(t))$ Hence every orbit $\gamma(t)$ of this flow can be uniquely written as $\gamma(t) = (x(t), \dot{x}(t))$, where $x(t)$ is a solution of (E-L).

It is well known that solutions of (E-L), and through them, orbits of the flow $f_t: TM \to TM$, are characterized by local variational properties.

Here we shall revisit an old subject: orbits of the flow f_t, selected in the intricate phase portrait of f_t, by requiring of them to satisfy global variational properties instead of the local ones that every orbit satisfies. Research on these special orbits goes back to Morse ([**Mo**], 1924) and Hedlund ([**H**], 1932), and has recently reappeared in the works of Bangert ([**B**]) and Mather ([**Ma1**], [**Ma2**]). Our approach while visibly conceptually indebted to those works, will be independent and self contained leading to new results and also to stronger forms of already known ones, like Mather's Graph Theorem ([**Ma**]) or the coboundary property ([**M**]).

Recall that the action of the Lagrangian L on an absolutely continuous curve $x: [a, b] \to M$ is defined by

$$S_L(x) = \int_a^b L(x(t), \dot{x}(t))dt.$$

Given two points $p_i \in M$, $i = 1, 2$ denote $Ac(p_1, p_2)$ the set of absolutely continuous curves $x: [0, T] \to M$, with $x(0) = p_1$, $x(T) = p_2$. For each $k \in \mathbb{R}$ we define the *action potential* $\Phi_k: M \times M \to \mathbb{R}$ by

$$\Phi_k(p_1, p_2) = \inf\{S_{L+k}(x) | x \in Ac(p_1, p_2)\}.$$

Theorem I. *There exists $c(L) \in \mathbb{R}$ (called the critical value of L) such that:*

a) *If $k < c(L)$, then $\Phi_k(p_1, p_2) = -\infty$, $\forall p_1, p_2 \in M$.*

b) *If $k \geq c(L)$, then $\Phi_k(p_1, p_2) > -\infty$, $\forall p_1, p_2 \in M$, and Φ_k is a Lipschitz function.*

c) *If $k \geq c(L)$, then*

$$\Phi_k(p_1, p_3) \leq \Phi_k(p_1, p_2) + \Phi_k(p_2, p_3), \qquad \forall p_1, p_2, p_3 \in M,$$
$$\Phi_k(p_1, p_2) + \Phi_k(p_2, p_1) \geq 0, \qquad \forall p_1, p_2 \in M.$$

d) *If $k > c(L)$, then $\Phi_k(p_1, p_2) + \Phi_k(p_1, p_2) > 0$, $\forall p_1 \neq p_2$ in M.*

Defining $d_k: M \times M \to \mathbb{R}$ by $d_k(p_1, p_2) = \Phi_k(p_1, p_2) + \Phi_k(p_2, p_1)$, the properties above say that $d_k(\cdot, \cdot)$ is a metric for $k > c(L)$ and a pseudometric for $k = c(L)$.

Denote $\mathcal{M}(L)$ the set of invariant probabilities of the flow f_t.

Theorem II. **(Ergodic Determination of** $c(L)$**.)**

$$c(L) = -\min\{\int L d\mu | \mu \in \mathcal{M}(L)\}.$$

Definition. *We say that* $\mu \in \mathcal{M}(L)$ *is a minimizing measure if*

$$c(L) = -\int L d\mu$$

We will denote by $\widehat{\mathcal{M}}(L)$ *the set of the minimizing measures in* $\mathcal{M}(L)$*.*

Generically, in the sense of [M], the structure of $\widehat{\mathcal{M}}(L)$ is simple.

Theorem III. *For a generic* L*,* $\widehat{\mathcal{M}}(L)$ *contains a single measure and this measure is uniquely ergodic. When this measure is supported by a periodic orbit, this orbit is hyperbolic.*

But we can hope even more:

Conjecture. *For a generic* L*,* $\widehat{\mathcal{M}}(L)$ *consist of a single measure supported in a periodic orbit.*

The prerequisite of the next definition is this remark: for every absolutely continuous $x: [a, b] \to M$ and all $k \geq c(L)$:

$$S_{L+k}(x) \geq \Phi_k(x(a), x(b)) \geq -\Phi_k(x(b), x(a)). \qquad (*)$$

Definition. *Set* $c = c(L)$*. We say that* $x: [a, b] \to M$ *is a semistatic curve if it is absolutely continuous and:*

$$S_{L+c}(x|_{[t_0,t_1]}) = \Phi_c(x(t_0), x(t_1)), \qquad (1)$$

for all $a < t_0 \leq t_1 < b$*; and that is a static curve if*

$$S_{L+c}(x|_{[t_0,t_1]}) = -\Phi_c(x(t_1), x(t_0)) \qquad (2)$$

for all $a < t_0 \leq t_1 < b$*.*

By $(*)$, equality (2) implies (1). Hence static curves are semistatic. Semistatic curves are solutions of (E-L). This follows from classic results that grant that absolutely continuous curves with much weaker variational properties are solutions.

Definition.

$$\Sigma(L) = \{w \in TM | x_w : \mathbb{R} \to M \textit{ is semistatic}\},$$
$$\widehat{\Sigma}(L) = \{w \in TM | x_w : \mathbb{R} \to M \textit{ is static}\},$$
$$\Sigma^+(L) = \{w \in TM | x_w|_{[0,\infty)} \textit{ is semistatic}\}.$$

Remarks.

a) Replacing c by any other real number in the definition of semistatic solution the set $\Sigma^+(L)$ (and then $\widehat{\Sigma}(L) \subset \Sigma(L) \subset \Sigma^+(L)$) becomes empty.

b) In [**Ma2**], Σ denotes what in our setting would be the closure of the union of the supports of the minimizing measures. This set is in general much smaller than the set called $\widehat{\Sigma}(L)$.

Theorem IV. (Characterization of Minimizing Measures). *A measure $\mu \in \mathcal{M}(L)$ is minimizing if and only if* $\text{supp}(\mu) \subset \widehat{\Sigma}(L)$.

Theorem V. (Recurrence Properties).

a) *$\Sigma(L)$ is chain transitive.*

b) *$\widehat{\Sigma}(L)$ is chain recurrent.*

c) *The ω-limit set of a semistatic orbit is contained in $\widehat{\Sigma}(L)$.*

Theorem VI. (Graph Properties).

a) *If $\gamma(t), t \geq 0$ is an orbit in $\Sigma^+(L)$ then denoting $\pi: TM \to M$ the canonical projection, the map $\pi|_{\{\gamma(t)|t\geq 0\}}$ is injective with Lipschitz inverse.*

b) *Denoting $\Sigma_0(L) \subset M$ the projection of $\widehat{\Sigma}(L)$, for every $p \in \Sigma_0(L)$ there exists a unique $\varphi(p) \in T_pM$ such that*

$$(p, \xi(p)) \in \Sigma^+(L).$$

Moreover

$$(p, \xi(p)) \in \widehat{\Sigma}(L),$$

and the vector field ξ is Lipschitz. Obviously $\widehat{\Sigma}(L) = \text{graph}(\varsigma)$.

The following result will imply the *covering property:*

$$\pi \Sigma^+(L) = M,$$

while also dealing with the *injectivity of π on certain subsets* of $\Sigma^+(L)$.

Define the pseudometric $d_c(\cdot,\cdot)$ on M by

$$d_c(a,b) = \Phi_c(a,b) + \Phi_c(b,a).$$

Denote by $\mathcal{G}$ these set of equivalence classes of the equivalence relation $d_c(a,b) = 0$ in $\widehat{\Sigma}(L)$. If $\Gamma \in \mathcal{G}$ set

$$\Gamma^+ = \{w \in \Sigma^+(L) | \omega(w) \subset \Gamma\}.$$

Obviously Γ^+ is forward invariant. Set:

$$\Gamma_0^+ = \bigcup_{t>0} \pi f_t \Gamma^+.$$

Theorem VII. (Covering Property). *If* $\Gamma \in \mathcal{G}$

a) $\pi\Gamma^+ = M$.

b) *For all* $p \in \Gamma_0^+$*, there exists a unique* $\xi\Gamma(p) \in T_pM$ *such that*

$$(p, \xi\Gamma(p)) \in \Gamma^+.$$

Moreover Γ_0^+ *is an open and dense subset of* M *and* $\xi\Gamma$ *is Lipschitz.*

Remarks.

a) The solutions of $\dot{x} = \xi\Gamma(x)$, are defined in $[0,\infty)$ and are semistatic curves.

b) On $\widehat{\Sigma}(L) \cap \Gamma$, we have $\xi\Gamma = \xi$.

The next result is an stronger form of Theorem III.

Theorem VIII. Generic Structure of $\Sigma(L)$). *For a generic Lagrangian* L*,* $\widehat{\Sigma}(L)$ *is a uniquely ergodic set and, if it is a periodic orbit, it is a hyperbolic periodic orbit.*

Now we can state the extension to all $\widehat{\Sigma}(L)$ of a property proved in [**M**] for supports of ergodic minimizing measures.

Theorem IX. (Coboundary Property). *If* $c = c(L)$*, then* $(L+c)|_{\widehat{\Sigma}(L)}$ *is a Lipschitz coboundary. More precisely, taking any* $p \in M$ *and defining* $G: \widehat{\Sigma}(L) \to \mathbb{R}$ *by*

$$G(w) = \Phi_c(p, \pi(w)),$$

then

$$(L+c)|_{\widehat{\Sigma}(L)} = \frac{dG}{df},$$

where

$$\frac{dG}{df}(w) := \lim_{h\to 0}\frac{1}{h}(G(f_h(w)) - g(w)).$$

Exploiting that the energy, $E: TM \to \mathbb{R}$, defined as usual by

$$E(x,v) = \frac{\partial L}{\partial v}(x,v)v - L(x,v)$$

is a first integral of the flow generated by L, leads to the *information on the position of* $\Sigma^+(L)$. First observe that it is easy to check that a semistatic curve $x: [a,b] \to M$ satisfies:

$$E(x(t), \dot{x}(t)) = c. \qquad (**)$$

This follows from calculating the derivative at $\lambda = 1$ of the function $F: \mathbb{R} \to \mathbb{R}$ given by

$$F(\lambda) = \int_a^b (L+c)(x_\lambda(t), \dot{x}_\lambda(t))dt,$$

where $x_\lambda: [a,b] \to M$ is given by $x_\lambda(t) = x(\lambda t)$.

From $(**)$ follows that

$$\Sigma(L) \subset E^{-1}(c),$$

that together with $\pi\Sigma(L) = M$ implies: $\pi E^{-1}(c) = M$. Hence,

$$c \geq \max_q E(q,0).$$

Moreover $\widehat{\Sigma}(L) \subset E^{-1}(c)$ implies:

Corollary. *$\mu \in \mathcal{M}(L)$ is minimizing if and only if*

$$\int \frac{\partial L}{\partial v}(x,v)v d\mu = 0,$$
$$\mathrm{supp}(\mu) \subset E^{-1}(c(L)).$$

From the view point of the variational calculus, the relevance of the critical value appears in the following results.

Theorem X. *If $k > c(L)$, for all $a, b \in M$, there exists a solution of (E-L) such that $x(0) = a$, $x(T) = b$ for some $T \geq 0$, and*

$$S_{L+k}(x|_{[0,T]}) = \min S_{L+k}(y),$$

where the minimum is taken over all the absolutely continuous $y\colon [0,T_1] \to M$, $T_1 \geq 0$, $y(0) = a$, $y(T_1) = b$. *Moreover, the solution* $x(t)$ *is contained in the energy level* $E^{-1}(k)$.

Using that on $E^{-1}(k)$ we obtain $L + k = (\partial L/\partial v)v$, it follow that:

$$(L+k)(x,v) = \frac{\partial L}{\partial v}(x,v)v,$$

on $E^{-1}(k)$, it follows that:

Corollary.

a) *If* $k > c(L)$ *and* $a, b \in M$, *there exists a solution* $x(t)$ *of* (E-L) *such that* $x(0) = a$, $x(T) = b$ *for some* $T \geq 0$, $E(x(t), \dot{x}(t)) = k$ *for all* $t \in \mathbb{R}$, *and*

$$\int_0^T \frac{\partial L}{\partial v}(x(t), \dot{x}(t))\dot{x}\, dt = \min \int_0^T \frac{\partial L}{\partial v}(y(t), \dot{y}(t))\dot{y}\, dt, \tag{3}$$

where the minimum is taken over all the absolutely continuous $y\colon [0,T_1] \to M$, $T_1 \geq 0$, $y(0) = a$, $y(T_1) = b$, *and* $E(y(t), \dot{y}(t)) = k$ *for a.e.* $t \in [0, T_1]$.

b) *Conversely, if given* $k > c(L)$, *and* $a, b \in M$, *there exists an absolutely continuous* $x\colon [0,T] \to M$, *with* $x(0) = a$, $x(T) = b$, $E(x(t), \dot{x}(t)) = k$ *and satisfying the minimization property (3), then* $x(t)$ *is a solution of (E-L).*

An interesting characterization of the critical value, in terms of an analogous to Tonelli's Theorem ([**Ma**]) in a prescribed energy level is given by the following result.

Theorem XI. *Suppose that* $k \in \mathbb{R}$ *has the following property: for all* a, $b \in \pi E^{-1}(k)$ *there exists an absolutely continuous curve* $x\colon [0,T] \to M$, $T \geq 0$, *such that:*

a) $E(x(t), \dot{x}(t)) = k$ *for a.e.* $t \in [0,T]$,

b) $x(0) = a$, $x(T) = b$,

c)

$$\int_0^T \frac{\partial L}{\partial v}(x(t), \dot{x}(t))\dot{x}(t)\, dt = \min \int_0^T \frac{\partial L}{\partial v}(y(t), \dot{y}(t))\dot{y}(t)\, dt,$$

where the minimum is taken over all the absolutely continuous $y\colon [0,T_1] \to M$, $T_1 \geq 0$, $y(0) = a$, $y(T_1) = b$, *and* $E(y(t), \dot{y}(t)) = k$ *for* *a.e.* $t \in [0, T_1]$.

Then $k > c(L)$ *and* $x(t)$ *is a solution of* (E-L).

Now let us prove the characterization of the minimizing measures given in Theorem IV.

Assume $c(L) = 0$, define $A_n: TM \to \mathbb{R}$, $F_i: TM \to \mathbb{R}$, $i = 1, 2$ by:

$$A_n(\theta) = S_L(x|_{[0,n]}),$$
$$F_1(\theta) = \Phi_0(\pi(\theta), \pi(f_1(\theta))),$$
$$F_2(\theta) = \Phi_0(\pi(f_1(\theta)), \pi(\theta)),$$

where $x: \mathbb{R} \to M$ is the solution of (E-L) with $(x(0), \dot{x}(0)) = \theta$.

Then by Birkhoff's Theorem, for all $\mu \in \mathcal{M}(L)$, we have:

$$\int L d\mu = \int A_n d\mu, \text{ for all } n,$$

and

$$\int F_1 d\mu \geq 0,$$

this last property because

$$\int F_1 d\mu = \int \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} F_1(f_j(\theta)) d\mu$$
$$= \int \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \Phi_0(\pi(f_j\theta), \pi(f_{j+1}(\theta)) d\mu$$
$$\geq \int \lim_{n\to\infty} \frac{1}{n} \Phi_0(\pi(\theta), \pi(f_n(\theta)) d\mu = 0.$$

Moreover from the definition of Φ_0:

$$A_1 \geq F_1. \tag{4}$$

Now suppose that μ is minimizing. Then:

$$c(L) = 0 = \int L d\mu = \int A_1 d\mu \geq \int F_1 d\mu \geq 0.$$

Hence

$$\int A_1 d\mu = \int F_1 d\mu,$$

that, by (4), implies $A_1 = F_1$ μ-a.e. By the continuity of both functions, we get

$$A_1(\theta) = F_1(\theta), \text{ for all } \theta \in \text{supp}(\mu). \tag{5}$$

Then, if we prove:

$$A_1(\theta) = -F_2(\theta), \text{ for all } \theta \in \operatorname{supp}\mu, \tag{6}$$

the prove of $\operatorname{supp}(\mu) \subset \widehat{\Sigma}(L)$ will be complete. To prove (6) we use that

$$\int L d\mu = 0$$

implies that for a.e. θ, there exists a sequence $n_j \to \infty$ such that:

$$A_{n_j}(\theta) \to 0, \tag{7}$$

$$d(\theta, f_{n_j}(\theta)) \to 0. \tag{8}$$

From (8) follows that

$$F_2(\theta) \le \lim_{j\to+\infty} A_{n_j-1}(f_{n_j-1}(\theta)).$$

Since

$$A_{n_j}(\theta) = A_1(\theta) + A_{n_j-1}(f_{n_j-1}(\theta))$$

and (7), we obtain $F_2(\theta) \le -A_1(\theta)$. Using (5), we get $F_2(\theta) \le -F_1(\theta)$. Using that $F_2 + F_1 \ge 0$ (because $d_0(\cdot,\cdot)$ is a pseudometric), we obtain (6).

Now suppose $\operatorname{supp}(\mu) \subset \widehat{\Sigma}(L)$. On $\widehat{\Sigma}(L)$ we have $A_1 = F_1 = -F_2$ by definition of $\widehat{\Sigma}(L)$. Then

$$\int L d\mu = \int A_1 d\mu = \int F_1 d\mu \ge 0.$$

Moreover the same argument used to prove that the μ-average of F_1 is positive can be used to prove that the μ-average of F_2 is positive. Then:

$$\int L d\mu = \int A_1 d\mu = -\int F_2 d\mu \le 0. \qquad \square$$

Finally, let us recall a weaker concept of global minimization (taken from Bangert [**B**]) that, as the next three results will show, has many interesting connections with the forms of global minimization introduced above.

Definition. *We say that a solution $x(t)$ of* (E-L) *is a minimizer (resp. forward minimizer) if*

$$S_L(x|_{[t_0,t_1]}) \le S_L(y),$$

for every $t_0 \leq t_1$ *(resp.* $0 < t_0 \leq t_1$*) and every absolutely continuous* $y\colon [t_0, t_1] \to M$ *with* $y(t_i) = x(t_i)$, $i = 1, 2$.

Denote $\Lambda(L)$ (resp. $\Lambda^+(L)$) the set of $(p, v) \in TM$ such that the solution $x(t)$ of (E-L) with initial condition $(x(0), \dot{x}(0)) = (p, v)$ is a minimizer (resp. a forward minimizer).

Theorem XII. *The* ω*-limit set of an orbit in* $\Lambda^+(L)$ *is contained in* $\widehat{\Sigma}(L)$.

Theorem XIII. $f_t|_{\Lambda(L)}$ *is chain transitive.*

Theorem XIV.

a) *There exists* $C > 0$ *such that setting* $c = c(L)$,

$$\left|S_L(x|_{[t_0,t_1]})\right| \leq C,$$

for every forward minimizer $x(t)$ *and all* $0 \leq t_0 \leq t_1$.

b) *If* $x(t)$ *is a forward minimizer and* $p \in M$ *is such that*

$$p = \lim_{n \to +\infty} x(t_n),$$

for some sequence $t_n \to +\infty$, *then the limit*

$$\lim_{n \to +\infty} S_L(x|_{[t_0,t_n]}),$$

exists and depends only on p.

Now let us recall a device, exploited at length and in a protagonic role in [**B**], [**Ma1**], [**Ma2**] to enlarge the scope of the methods presented above. It consists in observing that the Lagrangians L and $L - \theta$, where θ is a closed 1-form on M, generate the *same* flow. Then the set of minimizing measures of $L - \theta$, to be denoted by $\widehat{\mathcal{M}}^\theta(L)$ is contained in $\widehat{\mathcal{M}}(L)$, and the subsets of TM given by $\Sigma(L - \theta)$, $\widehat{\Sigma}(L - \theta)$, $\Lambda(L - \theta)$, $\Lambda^+(L - \theta)$ are invariant sets of f_t. All these sets, as well as the critical value $c(L - \theta)$ of $L - \theta$ depend only on the cohomology class $[\theta] \in H^1(M, \mathbb{R})$ of θ. It is not difficult to check the convexity of the function $\beta^*\colon H^1(M, \mathbb{R}) \to \mathbb{R}$ defined by

$$\beta^*([\theta]) = c(L - \theta).$$

(This is the dual of Mather's action function $\beta\colon H_1(M, \mathbb{R}) \to \mathbb{R}$). Define the *strict critical value* $c_0(L)$ by

$$c_0(L) = \min_{\theta} c(L - \theta).$$

Using the concept of homology (or asymptotic cycle) of a $\mu \in \mathcal{M}(L)$, it can be proved that:

$$-c_0(L) = \min \left\{ \int L d\mu \,\middle|\, \mu \in \mathcal{M}(L), \rho(\mu) = 0 \right\}.$$

This is part of the duality between β^* and Mather's action function.

Observe that in an energy level $E^{-1}(c)$ with $c > c_0(L)$, the results of Theorem X and its Corollary can be applied replacing L by $L - \theta$ where θ is a closed form satisfying:

$$c > c(L - \theta) > c_0(L).$$

Moreover, observe that

$$c_0(L) \geq \max_x E(x, 0),$$

because the energy functions of L and $L - \theta$ coincide. The equality holds when L is a mechanical Legrangian, i.e. of the form

$$L(x, v) = \frac{1}{2}\langle v, v\rangle_x - V(x),$$

where $\langle \cdot, \cdot \rangle_x$ is a Riemannian structure on M, and $V\colon M \to \mathbb{R}$ is a potential. In fact, in this case $c(L) = \max_x E(x, 0)$ and the minimizing measures are linear combinations of the Dirac probabilities concentrated at the maximums of V.

But in general the equality doesn't hold. An example will be given after the following corollary of Theorem X.

Corollary. *If $k > c_0(L)$, for every free homotopy class of M there exists a periodic orbit in $E^{-1}(k)$ such that its projection on M belongs to that free homotopy class.*

Now let us exhibit a Lagrangian on the two dimensional torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ having energy levels $E^{-1}(k)$, $k \in [a, b]$ with $b > \max_x E(x, 0)$, such that the Corollary above doesn't hold in $E^{-1}(k)$; hence $c_0(L) > b > \max_x E(x, 0)$.

The example will be a Lagrangian of the form:

$$L_\lambda(x, v) = \frac{1}{2}\langle v, v\rangle_x + \lambda\,\psi_1(x_2)\,v_1 - \psi_2(x_2), \quad \lambda \in \mathbb{R},$$

where $\langle\cdot,\cdot\rangle_x$ is the Euclidean inner product and ψ_1, ψ_2 are bump functions as in the figure below.

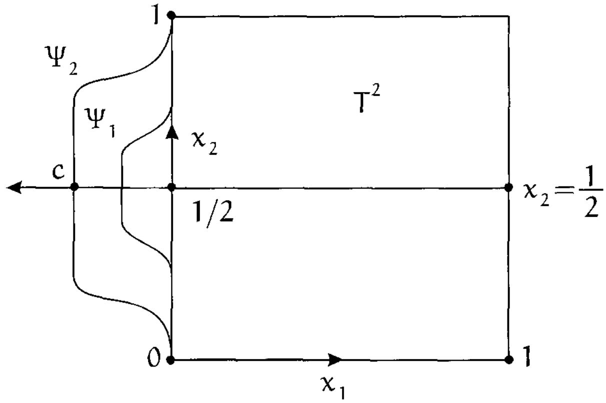

Figure 1.

Then the energy is

$$E(x, v) = \frac{1}{2}\|v\|^2 + \psi_2(x_2),$$

independently of λ. When $\lambda > 0$ the effect of the term $\lambda\psi_1(x_2)v_1$ is that of a magnetic field normal to the plane supported in the band $[0,1] \times \text{supp}(\psi_1) \simeq \mathbb{R}/\mathbb{Z} \times \text{supp}(\psi_1)$. Denote $c = \max\psi_2$ and $[k_1, k_2] = \psi_2^{-1}(c)$. Then $0 < k_1 < \frac{1}{2} < k_2 < 1$. Denote $S_\lambda \subset T^2$ the set of points that can be reached from a point (x_1, x_2) with $x_2 = 1$, through a solution of (E-L) with $L = L_\lambda$, contained in the level $E = c$ (i.e. the initial velocity satisfies $\frac{1}{2}\|v\|^2 = c$). For $\lambda = 0$, $S_0 = T^2 - [k_1, k_2] \times \mathbb{R}/\mathbb{Z}$. When $\lambda > 0$, S_λ diminishes, say $S_\lambda \subset [k_1(\lambda), k_2(\lambda)] \times \mathbb{R}/\mathbb{Z}$ with $0 < k_1(\lambda) < k_1 < k_2 < k_2(\lambda) < 1$. Then the level $E^{-1}(c)$, for $\lambda > 0$ doesn't contain an orbit whose projection is in the homotopy class of $(0, x_2)$, $x_2 \in \mathbb{R}$, because such orbits would intersect $x_2 = 1$.

Since the minimizing measures of $L - \theta$ share the same basic properties, it is convenient, following Mather [**Ma1**], to extend the term minimizing measures to all the measures in

$$\mathcal{M}^*(L) = \bigcup_\theta \mathcal{M}^\theta(L).$$

Similarly it is also convenient to extend the terms semistatic, static, and minimizing orbits to all the orbits having the corresponding property for some $L - \theta$. The sets

$$\Sigma^*(L) = \bigcup_\theta \Sigma(L-\theta), \quad \widehat{\Sigma}^*(L) = \bigcup_\theta \widehat{\Sigma}(L-\theta)$$

and

$$\Lambda(L) = \bigcup_\theta \Lambda(L-\theta),$$

are closed, $\widehat{\Sigma}(L-\theta)$ and $\Lambda(L-\theta)$ are upper semicontinuous functions of $[\theta]$. From this property follows that:

Corollary. $\Sigma^*(L)$ *is chain transitive.*

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Ricardo Mañé
IMPA
Estrada Dona Castorina, 110
22460-320 Jardim Botânico
Rio de Janeiro
Brasil