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THE ADVANTAGE OF CAPITALISM VS. SOCIALISM DEPENDS ON THE CRITERION

Abstract. Talk given at the Linnik symposium, St. Petersburg, April 2005.

We consider two distinct government tax policies towards companies: the republican policy gives tax breaks to the richer companies while the democratic policy would perhaps give breaks to the weaker companies in hopes to keep them alive and so reduce unemployment. Which policy is better? We show that it depends on the criterion of optimization, at least for the case of two companies, which is all that we can handle, in the stated mathematical formulation of the question.

Suppose that two standard Brownian motion processes are controllable by adding processes of nonnegative drifts with sum unity to each process non-anticipatively. Thus the controlled processes satisfy for $i = 1, 2$,

$$dX_i(t) = \mu_i(t)dt + dW_i(t), t \geqslant 0, X_i(0) = x_i,$$

where $\mu_i(t) \geqslant 0, i = 1, 2, \mu_1(t) + \mu_2(t) \leqslant 1, t \geqslant 0$, and we are allowed to change the drifts μ_i instantaneously but without knowing the future increments of the independent standard Brownian motions W_i. It is assumed that if $X_i(t)$ hits zero for some finite t then company i is bankrupt and and it disappears.

We denote by ν the number, 0, 1, or 2 of X_i which survive forever, i.e., which never hit zero. We show that to maximize $P(\nu = 2)$, the probability that $\nu = 2$, i.e. that both companies survive forever, the optimal policy is to always apply the drift to the bottom company. This is rigorously proven. We also argue that if the criterion is to maximize $E\nu$, the *expected number* of companies that survive forever, then the optimal policy sometimes pushes the upper company. We have no rigorous proof that this is correct but the result has been verified numerically in two different ways and the evidence seems quite convincing. The problem is much harder if $n > 2$ or if the diffusion constants of the two Brownian motions are unequal.

The original mathematical problem was formulated by David Aldous.[1]

Background and methodology involved

We begin with a review of stochastic optimization theory in order to explain some of the mathematical difficulties of the problem. Problems of stochastic optimization in finance and economics have a long history perhaps beginning with von Neumann and Morganstern's "Theory of Games and Economic Behavior" which motivated and initiated the theory of linear programming. Optimal control and optimal stopping problems came along later. Early examples are:

Pricing the perpetual American option, posed by Samuelson, solved by McKean;

Maximizing the probability of getting to a specific wealth in a sub-fair casino, posed and solved by Dubins and Savage.

Such examples involved martingales and thus appeared to be a new problem direction but Dynkin understood clearly that it reduces to linear programming. Though these problems were infinite dimensional, the technique of smooth fit (which I learned in Russia 40 years ago) sometimes enabled optimal solutions to be guessed. Ito calculus played an important role in giving a simple criterion for the guessed optimal process to be a martingale or a submartingale. The technique was used in a number of cases to solve problems relevant to finance and economics, without having to solve the nonlinear differential equations which arise in the dynamic programming approach.

Difficulty number one: some problems are too discrete

Suppose the mafia will kill you if you do not repay a debt to them. That is you must turn a fortune $0 \leqslant f_0 < 1$ into one of size 1. Suppose 1 means one million rubles in a casino where there is only one *subfair* "double-or-nothing" bet. One can stake any amount $s \leqslant f_0$ and reach the fortune $f_1 = f_0 + s$ with probability $p < \frac{1}{2}$, or $f_1 = f_0 - s$ with probability $q = 1 - p > \frac{1}{2}$.

It was proved that *bold play*, where one stakes $s = \min(f_0, 1 - f_0)$ at each bet until one either goes broke or reaches the goal is optimal.

Does it really need proof?

[1] We consider also the more general version, suggested by Oded Palmon, where $\mu_i \geqslant -\delta$. In this version, money may be "stolen" from the rich and given to the poor.

If, instead, after every bet an interest payment is extracted so that $f_1 = \frac{f_0+s}{1+a}$ with probability $p < \frac{1}{2}$, or $f_1 = \frac{f_0-s}{1+a}$ with probability $q = 1-p$, by dividing by $1+a > 1$ then it is even more "evident" that bold play is optimal, *but this is in fact false!* The optimal strategy appears to be very difficult to find.

The reason that this class of problems is difficult is that there is too much discreteness and Ito or differential calculus is not involved.

Difficulty number two: some problems are higher dimensional

An early one-dimensional problem, now "easy", is to price the perpetual American option, posed by Samuelson and solved by McKean in 1965:

$$V(x) = \sup_{\tau} E_x(X(\tau) - k)^+ e^{-r\tau}$$

where

$$X = X(t) = x_0 e^{\sigma W(t) + (\frac{\sigma^2}{2} - \mu)t},$$

or

$$dX(t) = X(t)(\mu dt + \sigma dW(t)), X(0) = x_0.$$

Using Ito calculus and the principle of smooth fit, the value of the option was found and the optimal $\tau = \tau_a$ was shown to be the first passage time to a specific level $a > k$.

Xin Guo generalized this result to models involving arbitrage and insider information in her (2000) thesis.

For problems in *economics* rather than finance, one makes corporate decisions, and to study this Radner introduced a model under which optimal decisions of direction and profit taking can be studied. In Radner's model, the net worth of the company at time t follows the sde

$$dX(t) = \mu_{I(t)}dt + \sigma_{I(t)}dW(t) - dZ(t), x(0) = x_0$$

where $I(t) \in \{1, \ldots, n\}$ and $Z(t) \uparrow$ is the total dividends or profit taken up to time t. The set $\mathcal{S} = \{(\mu_i, \sigma_i), i = 1, \ldots, n\}$ are the available corporate directions and one seeks to maximize the total discounted expected profit

$$V(x_0, \mathcal{S}) = \sup_{Z,I} E_{x_0} \int_0^{\tau_0} e^{-rt} dZ(t)$$

until bankruptcy at time $\tau_0 \leqslant \infty$ by choosing a nonanticipating corporate direction $I(t)$ and a nonanticipating profit-taking policy $Z(t)$. The value $V(x_0)$ is the true worth of the company and the complete solution was found.

The company in the Radner model goes bankrupt in a finite time w.p. 1.

This may be thought of as the consequence of unrestrained profit maximization.

The above problems that have been solved up to now in finance, economics, or engineering, have always involved a single Brownian motion. This effectively reduces the guess of the optimal control boundary to guesssing a single threshold, possibly one for each value of a parameter, and smooth-fit enables the threshold to be guessed because the Ito equation is one-dimensional and it's only necessary to solve an ordinary differential equation to apply smooth fit. But the problem of this paper involves two independent Brownian motions and hence its solution breaks new ground.

Difficulty number 3: one needs an explicit solution to make a proof

If one manages to guess the optimal control strategy *and one can find the value of the optimal payoff under this strategy* then giving a proof that the optimum is the guessed answer is quite straightforward. One can verify easily from the explicit formula that the process $Z(t) = \hat{V}(X_t, Y_t)$ is a supermartingale for any policy.

The supermartingale inequality then gives that no policy can do better than pushing the bottom company and since pushing the bottom company is a legitimate policy this proves optimality and solves the problem.

We can only find $\hat{V}$ in the one case when we are maximizing under the "democratic party criterion" of maximizing $P(\nu = 2)$, and even then, only when $\delta = 0$.

The optimal control policy, republican vs. democrat, for two companies

$$dX_i = \mu_i(t)dt + dW_i, \quad X_i(0) = x_i, i = 1, 2, t \geqslant 0,$$

where $W_i(t)$ are independent Brownian motions, $x_i \geqslant 0, i = 1, 2$ are given, and for every $t \geqslant 0$ we must choose $\mu_= \mu_i(t), t \geqslant 0$,

$$\mu_i \geqslant -\delta, i = 1, 2, \mu_1 + \mu_2 = 1,$$

where $\infty > \delta \geqslant -\frac{1}{2}$ is given. Note that this model also applies to Russian tax policy where large firms like Yukos may be subject to a rather large δ penalty.

Of course better results can be obtained if one knows the future and this is the reason that one cannot avoid martingale methods since martingales are the only way to use the restriction to nonanticipative strategies.

We will prove rigorously (but only in the case $\delta = 0$) that the democratic party policy (or the policy of Putin vis-vis Yukos) is optimal by explicitly constructing the probability that both companies survive forever under this "Robin Hood" policy. A proof that this Robin Hood policy is actually optimal for all δ (but without an explicit formula for the value function) has been found by (appropriately) Robin Pemantle using a very simple coupling argument. In fact Robin's argument shows that for n companies with the same diffusion constants that pushing the bottom company is best to maximize the chance that all companies survive.

Note that this guess of the optimal policy is completely similar to the guess of the particular extreme point of a convex set on which one solves a linear programming problem as was mentioned.

Consider then the particular function,

$$V(x_1, x_2) = 1 - e^{-2\min(x_1,x_2)} - 2\min(x_1, x_2)e^{-x_1-x_2}.$$

Suppose π is any policy of pushing one or the other company which is properly nonanticipating and let $V^\pi(x_1, x_2)$ denote the probability that, under policy π, both companies survive forever. We want to show that $V^\pi \leqslant V$, and that V is the probability obtained when the bottom company is pushed.

The process $Z(t) = V(X_1^\pi(t), X_2^\pi)$ is a supermartingale if and only if its Ito differential is locally expectation decreasing, i.e.,

$$EdZ(t) = \frac{1}{2}(V_{11} + V_{22}) + \max(V_1, V_2) \leqslant 0 \qquad (1)$$

with subscripts denoting partial derivatives, at every (x_1, x_2) in the first quadrant. We also need that $V \in C_2$ on the diagonal $x_1 = x_2$, which it is.

The drift of Y is

$$-2e^{-2x_1} + 2^{-x_1-x_2} - 2x_1e^{-x_1-x_2} \leqslant 0.$$

Thus $EY(\infty) \leqslant Y(0) = V(x_1, x_2)$.

But at $t = \infty$, either both companies are at infinity or one has gone broke. In the first case, $Y(\infty) = 1$ and in the second case $Y(\infty) = 0$ because $V(0, \cdot) = 0$, and $V(\infty, \infty) = 1$. Thus the supermartingale inequality gives that for any π, the probability that both companies survive forever is at most $V(x_1, x_2)$. This chance will be strictly less than $V(x_1, x_2)$ unless one never pushes the maximum company.

Finally, since V is bounded, the policy π_0 where the bottom company is always pushed achieves equality throughout the proof and so we see that V is the value achieved under π_0. This completes the proof.

How was V guessed?

With great luck and also great difficulty we (Henry McKean) solved the pde, the Bellman equation. It was fortunate that it had an elementary solution which was guessable. We thought it might be possible to "organize" the guess using the Kontorovich-Lebedev transform, because in the region $x_1 \leqslant x_2$, if we make the transformation

$$H(x_1, x_2) = e^{x_1} V(x_1, x_2)$$

then H satisies the Helmholtz equation:

$$H_{11} + H_{22} = H.$$

We can write with polar coordinates (r, θ) for (x_1, x_2), the representtation,

$$V = e^{-x_1} \int_0^\infty K_{i\nu}(r)[A(\nu)\cosh(\nu\theta) + B(\nu)\sinh(\nu\theta)]d\nu,$$

but it does not seem to be possible to derive the answer this way even for the case $\delta = 0$ where we know it.

This raises the question of avoiding having to actually solve the pde and just relying on some existence theorem. Is it possible to utilize the martingale/submartingale proof above when one only has an existence proof rather than an explicit guessed solution, as above. Some think not (including me); there are no complete examples of this possibility that we are aware of.

The main result would seem to prove that socialism has some advantages since it is certainly better for society if everyone remains employed and companies remain solvent. But George Bush does not want to maximize the probability that *all* companies remain solvent but instead he wants to maximize say the *expected* number of companies that remain solvent. (What Putin wants is less clear.) It is tempting to conjecture that pushing the bottom company would still be optimal but it does not seem to be so.

We first generalize the criterion. Suppose the problem is to maximize the expression

$$(1-\alpha)P(\nu_\infty = 2) + \alpha P(\nu_\infty = 1)$$

where $\nu_\infty = \nu_\infty^\pi$ is the number of companies that never go bankrupt under the policy π. We want to maximize the expression over all nonanticipating policies π. Here α is a parameter and we note that when $\alpha = 0$ the problem reduces to maximizing the probability that both companies survive forever, while when $\alpha = \frac{1}{3}$ the problem is equivalent (up to a multiplicative factor of 2) to maximizing the expected number of companies that survive forever.

We calculated numerically a direct solution of the following nonlinear pde in a 500 x 500 pixel array, and obtained an approximate optimal payoff function.

$$\frac{1}{2}(V_{11} + V_{22}) + \max(V_1(x_1, x_2), V_2(x_1, x_2)) = 0,$$

with the boundary condition (coming from the fact that a Brownian motion with unit drift and diffusion stating at x_2 has probability $1 - e^{-2x_2}$ of never hitting zero), along the axes,

$$V(0, x_2) = 1 - e^{-2x_2}, V(x_1, 0) = 1 - e^{-2x_1},$$

and at infinity,

$$V(x_1, \infty) = (1-\alpha)(1 - e^{-2x_1}) + \alpha e^{-2x_1},$$

$$V(\infty, x_2) = (1-\alpha)(1 - e^{-2x_2}) + \alpha e^{-2x_2},$$

which provides enough data to solve the numerical equation if ∞ is replaced by a finite value and a discrete matrix is used instead of a continuous square.

We calculated the approximate optimal V and then indicated with a black dot each pixel in which the maximum in the nonlinear pde occurs at the smaller of x_1, x_2, i.e. at the bottom point. We left the pixel white, otherwise.

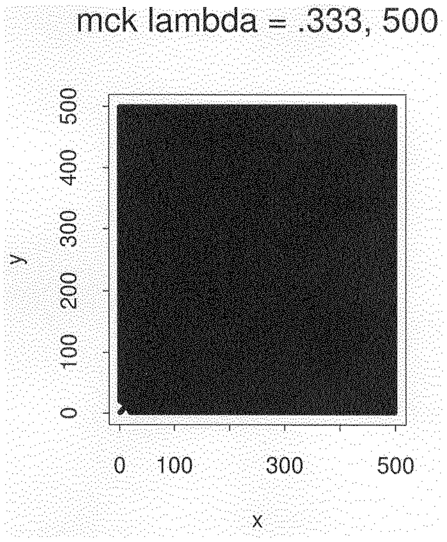

Fig. 1. The black dots indicates those values x_1, x_2 of the companies at which the bottom company is optimally pushed. Note that on the diagonal $x_1 = x_2$ either company can be pushed. Note that near the origin, i.e. when both companies are in danger of going bankrupt it is optimal to push the top company under this "republican" criterion of optimization. Here $\alpha = \frac{1}{3}$, corresponding to maximizing the expected number of companies surviving forever. .

Show pictures.

The figures indicate that the boundary of the region where the top company gets optimally pushed intersects the axes, but I think this is an artifact of the discrete approximation made in solving the pde. Robin Pemantle kindly pointed this out to me.

We are grateful to Jerry Han for checking the above result by writing a linear program as an alternative numerical check on our direct solution of the above nonlinear pde. The results were in agreement. Jerry also considered the case when there are 3 companies and he will publish his results separately. It seems extremly likely that with any number of companies (all with the same diffusion coefficient) and with drifts $\geqslant -\delta$ summing to one, one should push the bottom company in order to maximize the chance that all companies survive forever. But we could note

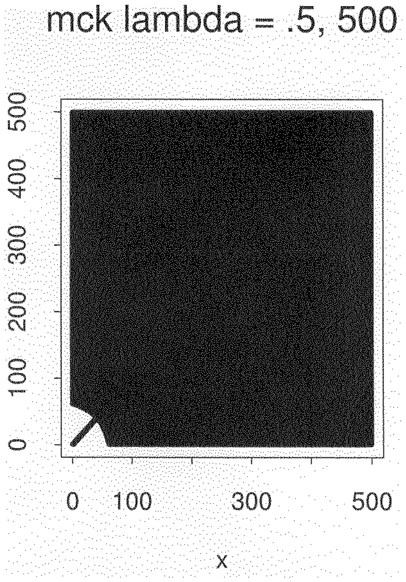

Fig. 2. Now $\alpha = \frac{1}{2}$. The black dots indicates those values x_1, x_2 of the companies at which the bottom company is optimally pushed. Note that on the diagonal $x_1 = x_2$ either company can be pushed. Note that again near the origin, but now the region gets larger, i.e. when both companies are in danger of going bankrupt it is optimal to push the top company under this 'republican" criterion of optimization. .

solve the pde except for $n = 2$ and $\delta = 0$.

Mathematics is a stingy god.

Consider two distinct government tax policies towards companies: the republican policy gives tax breaks to the richer companies while the democratic policy would perhaps give breaks to the weaker companies in hopes to keep them alive and so reduce unemployment.

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