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FORMALITY OF THE COMPLEMENTS OF SUBSPACE ARRANGEMENTS WITH GEOMETRIC LATTICES

Abstract. We show that, for an arrangement of subspaces in a complex vector space with geometric intersection lattice, the complement of the arrangement is formal. We prove that the Morgan rational model for such an arrangement complement is formal as a differential graded algebra.

1. Introduction

Let $\mathcal{A}$ be an arrangement of linear subspaces in a complex vector space V; we denote the complement of $\mathcal{A}$ by $\mathcal{M}(\mathcal{A}) := V \setminus \bigcup \mathcal{A}$. In this note we address the question whether $\mathcal{M}(\mathcal{A})$ is a formal space. If $\mathcal{A}$ is an arrangement of hyperplanes, then the formality of the arrangement complement follows immediately from Brieskorn's result [1, Lemma 5], which gives an embedding of $H^*(\mathcal{M}(\mathcal{A}))$ into the deRham complex of $\mathcal{M}(\mathcal{A})$. In the general case, however, the question is more subtle.

Proving formality can be reduced to an algebraic question. Indeed, using results of Morgan from [5], to prove that $\mathcal{M}(\mathcal{A})$ is formal it suffices to prove that the Morgan rational model of $\mathcal{M}(\mathcal{A})$ is formal as a differential graded algebra.

Formality of a differential graded algebra, i.e., the property that it is quasi-isomorphic to its cohomology algebra, is preserved under the equivalence relation on differential graded algebras that is generated by quasi-isomorphisms. Using this fact, we construct a sequence of quasi-isomorphic differential graded algebras, starting from the Morgan rational model of $\mathcal{M}(\mathcal{A})$ and finishing on a differential graded algebra with a quasi-isomorphism to its cohomology algebra.

Morgan rational models for arrangement complements have not only been described explicitly in work of De Concini and Procesi [2], there are also a number of quasi-isomorphic variations at hand. The first quasi-isomorphism that we rely on is the quasi-isomorphism between the De Concini-Procesi rational model for arrangement complements and a differential graded algebra $\mathrm{CM}_{\mathcal{A}}$ introduced by the second author in [10].

Its underlying chain complex is the flag complex of the intersection lattice of $\mathcal{A}$. We go further and prove that $\mathrm{CM}_{\mathcal{A}}$ in turn is quasi-isomorphic to a differential graded algebra $D_{\mathcal{A}}$, which as well has been introduced by the second author [7]. In contrast to $\mathrm{CM}_{\mathcal{A}}$, the underlying chain complex of $D_{\mathcal{A}}$ is the relative atomic complex of the intersection lattice of $\mathcal{A}$.

It is for $D_{\mathcal{A}}$ that we prove formality in the case that the intersection lattice of the arrangement is a geometric lattice. Only in this last part of our work, we have to rely on specific properties of geometric intersection lattices. We remark here that formality of arrangements with geometric intersection lattices was mentioned without a proof in [10, Remark 7.3(ii)], and was used recently in work of Papadima and Suciu [6]. Our goal here is to present all the details of the proof.

The paper is organized as follows: We start out by recalling the definition of quasi-isomorphism and formality for differential graded algebras. We then introduce the central for our approach object, the differential graded algebra $D_{\mathcal{A}}$ for a given subspace arrangement. In Section 2.3 we explain its role for arrangement cohomology by showing that it is quasi-isomorphic to the arrangement model $\mathrm{CM}_{\mathcal{A}}$ constructed in [10]. Finally, Section 3 is devoted to proving that $D_{\mathcal{A}}$ is formal if the arrangement has a geometric intersection lattice. In the last section, we discuss the perspectives and limitations of our approach towards proving formality for more general subspace arrangements.

2. Preliminaries

2.1. Formality of differential graded algebras. We recall the definitions of quasi-isomorphisms of differential graded algebras (d.g.a.), and of the equivalence relation they generate. Henceforth, we give the concept of formality that we will use in the sequel.

Definition 2.1. *Let (E, d_E) and (F, d_F) be differential graded algebras. A d.g.a. morphism $h : (E, d_E) \to (F, d_F)$ is called quasi-isomorphism if the induced map in cohomology, $h^* : H^*(E) \to H^*(F)$, is an isomorphism.*

The existence of a quasi-isomorphism from one differential graded algebra to another defines a relation which is not in general symmetric or transitive. We consider the equivalence relation of d.g.a.'s that is generated by quasi-isomorphisms, and we say that two d.g.a.'s E and F are *quasi-isomorphic* if there exists a finite sequence of d.g.a.'s starting on E and finishing on F, each neighboring pair being connected by a quasi-isomorphism in at least one direction.

Definition 2.2. *A differential graded algebra (E, d_E) is called formal if it is quasi-isomorphic to its cohomology algebra $H^*(E)$ with zero differential.*

Notice that formality of a differential graded algebra is invariant under the relation of being quasi-isomorphic.

2.2. The relative atomic differential graded algebra of an arrangement. Let $\mathcal{A}$ be an arrangement of linear subspaces in a complex vector space, and let $\mathcal{L}$ denote its intersection lattice, i.e., the poset of intersections among subspaces in $\mathcal{A}$ ordered by reversed inclusion. We will frequently refer to the (complex) codimension, $\operatorname{cd} A$, of elements A in $\mathcal{L}$ as subspaces in the ambient space of $\mathcal{A}$. We assume that spaces in $\mathcal{A}$ are inclusion maximal, hence in one-to-one correspondence with the atoms $\mathfrak{A}(\mathcal{L})$ in $\mathcal{L}$. Furthermore, we fix a linear order on the set of atoms $\mathfrak{A}(\mathcal{L})$.

We define the *relative atomic differential graded algebra* $D_{\mathcal{A}}$ associated with an arrangement $\mathcal{A}$ as follows. The underlying chain complex (D, d) is the *relative atomic complex*, with coefficients in $\mathbb{Q}$. We recall its definition from [7, Def. 2.2]. The complex (D, d) is generated by all subsets σ of $\mathfrak{A}(\mathcal{L})$, and for $\sigma = \{i_1, \dots, i_k\}$ in $\mathfrak{A}(\mathcal{L})$, the differential d is defined by

$$d\sigma \;=\; \sum_{j:\bigvee \sigma_j = \bigvee \sigma} (-1)^j \, \sigma_j \tag{2.1}$$

where $\sigma_j = \sigma \setminus \{i_j\}$ for $j = 1, \dots, k$, and the indexing of elements in σ follows the linear order imposed on $\mathfrak{A}(\mathcal{L})$. With $\deg(\sigma) = 2 \operatorname{cd} \bigvee \sigma - |\sigma|$, (D, d) is a cochain complex.

We define a multiplication on (D, d) as follows. For subsets σ and τ in $\mathfrak{A}(\mathcal{L})$,

$$\sigma \cdot \tau \;=\; \begin{cases} (-1)^{\operatorname{sgn}\epsilon(\sigma,\tau)} \sigma \cup \tau & \text{if} \quad \operatorname{cd}\bigvee \sigma + \operatorname{cd}\bigvee \tau = \operatorname{cd}\bigvee(\sigma \cup \tau) \\ 0 & \text{otherwise} \end{cases}, \tag{2.2}$$

where $\epsilon(\sigma, \tau)$ is the permutation that, applied to $\sigma \cup \tau$ with the induced linear order, places elements of τ after elements of σ, both in the induced linear order.

Theorem 2.3 ([8, Prop. 3.1]). *For any arrangement $\mathcal{A}$ of complex subspaces, $D_{\mathcal{A}}$ with underlying chain complex (D, d) and multiplication defined in (2.2) is a differential graded algebra.*

Remark 2.4. **(1)** A differential graded algebra similar to the one discussed here, can be defined for arbitrary lattices with a labelling of elements that satisfies certain rank-like conditions. For a detailed discussion of the general context, see [8, Section 3].

(2) Recall that there are two abstract simplicial complexes associated with any finite lattice $\mathcal{L}$: the *flag complex* or *order complex* $F(\mathcal{L})$ and the *atomic complex* $C(\mathcal{L})$. The flag complex is formed by all flags, i.e., linearly ordered subsets, in the reduced lattice $\bar{\mathcal{L}}=\mathcal{L}\setminus\{\hat{0},\hat{1}\}$. The atomic complex consists of all subsets σ of $\mathfrak{A}(\mathcal{L})$ with $\bigvee\sigma<\hat{1}$. The abstract simplicial complexes $F(\mathcal{L})$ and $C(\mathcal{L})$, in fact, are homotopy equivalent. To simplify notation, we will not distinguish between an abstract simplicial complex and its simplicial chain complex with rational coefficients.

We can now give an explanation for the terminology chosen for the differential graded algebra $D_{\mathcal{A}}$. Observe that the complex (D,d) naturally decomposes as a direct sum

$$D \;=\; \bigoplus_{A\in\mathcal{L}} D(A), \tag{2.3}$$

where $D(A)$, for $A\in\mathcal{L}$, is generated by all subsets σ in $\mathfrak{A}(\mathcal{L})$ with $\bigvee\sigma=A$.

In fact, for any $A\in\mathcal{L}$, and $p\geq 0$, there is a natural isomorphism

$$H^p(D(A)) \;\cong\; \widetilde{H}_{2\operatorname{cd}A-p-2}(C(\hat{0},A)), \tag{2.4}$$

where $C(\hat{0},A)$ denotes the atomic complex of the interval $[\hat{0},A]$ in $\mathcal{L}$.

It is an easy observation that $(D(A),d)$, graded by cardinality of generators $\sigma\subseteq\mathfrak{A}(\mathcal{L})$, is the same as the relative simplicial chain complex $\Sigma(A)/C(\hat{0},A)$, where $\Sigma(A)$ is the simplicial chain complex of the full simplex on the vertex set $\{X\in\mathfrak{A}(\mathcal{L})\,|\,X\leq A\}$. The isomorphism stated above is part of the exact homology pair sequence of $(\Sigma(A),C(\hat{0},A))$; compare [9, Sect. 3.1.2] for details. We will later write out a chain map that induces the isomorphism in (2.4) as part of our proof of Proposition 2.6.

(3) For an arrangement with geometric intersection lattice, $H^*(D,d)$ is generated by the classes $[\sigma]$ of independent subsets $\sigma\subseteq\mathfrak{A}(\mathcal{L})$. Indeed, for any independent set σ in $\mathcal{L}$, σ is a cocycle in (D,d) by definition of the differential. Moreover, from the description of homology in (2.4), and using the well-known results of Folkman on lattice homology [3], we see that $H^p(D(A))=0$ unless $p=2\operatorname{cd}A-\operatorname{rk}A$. For $\sigma\subseteq\mathfrak{A}(\mathcal{L})$ to be a generator of $D^{2\operatorname{cd}A-\operatorname{rk}A}(A)$, it is a necessary condition that $\bigvee\sigma=A$ and $|\sigma|=\operatorname{rk}A$, hence σ has to be independent. It follows that the classes $[\sigma]$, σ independent, generate $H^*(D_{\mathcal{A}})$.

2.3. Arrangement cohomology. We here recall the definition of another differential graded algebra $\mathrm{CM}_{\mathcal{A}}$ associated with any complex subspace arrangement. This differential graded algebra is the main character in [10], where it is shown to be a rational model for the arrangement complement. It is a considerable simplification of the rational model presented earlier in work of De Concini and Procesi [2], and relates their results in an elucidating way to the much earlier results of Goresky and MacPherson [4] on the linear structure of arrangement cohomology.

The underlying chain complex of $\mathrm{CM}_{\mathcal{A}}$ is the $\mathcal{L}$-graded complex

$$\mathrm{CM}_{\mathcal{A}} \;=\; \bigoplus_{A\in\mathcal{L}} F(\hat{0},A),$$

where $F(\hat{0},A)$, for $A\in\mathcal{L}$, is the flag complex of the open interval $(\hat{0},A)$ of elements in $\mathcal{L}$ below A.

To describe the multiplication we need to fix some notation. Given a flag $T \in F(\hat{0},A)$, denote by $\overline{T}$ the flag extended by the lattice element A, and by $\underline{T}$ the flag with its maximal element removed. For an ordered collection of lattice elements $\mathcal{C}=\{C_1,\ldots,C_t\}$, denote by $\lambda(\mathcal{C})$ the chain in $\mathcal{L}$ obtained by taking successive joins,

$$\lambda(\mathcal{C}):\; C_1 < C_1\vee C_2 < \ldots < \bigvee_{i=1}^{t} C_i,$$

and set $\lambda(\mathcal{C})$ to zero in case there are repetitions occurring among the joins.

For $A,B\in\mathcal{L}$, let $T_A\,:\,A_1\,<\,\ldots\,<\,A_p$, $T_B\,:\,B_1\,<\,\ldots\,<\,B_q$, be flags in $F(\hat{0},A)$, $F(\hat{0},B)$, respectively. Let $\mathfrak{S}_{p+1,q+1}$ denote the shuffle permutations in the symmetric group $\mathfrak{S}_{p+q+2}$, i.e., permutations of $[p+q+2]$ that respect the relative order of the first $p{+}1$ and the relative order of the last $q{+}1$ elements. Denote by $(\overline{T}_A,\overline{T}_B)^{\pi}$ the result of applying $\pi\in\mathfrak{S}_{p+1,q+1}$ to the pair of chains, with elements of $\overline{T}_A$ in ascending order preceding elements of $\overline{T}_B$ in ascending order, and thereafter applying λ,

$$(\overline{T}_A,\overline{T}_B)^{\pi} \;=\; \lambda\pi(\overline{T}_A,\overline{T}_B).$$

We are now ready to describe the product on $\mathrm{CM}_{\mathcal{A}}$. For T_A and T_B as above, we define

$$T_A\cdot T_B = \begin{cases} \displaystyle\sum_{\pi\in\mathfrak{S}_{p+1,q+1}} (-1)^{\operatorname{sgn}\pi}\;\underline{(\overline{T}_A,\overline{T}_B)^{\pi}} \quad \text{if} \\ \qquad \operatorname{cd}A+\operatorname{cd}B=\operatorname{cd}(A\vee B), \\ 0 \quad \text{otherwise.} \end{cases} \tag{2.5}$$

As we mentioned in the introduction, $\mathrm{CM}_{\mathcal{A}}$ is our link between the relative atomic differential graded algebra $D_{\mathcal{A}}$ and the De Concini-Procesi rational model for arrangement complements given in [2].

Theorem 2.5 [10, Cor. 4.7 and 5.3]**.** *The differential graded algebra $\mathrm{CM}_{\mathcal{A}}$ associated with an arrangement $\mathcal{A}$ is quasi-isomorphic to the De Concini-Procesi rational model for the arrangement complement.*

To complete the sequence of quasi-isomorphisms, we are left to show the following.

Proposition 2.6. *The relative atomic differential graded algebra $D_{\mathcal{A}}$ of an arrangement $\mathcal{A}$ is quasi-isomorphic to the differential graded algebra $\mathrm{CM}_{\mathcal{A}}$.*

Proof. We define a homomorphism of differential graded algebras $h : D_{\mathcal{A}} \longrightarrow \mathrm{CM}_{\mathcal{A}}$. The homomorphism respects the $\mathcal{L}$-grading of both algebras, hence it will be sufficient to define $h_A : D(A) \longrightarrow F(\hat{0}, A)$ for any $A \in \mathcal{L}$. The ingredients are two maps, $g_A : D(A) \longrightarrow C(\hat{0}, A)$ and $f_A : C(\hat{0}, A) \longrightarrow F(\hat{0}, A)$, where $C(\hat{0}, A)$ is the atomic complex of the interval $[\hat{0}, A]$ in $\mathcal{L}$. For $\sigma \subseteq \mathfrak{A}(\mathcal{L})$ with $\bigvee \sigma = A$, we define

$$g_A(\sigma) \;=\; \sum_{j:\bigvee \sigma_j < \bigvee \sigma} (-1)^j \, \sigma_j ,$$

where, as in the definition of the differential in $D_{\mathcal{A}}$ in (2.1), the indexing of elements in σ follows the linear order imposed on $\mathfrak{A}(\mathcal{L})$.

The map f_A is the standard chain homotopy equivalence between the atomic complex and the flag complex of a given lattice, which we recall from [10, Lemma 6.1] for completeness. For $\sigma \subseteq \mathfrak{A}(\mathcal{L})$ with $\bigvee \sigma < A$, we define

$$f_A(\sigma) \;=\; \sum_{\pi \in \mathfrak{S}_{|\sigma|}} (-1)^{\operatorname{sgn} \pi} \lambda(\sigma).$$

We now define for $\sigma \subseteq \mathfrak{A}(\mathcal{L})$ with $\bigvee \sigma = A$,

$$h_A(\sigma) \;=\; (-1)^{2\,\mathrm{cd}\,A - |\sigma|} \, f_A \, g_A(\sigma). \qquad (2.6)$$

We claim that $h = \sum_{A \in \mathcal{L}} h_A$ is a quasi-isomorphism of differential graded algebras, and we break our proof into 4 steps.

(1) *h is a homogeneous map.* This is obvious from the definition of the gradings on $D_{\mathcal{A}}$ and $\mathrm{CM}_{\mathcal{A}}$.

(2) *h_A is a map of chain complexes.* Due to the sign in (2.6), we need to show that $f_A\, g_A(d\sigma) = -\, d(f_A\, g_A(\sigma))$ for $\sigma \subseteq \mathfrak{A}(\mathcal{L})$ with $\bigvee \sigma = A$. By definition of d and g_A, we have

$$(d + g_A)(\sigma) \;=\; \sum_{j\in\sigma} (-1)^j\, \sigma_j ,$$

which implies that $(d+g_A)^2 = 0$. With $d^2 = 0$ and $g_A^2 = 0$, we conclude that $dg_A = -\, g_A d$, and since f_A is a chain map, our claim follows.

(3) *h is multiplicative.* Let σ, τ be subsets in $\mathfrak{A}(\mathcal{L})$ and denote $\bigvee \sigma = A$ and $\bigvee \tau = B$. We can assume that $\mathrm{cd}\, A + cdB = \mathrm{cd}(A \vee B)$, in particular, $\sigma \cap \tau = \varnothing$, since otherwise both sides of the equation

$$h_{A\vee B}(\sigma \cdot \tau) = h_A(\sigma) \cdot h_B(\tau)$$

are 0 by definition of products in $D_{\mathcal{A}}$ and $\mathrm{CM}_{\mathcal{A}}$, respectively. The sign in the definition of h is chosen so that here we are left to show that

$$f_{A\vee B}\, g_{A\vee B}(\sigma \cdot \tau) \;=\; f_A g_A(\sigma) \cdot f_B g_B(\tau). \tag{2.7}$$

Now we make two claims.

Claim 1. The set of nonzero flags occurring on the left hand side of (2.7) coincides with the set of flags on the right hand side.

Claim 2. The signs of a flag occurring on the left hand side and on the right hand side of (2.7) are the same.

To prove Claim 1, fix a nonzero flag F occurring on the left hand side of (2.7). It is constructed by applying λ to some linear order on $(\sigma \cup \tau) \setminus \{a\}$ for an atom a. Suppose $a \in \sigma$, the other case being similar. Denote by b the last element of τ in this ordering. Since F does not have repetitions, we find $\pm\tau \setminus \{b\}$ as a summand in $g_B(\tau)$. The induced orderings on $\sigma \setminus \{a\}$ and $\tau \setminus \{b\}$ produce flags F_1 and F_2 in $F(\hat{0}, A)$ and $F(\hat{0}, B)$, respectively, whose product occurs on the right hand side of (2.7). The linear order that gave rise to F prescribes a shuffle permutation that generates $\pm F$ as a summand of $F_1 \cdot F_2$ on the right hand side. The opposite inclusion can be shown by inverting all the steps of the proof.

To prove Claim 2, fix again a flag F occurring on the left hand side of (2.7) as above, and keep the notation from the proof of Claim 1. Denote the coefficients of F on the left hand side and on the right hand side by $(-1)^\ell$ and $(-1)^r$, respectively.

Then we have

$$\ell \;= \mathrm{sgn}\; \epsilon(\sigma, \tau) \,+\, [a \in \sigma \cup \tau] \,+\, [(\sigma \setminus \{a\}) \cup \tau], \tag{2.8}$$

where the second term is the numerical position of a in $\sigma \cup \tau$ (in the initial ordering) and $[\rho]$ is the parity of the permutation induced by the ordering that gives rise to F on any subset ρ of $(\sigma \setminus \{a\}) \cup \tau$. Using similar notation, we have

$$r \;=\; [a \in \sigma] + [b \in \tau] + [\sigma \setminus \{a\}] + [\tau \setminus \{b\}] + [\sigma \setminus \{a\}, \tau] + |\tau|, \quad (2.9)$$

where $[\rho_1, \rho_2]$ is the parity of the shuffle permutation, induced by the ordering that gives rise to F, of two disjoint subsets ρ_1 and ρ_2 of $(\sigma \setminus \{a\}) \cup (\tau \setminus \{b\})$ (from the starting position of ρ_2 after ρ_1). Recall that, to obtain F, we need first to augment $f_A(\sigma \setminus \{a\})$ by A, and $f_B(\tau \setminus \{b\})$ by B, respectively, and then apply the needed shuffle. It is easy to see that the shuffle should have A at the end of the set. Thus the last summand in (2.9) comes from moving A over τ. The augmentation by B amounts just to the substitution of b by B in τ.

Now we need the following straightforward equalities (modulo 2).

(i) $\operatorname{sgn}\, \epsilon(\sigma, \tau) = \operatorname{sgn}\, \epsilon(\sigma \setminus \{a\}, \tau) + |\tau_{<a}|$,
where the new symbols are self-explanatory (e.g., $\tau_{<a} = \{c \in \tau | c < a\}$);

(ii) $[a \in \sigma \cup \tau] = [a \in \sigma] + |\tau_{<a}|$;

(iii) $[\sigma \setminus \{a\}] + [\tau \setminus \{b\}] + [\sigma \setminus \{a\}, \tau] = \operatorname{sgn}\, \epsilon(\sigma \setminus \{a\}, \tau) + |\tau_{>b}| + [(\sigma \setminus \{a\}) \cup \tau]$;

(iv) $[b \in \tau] + |\tau_{>b}| = |\tau|$.

Substituting (i)–(iv) in (2.8) and (2.9), we obtain the needed equality of the coefficients of F on the left and right hand sides of (2.7).

(4) *h is a quasi-isomorphism.* Both maps g_A and f_A induce isomorphisms in homology, which completes our proof. □

To prove that the complement of an arrangement $\mathcal{A}$ is formal, we are left to show that the relative atomic differential graded algebra $(D_{\mathcal{A}}, d)$ is formal. Note that so far (with the exception of Remark 2.4(3) we did not refer to any specific properties of the arrangement or its intersection lattice. It is only in the next section that we will restrict ourselves to arrangements with geometric intersection lattices.

3. Formality of $D_{\mathcal{A}}$ for geometric lattices

Theorem 3.1. *Let $\mathcal{A}$ be a complex subspace arrangement with geometric intersection lattice. The linear map $\Psi \,:\, D_{\mathcal{A}} \longrightarrow H^*(D_{\mathcal{A}}, d)$*

$$\sigma \longmapsto \begin{cases} [\sigma] & \text{if } u\sigma \text{ is independent,} \\ 0 & \text{otherwise} \end{cases}$$

is a quasi-isomorphism of differential graded algebras, i.e., $D_{\mathcal{A}}$ is formal.

Proof. **(1)** Ψ *is multiplicative.* We need to check that for any two subsets σ, τ in $\mathfrak{A}(\mathcal{L})$,

$$\Psi(\sigma\tau) \;=\; \Psi(\sigma)\Psi(\tau). \tag{3.1}$$

(i) First assume that both σ and τ are independent in $\mathcal{L}$, and $\mathrm{cd}\bigvee\sigma + \mathrm{cd}\bigvee\tau = \mathrm{cd}\bigvee\sigma\cup\tau$.
Claim. $\sigma\cup\tau$ is independent in $\mathcal{L}$.
We first observe that $\sigma\cap\tau=\varnothing$. For, if $c\in\sigma\cap\tau$, then $\mathrm{cd}((\bigvee\sigma)\wedge(\bigvee\tau))\geq \mathrm{cd}\; c>0$, and

$$\mathrm{cd}\bigvee(\sigma\cup\tau) < \mathrm{cd}\bigvee(\sigma\cup\tau) + \mathrm{cd}((\bigvee\sigma)\wedge(\bigvee\tau)) \leq \mathrm{cd}\bigvee\sigma + \mathrm{cd}\bigvee\tau,$$

contrary to our assumption.

Assume that $\sigma\cup\tau$ were dependent, hence there existed a $c\in\sigma\cup\tau$ with $\bigvee(\sigma\cup\tau\setminus\{c\}) = \bigvee(\sigma\cup\tau)$. Without loss of generality, we can assume that $c\in\tau$. Then

$$\begin{aligned}\mathrm{cd}\bigvee(\sigma\cup\tau) &= \mathrm{cd}(\bigvee\sigma\;\vee\;\bigvee(\tau\setminus\{c\}))\\ &\leq \mathrm{cd}\bigvee\sigma + \mathrm{cd}\bigvee(\tau\setminus\{c\}) < \mathrm{cd}\bigvee\sigma + \mathrm{cd}\bigvee\tau,\end{aligned}$$

where the strict inequality holds since τ is independent. Again, we reach a contradiction to our assumption.

With $\sigma\cup\tau$ being independent, equality (3.1) holds by definition of multiplication in $H^*(D_{\mathcal{A}},d)$.
(ii) Now assume that σ and τ are independent in $\mathcal{L}$, but

$$\mathrm{cd}\bigvee\sigma + \mathrm{cd}\bigvee\tau \neq \mathrm{cd}\bigvee(\sigma\cup\tau).$$

In this case $\sigma\tau=0$ in D, hence both sides in (3.1) are zero, the right hand side again by definition of multiplication in $H^*(D,d)$.
(iii) We conclude by assuming that at least one of σ or τ are dependent sets, say σ. With $\psi(\sigma)=0$ by definition, the right hand side in (3.1) is zero. As σ is dependent, so is $\sigma\cup\tau$. Either $\sigma\tau=0$ in D, then so is its image under Ψ, or codimensions add up and $\sigma\tau$ equals $\sigma\cup\tau$ up to sign. Then, the left hand side is zero by definition of Ψ.

(2) Ψ *is a homomorphism of differential graded algebras.* We need to show that $\Psi(d\sigma)=0$ for any subset σ in $\mathfrak{A}(\mathcal{L})$.

For an independent set σ in $\mathcal{L}$, $d\sigma = 0$ by definition, hence $\Psi(d\sigma) = 0$. Let σ be a dependent set in $\mathcal{L}$. We recall that

$$d\sigma \;=\; \sum_{j:\bigvee \sigma_j = \bigvee \sigma} (-1)^j \, \sigma_j ,$$

and we observe that $d\sigma$ maps to zero under Ψ if either all summands σ_j in (2.1) are dependent or if all summands in (2.1) are independent. In the latter case, $\Psi(d\sigma) = [d\sigma]$ is the homology class of a boundary in D, hence is zero. We need to prove that these two cases exhaust all possibilities.

Assume that σ_i is an independent, σ_j a dependent set in $\mathcal{L}$, both occurring as summands in $d\sigma$, hence $\bigvee \sigma_i = \bigvee \sigma_j = \bigvee \sigma$. We consider $\tau = \sigma_i \setminus \{j\} = \sigma_j \setminus \{i\}$; τ is independent as a subset of σ_i, hence it is a maximal independent set in σ_j. We see that $\bigvee \tau = \bigvee \sigma_j = \bigvee \sigma$, in contradiction to σ_i being independent.

(3) Ψ *is a quasi-isomorphism.* In the case of a geometric lattice, the classes $[\sigma]$ for independent sets σ in L generate $H^*(D,d)$, compare Remark 2.4. Hence, the induced map Ψ^* is surjective, and, since $H^*(D,d)$ is finite-dimensional, this suffices for Ψ^* to be an isomorphism.

4. An outlook

With the purpose of going beyond the case of geometric lattices, one might be tempted to replace the notion of independent sets in a geometric lattice with the following (compare [7, Sect. 3]).

Definition 4.1. *Let $\mathcal{L}$ be a finite lattice and $\mathfrak{A}(\mathcal{L})$ its set of atoms. A subset σ in $\mathfrak{A}(\mathcal{L})$ is called independent if*

$$\bigvee \sigma \setminus \{s\} \;<\; \bigvee \sigma \qquad \text{for any} \quad s \in \sigma .$$

We remark again that prior to Section 3 we have not been referring to any specific property of geometric lattices. A careful reading of the proof of Theorem 3.1 shows that there are two points where we had to rely on the intersection lattice being geometric.

(1) $H^*(D,d)$ is generated by classes $[\sigma]$ of independent sets σ in $\mathcal{L}$.
(2) The generators σ_i occurring in $d\sigma$ for $\sigma \subseteq \mathfrak{A}(\mathcal{L})$ are either all independent or all dependent sets in $\mathcal{L}$.

Example 4.2. Consider the arrangement $\mathcal{A}$ given by the following four subspaces in $\mathbb{C}^4$:

$$\begin{aligned} U_1 &= \{x = u = 0\}, & U_2 &= \{y = u = 0\}, \\ U_3 &= \{z = u = 0\}, & U_4 &= \{x = y, z = 0\}. \end{aligned}$$

Its intersection lattice is *not* geometric, compare the Hasse diagram in Figure 1. The independent sets according to Definition 4.1 are

$$1,\ 2,\ 3,\ 4,\ 12,\ 13,\ 14,\ 23,\ 24,\ 34,\ 123,$$

where, for brevity, we denote atoms in $\mathcal{L}$ by their indices.

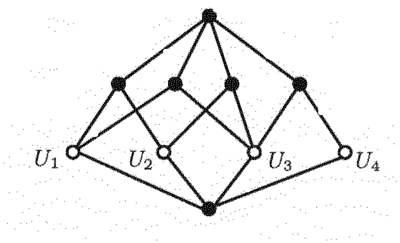

Fig. 1. The intersection lattice of $\mathcal{A}$ in Example 4.2.

Consider the relative atomic complex $(D_{\mathcal{A}}, d)$. Its $\mathcal{L}$-homogeneous components $D(A)$ consist of a single rank 1 cochain group each, for any A in $\mathcal{L}$ except $A = \hat{0}$. The latter reads as follows.

$$D(\hat{0})\ :\quad 0 \longrightarrow \langle 1234\rangle \longrightarrow \langle 123, 124, 134, 234\rangle \longrightarrow \langle 14, 24\rangle \longrightarrow 0,$$

with non-trivial cochain groups in degrees 4, 5, and 6. Applying d, we see that $H^p(D(\hat{0})) = 0$, unless $p = 4$, and $H^4(D(\hat{0})) = \langle [123]\rangle$.

Hence, we find that $H^*(D)$ is generated by the classes of the independent sets in $\mathcal{L}$. However, contrary to the geometric case, these classes can be zero, as are [14] and [24] in the present example.

Since $d(1234) = 234-134+124-123$ (observe that 123 is independent whereas the other classes are dependent!), we have

$$[123] \;=\; [234]-[134]+[124]. \tag{4.1}$$

This shows that the proof of Theorem 3.1 does not extend to the present arrangement – not all of the classes on the right hand side of (4.1), induced by dependent sets in $\mathcal{L}$, can be mapped to 0 under a quasi-isomorphism between D and $H^*(D)$.

However, it is easy to check that the arrangement $\mathcal{A}$ is formal – a quasi-isomorphism $D_{\mathcal{A}} \to H^*(D_{\mathcal{A}})$ can be constructed directly by mapping all 3 dependent sets above to the same generating class.

A slight variation of this example shows that the proof of Theorem 3.1 does extend to *some* non-geometric lattices. Consider the arrangement of subspaces in $\mathbb{C}^4$ given by

$$U_1 = \{x = u = 0\}, \quad U_2 = \{y = u = 0\},$$
$$U_3 = \{x = y, u = 0\}, \quad U_4 = \{x = y, z = 0\}.$$

Again, $H^*(D)$ is generated by classes of independent sets, and $d(1234) = 234 - 134 + 124$, hence it is a sum of dependent sets only.

Example 4.3. For $n > k \geq 2$, consider the *k-equal arrangement* $\mathcal{A}_{n,k}$ given by the codimension $k-1$ subspaces in $\mathbb{C}^n$ of the form

$$z_{i_1} = z_{i_2} = \ldots = z_{i_k} \quad \text{for any} \quad 1 \leq i_1 < \ldots < i_k \leq n.$$

Its intersection lattice is the subposet $\Pi_{n,k}$ of the lattice of set partitions of $\{1, \ldots, n\}$ formed by partitions with non-trivial block sizes larger or equal to k. Observe that $\Pi_{n,k}$ is *not* geometric for $k > 2$.

However, in various respects, k-equal arrangements do have properties that are similar to arrangements with geometric intersection lattices. For example, cohomology is generated by classes of independent sets of atoms in $\Pi_{n,k}$ in the sense of Definition 4.1 [10, Thm. 8.8(i)]. Our formality proof in Section 3, though, does not extend to $D_{\mathcal{A}_{n,k}}$; condition (2) mentioned above is violated.

Consider $\mathcal{A}_{7,3}$ and set

$$\sigma = \{123, 234, 345, 456, 567\}.$$

Here, the triples ijk are shorthand notation for partitions of $\{1, \ldots, 7\}$ with only nontrivial block ijk.

The set σ is dependent, since removing 234, 345 or 456 preserves the join. We see that $\sigma \setminus \{345\}$ is independent, whereas $\sigma \setminus \{234\}$ is dependent, e.g., $\bigvee \sigma \setminus \{234, 456\} = \bigvee \sigma \setminus \{234\}$.

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