Записки научных
семинаров ПОМИ
Том 326, 2005 г.

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UNITARY REPRESENTATIONS AND MODULAR ACTIONS

ABSTRACT. We call a measure-preserving action of a countable discrete group on a standard probability space tempered if the associated Koopman representation restricted to the orthogonal complement to the constant functions is weakly contained in the regular representation. Extending a result of Hjorth, we show that every tempered action is antimodular, i.e., in a precise sense "orthogonal" to any Borel action of a countable group by automorphisms on a countable rooted tree. We also study tempered actions of countable groups by automorphisms on compact metrizable groups, where it turns out that this notion has several ergodic theoretic reformulations and fits naturally in a hierarchy of strong ergodicity properties strictly between ergodicity and strong mixing.

Dedicated to Professor Anatoly Vershik on the occasion of his 70th birthday

In the study of countable Borel equivalence relations on standard Borel spaces (i.e., Polish spaces equipped with the associated Borel structure), an important role is played by the class of treeable equivalence relations. A countable Borel equivalence relation E on X is called *treeable* if there is an acyclic Borel graph $G \subseteq X^2$ whose connected components are exactly the E-classes. For more about treeable equivalence relations, see [14].

In the hierarchy induced by the partial order of *Borel reducibility*, $E \leq_B F$, where

$$E \leq_B F \quad \Leftrightarrow \quad \text{there is a Borel map} \quad f : X \to Y$$
$$\text{such that} \quad xEy \;\Leftrightarrow\; f(x)Ff(y),$$

with $E \subseteq X^2, F \subseteq Y^2$, there is a largest treeable equivalence relation, called the *universal treeable equivalence relation* and denoted by $E_{\infty T}$. It is uniquely determined up to *Borel bireducibility*:

$$E \sim_B F \;\Leftrightarrow\; E \leq_B F \;\text{ and }\; F \leq_B E,$$

Research partially supported by NSF Grant DMS-9987437 and a Guggenheim Fellowship.

and one of its manifestations is the equivalence relation induced by the shift action ($\gamma \cdot x(\delta) = x(\gamma^{-1}\delta)$) of F_2, the free group on 2 generators, on 2^{F_2}, restricted to the free part of this action (i.e., the set $\{x \in 2^{F_2} : \gamma \cdot x \neq x$ for all $\gamma \in F_2 \setminus \{1\}\}$). On the other hand, excluding tame (i.e., having a Borel selector) equivalence relations, there is also a smallest treeable equivalence relation, denoted by E_0. It is again uniquely determined up to Borel bireducibility, and one of its manifestations is the equivalence relation induced by the shift action of $\mathbb{Z}$ on $2^{\mathbb{Z}}$. Thus the nontame treeable equivalence relations are exactly those in the interval

$$E_0 \leq_B E \leq_B E_{\infty T}.$$

It has been known for some time (see, e.g., [14]) that

$$E_0 <_B E_{\infty T},$$

where

$$E <_B F \;\Leftrightarrow\; E \leq_B F \text{ and } F \not\leq_B F,$$

but it had remained open whether the interval $[E_0, E_{\infty T}]$ is nontrivial, i.e., whether there exists a treeable equivalence relation E with

$$E_0 <_B E <_B E_{\infty T}.$$

This was recently resolved affirmatively in Hjorth [12], who made crucial use of the notion of a modular action of a countable group.

Following Hjorth [12], call a Borel action of a countable (discrete) group Γ on a standard Borel space Y *modular* if there is a sequence of countable Borel partitions $\{\mathcal{P}_n\}$ of Y ($\mathcal{P}_n = \{A_i^{(n)}\}_{i \in I_n}$ with $|I_n| \leq \aleph_0, A_i^{(n)} \cap A_j^{(n)} = \emptyset$ if $i \neq j, \bigcup_{i \in I_n} A_i^{(n)} = Y$), so that $\{\mathcal{P}_n\}$ generates the Borel sets or, equivalently, separates points (i.e., for any $x \neq y \in Y$ there is n and $i \neq j$ with $x \in A_i^{(n)}, y \in A_j^{(n)}$), such that each $\mathcal{P}_n$ is Γ-invariant (i.e., for any $\gamma \in \Gamma$, $n, i \in I_n$, there is $j \in I_n$ with $\gamma \cdot A_i^{(n)} = A_j^{(n)}$).

Hjorth [12], proved that if Γ acts in a modular way on Y, then for any F_2-invariant Borel set $A \subseteq 2^{F_2}$ with $\mu(A) = 1$, where μ is the usual product measure on 2^{F_2}, there is no $\aleph_0$-to-1 Borel homomorphism $\rho : A \to Y$ of $E_{F_2}^A$ to E_Γ^Y, where $E_{F_2}^A$ denotes the equivalence relation induced by the shift action of F_2 on A, and similarly for E_Γ^Y, and a *homomorphism* is a map which sends $E_{F_2}^A$-equivalent elements to E_Γ^Y-equivalent elements, i.e.,

$$aE_{F_2}^A b \Rightarrow \rho(a) E_\Gamma^Y \rho(b).$$

Since one can easily find examples of free modular actions of F_2 with invariant probability Borel measure, this easily implies the existence of a treeable equivalence relation E with

$$E_0 <_B E <_B E_{\infty T}.$$

In this paper, we consider the following question motivated by Hjorth's result:

Given a countable group Γ and a Borel action of Γ on a standard Borel space X with invariant measure μ, under what circumstances is this action "antimodular" in the sense that Hjorth's theorem holds for it (i.e., for any modular action of a countable group Δ on a standard Borel space Y, there is no Borel $\aleph_0$-to-1 homomorphism from E_Γ^A to E_Δ^Y, where $A \subseteq X$ is Borel Γ-invariant with $\mu(A) = 1$)? Here and throughout the paper, *measure* means *probability Borel measure.*

We isolate a representation theoretic property of the Γ-action on X (i.e., a property of the associated unitary representation of Γ on $L^2(X, \mu)$), and verify that Hjorth's proof can be appropriately adapted in this generality to show that any action satisfying this property is "antimodular." More precisely, we call the action of Γ on X *tempered* if the associated unitary representation of Γ on $L^2(X, \mu)$, restricted to the orthogonal complement of the constants, is weakly contained in the (left-) regular representation of Γ. We also say that this action is *antimodular* if for any modular action of a countable group Δ on a standard Borel space Y, there is no Borel $\aleph_0$-to-1 homomorphism from E_Γ^X to E_Δ^Y. We now have:

Theorem. *Let Γ be a countable group with $F_2 \leq \Gamma$. Suppose Γ acts in a Borel way on a standard Borel space X with non-atomic invariant measure μ. If this action is tempered, then it is antimodular.*

Of course the action of F_2 on 2^{F_2}, with the usual product measure, is tempered, but this also holds for several other actions of interest, and thus provides many new examples of antimodular actions. In particular, we study the question of when this property is true for actions of infinite countable groups by (topological group) automorphisms on compact Polish groups. It turns out that being tempered in this case is a natural strong ergodicity property sandwiched strictly between mixing (= mildly mixing, in this case) and weakly mixing (=ergodicity, in this case).

For convenience, we start from scratch and we do not assume that the reader has already mastered Hjorth [12].

1. Modular actions

We first note some simple reformulations of modularity.

Given two partitions $\mathcal{P} = \{A_i\}$, $\mathcal{Q} = \{B_j\}$ of Y, we define their refinement:

$$\mathcal{P} \wedge Q = \{A_i \cap B_j\}.$$

We also put

$$\mathcal{P} \leq \mathcal{Q} \ \Leftrightarrow \ \text{ each } \ A_i \ \text{ is contained in some } \ B_j,$$

i.e., $\mathcal{P}$ refines $\mathcal{Q}$. If now Γ acts in a modular way on Y, $\{\mathcal{P}_n\}$ is a sequence of partition witnessing the modularity of the action, and

$$\mathcal{Q}_n = \mathcal{P}_0 \wedge \mathcal{P}_1 \wedge \cdots \wedge \mathcal{P}_n,$$

then clearly $\mathcal{Q}_0 \geq \mathcal{Q}_1 \geq \ldots$, the sequence $\{\mathcal{Q}_n\}$ generates the Borel sets, and each $\mathcal{Q}_n$ is Γ-invariant. Thus, in the definition of modularity, we can assume, if necessary, that $\mathcal{P}_0 \geq \mathcal{P}_1 \geq \ldots$.

Let us also note two simple stability properties of modularity.

(i) If Γ acts in a modular way on Y with witnessing partitions $\{\mathcal{P}_n\}$, then for any Γ-invariant Borel set $Z \subseteq Y$, the Γ-action on Z is also modular, with witness $\{\mathcal{P}_n|Z\}$, where, for $\mathcal{P}_n = \{A_i^{(n)}\}$, $\{\mathcal{P}_n|Z\} = \{A_i^{(n)} \cap Z\}$.

(ii) If Γ acts in a modular way on Y and Δ acts in a modular way on Z with witnessing partitions $\{\mathcal{P}_n\}, \{\mathcal{Q}_n\}$, resp., then the product action of $\Gamma \times \Delta$ on $Y \times Z$ is also modular with witnessing partitions $\{\mathcal{P}_n \times \mathcal{Q}_n\}$, where if $\mathcal{P}_n = \{A_i^{(n)}\}, \mathcal{Q}_n = \{B_j^{(n)}\}, \mathcal{P}_n \times \mathcal{Q}_n = \{A_i^{(n)} \times B_j^{(n)}\}$.

Canonical examples of modular actions arise from actions of groups by automorphisms on countable rooted trees. For each countable rooted tree T, with root v_0, we denote by $[T]$ the space of all infinite paths $\alpha = (v_0, v_1, v_2, \ldots)$ starting from the root. For each finite path $s = (v_0, v_1, \ldots, v_n)$, let $N_s = \{\alpha \in [T] : \forall i \leq n(\alpha(i) = v_i)\}$. With the topology whose basic open sets are the N_s, $[T]$ becomes a Polish space. If a countable group Γ acts by automorphisms on T (so that, in particular, $\gamma \cdot v_0 = v_0, \forall \gamma \in \Gamma$), Γ acts on $[T]$ continuously by $\gamma \cdot (v_0, v_1, v_2, \ldots) = (\gamma \cdot v_0, \gamma \cdot v_1, \gamma \cdot v_2, \ldots)$. Clearly this is a modular action with witness $\{\mathcal{P}_n\}$, where

$$\mathcal{P}_n = \{N_s : |s| = n\}.$$

A closely related type of example comes from actions of groups by isometries on ultrametric Polish spaces. If (X, d) is an ultrametric Polish space and a countable group Γ acts by isometries on X, then the action is modular with witness $\{\mathcal{P}_n\}$, where $\mathcal{P}_n$ is the collection of all open balls of radius $1/(n+1)$ in X.

We now have:

Fact 1.1. Suppose that a countable group Γ acts in a Borel way on a standard Borel space Y. Then the following are equivalent:

(i) The Γ-action on Y is modular.

(ii) There is an action of Γ by automorphisms on a countable rooted tree T and a Borel Γ-embedding of the action of Γ on Y to the induced Γ-action on $[T]$, i.e., a Borel injection $\varphi : Y \to [T]$ with $\gamma \cdot \varphi(y) = \varphi(\gamma \cdot y)$.

(iii) There is an action of Γ by isometries on a Polish ultrametric space (X, d) and a Borel Γ-embedding of the action of Γ on Y into the action of Γ on X.

Proof. (i)$\Rightarrow$ (ii): Let $\mathcal{P}_0 \geq \mathcal{P}_1 \geq \dots$ be a sequence of Borel partitions witnessing that the Γ-action on Y is modular. Assume, without loss of generality, that $\mathcal{P}_0 = \{Y\}$. Let T be the rooted tree defined as follows:

(a) The root of T is Y.

(b) The children of Y are the elements of $\mathcal{P}_1$.

(c) If $A \in \mathcal{P}_1$, the children of A are the $B \in \mathcal{P}_2$, $B \subseteq A$, etc.

Clearly Γ acts on T by automorphisms. Define then

$$\varphi : Y \to [T]$$

by

$$\varphi(y) = (A_0, A_1, \dots) \text{ if and only if } \forall n(y \in A_n).$$

Since $\{\mathcal{P}_n\}$ separates points, φ is a Borel embedding of Y to $[T]$ and clearly it is a Γ-map.

(ii) $\Rightarrow$(iii): Notice that $[T]$ with the metric

$$d(\alpha, \beta) = 2^{-n-1},$$

where $\alpha \neq \beta$ and n is the least with $\alpha(n) \neq \beta(n)$, is an ultrametric Polish space, and if Γ acts on T by automorphisms, then the induced action of Γ on $([T], d)$ is by isometries.

(iii) $\Rightarrow$ (i): This is immediate from the fact that the action of Γ on X is modular and the restriction of a modular action to an invariant Borel set is modular. $\square$

Let us also note the following fact, which will be useful later on:

Fact 1.2. Let E be a countable Borel equivalence relation on a standard Borel space X. Let the countable group Γ act in a modular way on a standard Borel space Y. If there is an $\aleph_0$-to-1 Borel homomorphism ρ of E to E_Γ^Y (i.e., $xEy \Rightarrow \rho(x)E_\Gamma^Y\rho(y)$), then there is a modular action of a countable group Γ_1 on some standard Borel space Y_1 and a 1-1 Borel homomorphism of E to $E_{\Gamma_1}^{Y_1}$.

Proof. Fix a countable group Δ and a modular Borel action of Δ on some standard Borel space Z, so that the Δ-action has an infinite orbit. For example, we can take $\Delta = \mathbb{Z}_2^{<\mathbb{N}} = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \dots$, and its action on $Z = 2^\mathbb{N} = \mathbb{Z}_2^\mathbb{N}$ by coordinate addition. Let $\Gamma_1 = \Gamma \times \Delta, Y_1 = Y \times Z$ and consider the product action of Γ_1 on Y_1, which is modular.

Let $\{z_n\}_{n\in\mathbb{N}}$ be a 1-1 enumeration of an infinite orbit of the Δ-action on Z. Fix also a Borel map $f : \rho(X)\times\mathbb{N} \to X$ with $\rho^{-1}(\{y\}) = \{f(y,n) : n \in \mathbb{N}\}$. Put then

$$\rho_1(x) = (\rho(x), z_{n_x}),$$

where $n_x =$ the least n such that $f(\rho(x), n) = x$. Clearly f is 1-1, so it is enough to check that

$$xEy \Rightarrow \rho_1(x)E_{\Gamma_1}^{Y_1}\rho_1(y).$$

If xEy, then $\rho(x)E_\Gamma^Y\rho(y)$, so fix $\gamma \in \Gamma$ with $\gamma\cdot\rho(x) = \rho(y)$. Let also $\rho_1(x) = (\rho(x), n), \rho_1(y) = (\rho(y), m)$. Then fix $\delta \in \Delta$ such that $\delta\cdot z_n = z_m$. Clearly $(\gamma,\delta)\cdot\rho_1(x) = \rho_1(y)$. □

We next discuss some examples of modular actions.

(A) First, it is clear that any countable group Γ admits a free modular action. Simply let Γ act freely on a countable set A and consider the tree $T = A^{<\mathbb{N}}$ (with root the empty sequence) and in which the children of $(a_0,\dots,a_{n-1})$ are all $(a_0,\dots,a_{n-1},a_n)$, $a_i \in A$. The group Γ acts in the obvious way on T, $\gamma\cdot(a_0,\dots,a_{n-1}) = (\gamma\cdot a_0,\dots,\gamma\cdot a_{n-1})$, and clearly the induced action on $[T]$ is free and modular.

(B) Let us call a Borel action of Γ on Y *finitely modular* if the witnessing partitions $\{\mathcal{P}_n\}$ can be all taken to be finite: $\mathcal{P}_n = \{A_1^{(n)},\dots,A_{k_n}^{(n)}\}, n = 0,1,2,\dots$. This is equivalent to saying that the Γ-action on Y can be Borel embedded to the Γ-action on some $[T]$, where Γ acts by automorphisms on the rooted tree T, and T is *locally finite* (i.e., every vertex in T has only finitely many children). Recall that a group Γ is called *residually finite* if $\forall\gamma\in\Gamma,\gamma\neq 1, \exists N \trianglelefteq \Gamma([\Gamma : N] < \infty$

and $\gamma \notin N$). Poincaré's Lemma asserts that the intersection of two finite index subgroups of a group has also finite index, from which it also follows that every finite index subgroup of a group contains a finite index *normal* subgroup (namely the intersection of its conjugates). Then it is clear that the following are equivalent for each countable group Γ:

(a) Γ is residually finite.

(b) There is a decreasing sequence $\Gamma = \Gamma_0 \geq \Gamma_1 \geq \dots$ of finite index subgroups with $\bigcap_n \Gamma_n = \{1\}$.

(c) There is a decreasing sequence $\Gamma = \Gamma_0 \geq \Gamma_1 \geq \dots$ of finite index *normal* subgroups with $\bigcap_n \Gamma_n = \{1\}$.

It should be pointed out that every finitely generated linear group (i.e., a finitely generated subgroup of $\mathrm{GL}_n(F)$, for some field F) is residually finite. See Lyndon–Schupp [19, III.7.11].

The next fact is well-known (although not in this terminology).

Fact 1.3. A countable group Γ admits a free finitely modular action if and only if Γ is residually finite.

Proof. If Γ is residually finite, fix a sequence $\Gamma = \Gamma_0 \geq \Gamma_1 \geq \dots$ as in (c) above and consider the tree T whose root is Γ, the children of the root are all the cosets of Γ_1, the children of each coset of Γ_1 are all the cosets of Γ_2 contained in it, etc. Γ acts by automorphisms on this tree by left-multiplication (which is the same as right-multiplication, by normality). The action of Γ on $[T]$ is of course modular and it is free, since if $\gamma \cdot (\Gamma_0, \gamma_1\Gamma_1, \gamma_2\Gamma_2, \dots) = (\Gamma_0, \gamma_1\Gamma_1, \gamma_2\Gamma_2, \dots)$, where $\Gamma_0 \supseteq \gamma_1\Gamma_1 \supseteq \gamma_2\Gamma_2, \dots$, then $\gamma_n^{-1}\gamma\gamma_n \in \Gamma_n$, for each n, or $\gamma \in \bigcap_n \Gamma_n = \{1\}$.

Conversely, let Γ act freely on Y in a finitely modular way, with witness the finite partitions $\mathcal{P}_0 \geq \mathcal{P}_1 \geq \mathcal{P}_2 \geq \dots$. Fix $y \in Y$. Then $\gamma \cdot y = y \Rightarrow \gamma = 1, \forall \gamma \in \Gamma$. Let $A_n \in \mathcal{P}_n$ be such that $y \in A_n$. Let $\Gamma_n =$ (stabilizer of A_n) $= \{\gamma \in \Gamma : \gamma \cdot A_n = A_n\}$. Since Γ acts on $\mathcal{P}_n, \Gamma \cdot A_n \subseteq \mathcal{P}_n$ is finite, so Γ_n has finite index in Γ. Since $A_n \supseteq A_{n+1}, \Gamma_n \geq \Gamma_{n+1}$. Finally, it is enough to check that $\bigcap_n \Gamma_n = \{1\}$. Indeed, let $\gamma \in \bigcap_n \Gamma_n$, so that $\gamma \cdot A_n = A_n$ and thus $\gamma \cdot (\bigcap_n A_n) = \bigcap_n A_n$. But $\bigcap_n A_n = \{y\}$, so $\gamma \cdot y = y$, thus $\gamma = 1$. $\square$

We also have the following:

Fact 1.4. A countable group Γ admits a free modular action with invariant (probability Borel) measure if and only if Γ is residually finite.

Proof. Suppose Γ admits a free modular action on Y with witness $\mathcal{P}_0 \geq \mathcal{P}_1 \geq \dots$ and invariant measure μ. By passing to a co-null set if necessary, we can assume that each $\mathcal{P}_n$ consists of sets of positive measure. It follows that for every n and $A \in \mathcal{P}_n$, the Γ-orbit of A is finite. Then we can repeat the argument in the second part of the proof of 1.3 to conclude that Γ is residually finite.

Conversely, assume Γ is residually finite and let $\Gamma_0 = \Gamma > \Gamma_1 > \Gamma_2 > \dots$ be a strictly decreasing sequence of normal subgroups of Γ with $\bigcap_n \Gamma_n = \{1\}$. Define then the tree T as in the first part of the proof of 1.3. A finite path $s = (v_0, \dots, v_n) = (\Gamma, \gamma_1\Gamma_1, \dots, \gamma_n\Gamma_n)$ in this tree is uniquely determined by the coset $\gamma_n\Gamma_n$ of Γ_n and, since Γ acts transitively on the set of such cosets, Γ acts transitively on the set of basic open sets

$$\{N_s : |s| = n+1\}, n \geq 0,$$

of $[T]$. Clearly this set has cardinality $[\Gamma : \Gamma_n] = k_n$, so there is a unique measure on $[T]$ for which $\mu(N_s) = 1/k_n$ and this is clearly the unique invariant measure for the Γ-action on $[T]$. □

(C) (Hjorth [12]) Let $(\mathbb{N})^{\mathbb{N}}$ be the set of all 1-1 functions from $\mathbb{N}$ into $\mathbb{N}$. This is a closed subset of $\mathbb{N}^{\mathbb{N}}$. Let S_∞ act on $\mathbb{N}^{\mathbb{N}}$ by

$$\pi \cdot \alpha(n) = \alpha(\pi^{-1}(n)).$$

Then $(\mathbb{N})^{\mathbb{N}}$ is invariant under this action. Let now Γ be a countable subgroup of S_∞. Then its action on $(\mathbb{N})^{\mathbb{N}}$ is modular. To see this, let for each $n, \mathcal{P}_n = \{A_k^{(n)}\}_{k \leq \infty}$, where, for $k \in \mathbb{N}, A_k^{(n)} = \{\alpha \in (\mathbb{N})^{\mathbb{N}} : \alpha(k) = n\}$, and $A_\infty^{(n)} = (\mathbb{N})^{\mathbb{N}} \setminus \bigcup_{k \in \mathbb{N}} A_k^{(n)}$. Since for $\gamma \in \Gamma, \gamma \cdot A_k^{(n)} = A_{\gamma(k)}^{(n)}$ for all $k \in \mathbb{N}$, each $\mathcal{P}_{n_S}$ is Γ-invariant and clearly $\{\mathcal{P}_n\}$ generates the Borel sets of $(\mathbb{N})^{\mathbb{N}}$. Since $S_\infty \subseteq (\mathbb{N})^{\mathbb{N}}$ is invariant under this action, it also follows that the right-translation action,

$$\gamma \cdot \alpha = \alpha\gamma^{-1},$$

of Γ on S_∞ is modular. Since the map $f(\alpha) = \alpha^{-1}$ from S_∞ to S_∞ is a Borel isomorphism of the right-translation action of Γ on S_∞ with its left-translation action,

$$\gamma * \alpha = \gamma\alpha,$$

it follows that the left-translation action of Γ on S_∞ is also modular.

(D) Consider the group $\Gamma = \mathrm{SL}_n(\mathbb{Z})$. Let also $G = (\mathbb{Q}/\mathbb{Z})^n$. Then Γ acts on the countable (discrete) Abelian group G by automorphisms in the usual way: if $\gamma = (a_{ij}) \in \Gamma$, then for

$$g = \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix} \in G, \ \gamma \cdot g = \begin{bmatrix} a_{11} \dots a_{1n} \\ \dots \\ a_{n1} \dots a_{nn} \end{bmatrix} \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}.$$

Denote by $K = \widehat{G}$ the dual group of G, a compact Polish group. Then Γ acts by (topological group) automorphisms on K, again in the usual way: if $\chi : G \to \mathbb{T}$ is a character and $\gamma \in \Gamma$, then $\gamma \cdot \chi : G \to \mathbb{T}$ is the character:

$$\gamma \cdot \chi(g) = \chi(\gamma^{-1} \cdot g).$$

One can explicitly compute K as follows (see [11, 25.5]). Consider first (the countable discrete group) $\mathbb{Q}$ and its dual $\widehat{\mathbb{Q}}$. Then, identifying a character $\chi : \mathbb{Q} \to \mathbb{T}$ with the sequence $\{\alpha_k\}_{k=1}^{\infty} \in \mathbb{T}^{\mathbb{N}^+}$ given by $a_k = \chi(\frac{1}{k!})$ (so that χ is recovered from this sequence via $\chi(m/k!) = a_k^m$), we see that $\widehat{\mathbb{Q}}$ can be identified with the subgroup of $\mathbb{T}^{\mathbb{N}^+}$ consisting of all $\{\alpha_k\} \in \mathbb{T}^{\mathbb{N}^+}$ such that $\alpha_{k+1}^{k+1} = \alpha_k$ for all $k \in \mathbb{N}^+$. Now $\widehat{(\mathbb{Q}/\mathbb{Z})}$ consists of all $\chi \in \widehat{\mathbb{Q}}$ such that $\chi(1) = 1$ and thus can be identified with the subgroup of all $\{\alpha_k\} \in \mathbb{T}^{\mathbb{N}^+}$ with $\alpha_{k+1}^{k+1} = \alpha_k, \alpha_k^{k!} = 1$ for all $k \in \mathbb{N}^+$, i.e., it consists of all $\{\alpha_k\} \in \prod_{k \in \mathbb{N}^+} R_{k!}$, with $\alpha_{k+1}^{k+1} = \alpha_k$ for all $k \in \mathbb{N}^+$, where $R_{k!}$ is the set of $(k!)$-th roots of unity. It follows that $K = \widehat{(\mathbb{Q}/\mathbb{Z})^n} \cong (\widehat{(\mathbb{Q}/\mathbb{Z})})^n$ can be identified with the set of all n-tuples $(\{\alpha_k^{(1)}\}, \dots, \{\alpha_k^{(n)}\})$ with each $\{\alpha_k^{(i)}\}$ as above. Tracing the definition of this dual, we see that the action of $\Gamma = \mathrm{SL}_n(\mathbb{Z})$ on K is given by the following formula.

If $\gamma = (a_{ij}) \in \mathrm{SL}_n(\mathbb{Z})$ and $x = (\{\alpha_k^{(1)}\}, \dots, \{\alpha_k^{(n)}\}) \in K$, then $\gamma \cdot x = y = (\{\beta_k^{(1)}\}, \dots, \{\beta_k^{(n)}\}) \in K$, where

$$\beta_k^{(i)} = (\alpha_k^{(1)})^{a_{1i}} (\alpha_k^{(2)})^{a_{2i}} \dots (\alpha_k^{(n)})^{a_{ni}}$$

for $k \in \mathbb{N}^+, 1 \le i \le n$. This shows that Γ acts by automorphisms on the rooted tree T consisting of $(\alpha_k^{(1)}, \dots, \alpha_k^{(n)})$, with $\alpha_k^{(i)} \in R_{k!}$ $(k = 1, 2, \dots)$, root $(1, \dots, 1)$, and in which the children of $(\alpha_k^{(1)}, \dots, \alpha_k^{(n)})$ are all $(\alpha_{k+1}^{(1)}, \dots, \alpha_{k+1}^{(n)})$ with $(a_{k+1}^{(i)})^{k+1} = a_k^{(i)}$. Since clearly K can be identified with $[T]$, this shows that the action of Γ on K is (finitely) modular.

By contrast, we will see later that the action of $\mathrm{SL}_n(\mathbb{Z})$ on the dual $\mathbb{T}^n$ of $\mathbb{Z}^n$ is antimodular, if $n \geq 2$. Similarly for the action of $\mathrm{SL}_n(\mathbb{Z})$ on the dual of $\mathbb{Q}^n$.

(E) (S. Thomas, private communication; see also [23]). Let K be a compact Polish group and let K act transitively in a Borel way on a standard Borel space Y. Let Γ be a countable dense subgroup of K. Assume that there is a decreasing sequence $\Gamma = \Gamma_0 \geq \Gamma_1 \geq \Gamma_2 \geq \dots$ of normal subgroups of Γ with $[\Gamma : \Gamma_n] < \infty$ such that $\bigcap_n \overline{\Gamma_n} = \{1\}$ (here bar denotes the closure in K). Then the Γ-action on Y is (finitely) modular.

To see this first let $x \in Y$ and let $L = K_x = \{k \in K : k \cdot x = x\}$ be its stabilizer. Then the K-space Y is Borel isomorphic to the K-space K/L, so we can assume without loss of generality that Y is compact and K acts continuously on Y.

Put $H_n = \overline{\Gamma_n}$, so that $H_n \trianglelefteq K, [K : H_n] < \infty$ and $\bigcap_n H_n = \{1\}$. Let $\mathcal{P}_n$ be the finite partition of Y consisting of the H_n-orbits of Y. Thus the elements of $\mathcal{P}_n$ are clopen subsets of Y. Since $H_n \trianglelefteq K, \Gamma$ acts on $\mathcal{P}_n$ (and since Γ is dense in K, it actually acts transitively on $\mathcal{P}_n$). It only remains to check that $\{\mathcal{P}_n\}$ separates points. Otherwise fix $x \neq y$ in Y, so that for each n, x, y belong to the same piece of the partition $\mathcal{P}_n$, i.e., there is some $h_n \in H_n$ with $h_n \cdot x = y$. Choose a subsequence $\{h_{n_i}\}$ which is converging, say $h_{n_i} \to h$. Clearly $h \in \bigcap_n H_n = \{1\}$, so as $h_{n_i} \cdot x \to h \cdot x = x$, $x = y$, a contradiction.

The following is an interesting instance of this situation. Fix $n \geq 2$ and a prime p, let $K = \mathrm{SL}_n(\mathbb{Z}_p)$, and consider the canonical action of $\mathrm{SL}_n(\mathbb{Z}_p)$ on the standard Borel space Y of k-dimensional subspaces of the n-dimensional vector space $\mathbb{Q}_p^n$, where $1 \leq k < n$. Then (see [23, 6.1]) this action is transitive. Consider now $\Gamma = \mathrm{SL}_n(\mathbb{Z})$, which is a dense subgroup of K. Let for $m = 0, 1, \dots, \Gamma_m$ be the kernel of the canonical surjection $\mathrm{SL}_n(\mathbb{Z}) \to \mathrm{SL}_n(\mathbb{Z}/p^m\mathbb{Z})$, so that $\overline{\Gamma_m}$ = the kernel of the canonical surjection $\mathrm{SL}_n(\mathbb{Z}_p) \to \mathrm{SL}_n(\mathbb{Z}_p/p^m\mathbb{Z}_p)$. It follows that $\Gamma = \Gamma_0 \geq \Gamma_1 \geq \dots, [\Gamma : \Gamma_m] < \infty$ and $\bigcap_m \overline{\Gamma_m} = \{1\}$. Thus the action of $\Gamma = \mathrm{SL}_n(\mathbb{Z})$ on the space of k-dimensional subspaces of $\mathbb{Q}_p^n, 1 \leq k < n$ is (finitely) modular.

Another example is the left-translation action of $\mathrm{SL}_n(\mathbb{Z}) \leq \mathrm{SL}_n(\mathbb{Z}_p)$ on $\mathrm{SL}_n(\mathbb{Z}_p)$.

2. Unitary representations

We review here some basic facts about unitary representations. A general reference is [1].

Let Γ be a countable (discrete) group. A *unitary representation* of Γ on a (separable, complex) Hilbert space H is an action of Γ on H by unitary operators or, equivalently, a homomorphism

$$\pi : \Gamma \to U(H),$$

where $U(H)$ is the unitary group of H. We write interchangeably

$$\gamma \cdot \xi = \pi(\gamma)\xi$$

for this action. We also denote H by H_π. Two representations $\pi : \Gamma \to U(H_\pi), \rho : \Gamma \to U(H_\rho)$ are *isomorphic*, in symbols $\pi \cong \rho$, if there is a Hilbert space isomorphism $T : H_\pi \to H_\rho$ such that $T(\gamma \cdot \xi) = \gamma \cdot T(\xi)$ for all $\gamma \in \Gamma, \xi \in H_\pi$. A *subrepresentation* of π is the restriction of π to a Γ-invariant closed subspace of H_π. If ρ is isomorphic to a subrepresentation of π, we say that ρ is (*strongly*) *contained* in π, in symbols $\rho \leq \pi$.

Given a countable family $\{\pi_i\}_{i \in I}$ of unitary representations of Γ with $H_i = H_{\pi_i}$, we define their *direct sum* $\bigoplus_i \pi_i$ as follows: Let $H = \bigoplus_i H_i$ be the direct sum of $\{H_i\}$ (i.e., the Hilbert space of all families $\{\xi_i\}$, also written as $\oplus_i \xi_i$ in this context, with $\xi_i \in H_i$ and $\sum ||\xi_i||_i^2 < \infty$, with inner product $\langle \oplus_i \xi_i, \oplus_i \eta_i \rangle = \sum_i \langle \xi_i, \eta_i \rangle_i$, where the subscript i refers to norm and inner product in H_i). Define the action of Γ on H by $\gamma \cdot \oplus_i \xi_i = \oplus_i(\gamma \cdot \xi_i)$. Clearly $H_i \leq \bigoplus_i H_i$. We denote, for each unitary representation π, by $n \cdot \pi$ the direct sum of n copies of π, $1 \leq n \leq \aleph_0$. Usually, we put $\infty \cdot \pi = \aleph_0 \cdot \pi$.

For each countable group Γ we denote by λ_Γ the *(left-) regular* representation of Γ, i.e., the (left-) shift action of Γ on $\ell^2(\Gamma)$:

$$\gamma \cdot \xi(\delta) = \xi(\gamma^{-1}\delta).$$

Finally, we will need the concept of weak containment of representations. The *matrix coefficients* of a representation π of Γ on H are the functions

$$f_{\xi,\eta}(\gamma) = \langle \gamma \cdot \xi, \eta \rangle, \ \xi, \eta \in H.$$

Then the diagonal matrix coefficients

$$\gamma \mapsto \langle \gamma, \xi, \xi \rangle$$

are positive definite functions on Γ (we recall that a function $f : \Gamma \to \mathbb{C}$ is *positive definite* if for all $\gamma_1, \ldots \gamma_n \in \Gamma, c_1, \ldots c_n \in \mathbb{C}$ we have

$\sum_{i,j=1}^{n} f(\gamma_j^{-1}\gamma_i)c_i\overline{c_j} \geq 0$). Conversely, given a positive definite f there is a unique (up to pointed isomorphism) triple (π_f, H_f, ξ_f) consisting of a unitary representation π_f of Γ on H_f and a *cyclic* vector $\xi_f \in H_f$ (i.e., a vector ξ_f such $\langle \Gamma \cdot \xi_f \rangle = H_f$, where $\langle \Gamma \cdot \xi_f \rangle$ is the closed linear span of $\Gamma \cdot \xi_f$), so that

$$f(\gamma) = \langle \gamma \cdot \xi_f, \xi_f \rangle,$$

(the GNS theorem).

We say that a positive definite function is *realized* in a unitary representation π if it is equal to a diagonal coefficient of π. Thus if f is realized in π, then $\pi_f \leq \pi$, since if $f(\gamma) = \langle \gamma \cdot \xi, \xi \rangle$, then π_f is isomorphic to the subrepresentation of π on the closed subspace $\langle \Gamma \cdot \xi \rangle$.

Definition 2.1. *Let π, ρ be two unitary representations of Γ on H_π, H_ρ, respectively. We say that π is* weakly contained *in ρ, in symbols*

$$\pi \prec \rho,$$

if every positive definite function realized in π is the pointwise limit of a sequence of finite sums of positive definite functions realized in ρ.

Explicitly, this means that if $\xi \in H_\pi$, and $f(\gamma) = \langle \gamma \cdot \xi, \xi \rangle$, then there is a sequence of functions f_n of the form $f_n(\gamma) = \sum_{i=1}^{k_n} \langle \gamma \cdot \eta_i, \eta_i \rangle$, $\eta_i \in H_\rho$, with $f_n(\gamma) \to f(\gamma)$ for all $\gamma \in \Gamma$. Equivalently, given any $\xi \in H_\pi, \epsilon > 0, F \subseteq \Gamma$ finite, there are $\eta_1, \dots, \eta_k \in H_\rho$ such that

$$|\langle \gamma \cdot \xi, \xi \rangle - \sum_{i=1}^{k} \langle \gamma \cdot \eta_i, \eta_i \rangle| < \epsilon, \quad \forall \gamma \in F.$$

Note that $\prec$ is transitive:

$$\pi \prec \rho \prec \sigma \Rightarrow \pi \prec \sigma.$$

We say that π, ρ are *weakly equivalent*, in symbols

$$\pi \sim \rho,$$

if $\pi \prec \rho$ and $\rho \prec \pi$. Clearly $\pi \leq \rho \Rightarrow \pi \prec \rho$ and $\pi \cong \rho \Rightarrow \pi \sim \rho$.

It is easy to check that

$$\forall i \in I(\pi_i \prec \rho) \;\Leftrightarrow\; \bigoplus_i \pi_i \prec \rho,$$

for any countable family $\{\pi_i\}$. It follows that

$$\pi \sim \infty \cdot \pi.$$

We are primarily interested in representations weakly contained in the regular one.

Proposition 2.2. *Let π be a unitary representation of a countable group Γ. Then the following are equivalent:*

(i) $\pi \prec \lambda_\Gamma$.

(ii) $\pi \prec \infty \cdot \lambda_\Gamma$.

(iii) *Every positive definite function realized in π is the pointwise limit of a sequence of positive definite functions realized in $\infty \cdot \lambda_\Gamma$.*

(iv) *For every $\xi_1, \dots, \xi_n \in H_\pi, \epsilon > 0, F \subseteq \Gamma$ finite, there are $\eta_1, \dots, \eta_n \in H_{\infty \cdot \lambda_\Gamma}$ such that*

$$|\langle \gamma \cdot \xi_i, \xi_j \rangle - \langle \gamma \cdot \eta_i, \eta_j \rangle| < \epsilon, \quad \text{for all} \quad \gamma \in F.$$

(v) *Same as* (iv) *for $\xi_1, \dots, \xi_n$ orthonormal and requiring that $\eta_1, \dots, \eta_n$ are also orthonormal.*

Proof. Note that if ρ is a unitary representation of Γ, then for $\theta_1, \dots, \theta_m \in H_\rho, \sum_{i=1}^m \langle \gamma \cdot \theta_i, \theta_i \rangle = \langle \gamma \cdot \bigoplus_{i=1}^m \theta_i, \bigoplus_{i=1}^m \theta_i \rangle$, in other words, the sum of finitely many positive definite functions realized in ρ is equal to a positive definite function realized in $\infty \cdot \rho$. Since $\infty \cdot (\infty \cdot \lambda_\Gamma) \cong \infty \cdot \lambda_\Gamma$, this immediately implies the equivalence of (i), (ii), (iii).

(v) $\Rightarrow$ (ii): For $n = 1$, (v) says that for any unit vector $\xi \in H_\pi, \epsilon > 0, F \subseteq \Gamma$ finite, there is a unit vector $\eta \in H_{\infty \cdot \lambda_\Gamma}$ with $|\langle \gamma \cdot \xi, \xi \rangle - \langle \gamma \cdot \eta, \eta \rangle| < \epsilon$ for all $\gamma \in F$. This immediately implies (ii).

(iii) $\Rightarrow$ (iv): First assume that π has a cyclic vector $\xi \in H_\pi$, i.e., $\langle \Gamma \cdot \xi \rangle = H_\pi$. Then given $\xi_1, \dots, \xi_n \in H_\pi, \epsilon > 0, F \subseteq \Gamma$ finite, find for $1 \leq i \leq n$, $\alpha_1^{(i)}, \dots, \alpha_{n_i}^{(i)} \in \mathbb{C}, \gamma_1^{(i)}, \dots, \gamma_{n_i}^{(i)} \in \Gamma$ such that

$$||\xi_i - \sum_{k=1}^{n_i} \alpha_k^{(i)} (\gamma_k^{(i)} \cdot \xi)|| < \epsilon, 1 \leq i \leq n.$$

If $M = \max \{||\xi_1||, \dots, ||\xi_n||\}$, we have

$$|\langle \gamma \cdot \xi_i, \xi_j \rangle - \langle \gamma \cdot \sum_{k=1}^{n_i} \alpha_k^{(i)} (\gamma_k^{(i)} \cdot \xi), \sum_{k=1}^{n_j} \alpha_k^{(j)} (\gamma_k^{(i)} \cdot \xi) \rangle| < 2\epsilon(\epsilon + M),$$

and

$$\begin{aligned}&\langle\gamma\cdot\sum_{k=1}^{n_i}\alpha_k^{(i)}(\gamma_k^{(i)}\cdot\xi),\sum_{k=1}^{n_j}\alpha_k^{(j)}(\gamma_k^{(j)}\cdot\xi)\rangle\\
&=\sum_{k=1}^{n_i}\sum_{\ell=1}^{n_j}\alpha_k^{(i)}\overline{\alpha}_\ell^{(j)}\langle\gamma\gamma_k^{(i)}\cdot\xi,\gamma_\ell^{(j)}\cdot\xi\rangle\\
&=\sum_{k=1}^{n_i}\sum_{\ell=1}^{n_j}\alpha_k^{(i)}\overline{\alpha}_\ell^{(j)}\langle(\gamma_\ell^{(j)})^{-1}\gamma\gamma_k^{(i)}\cdot\xi,\xi\rangle.\end{aligned}$$

Then find $\eta\in H_{\infty\cdot\lambda_\Gamma}$ such that

$$\sum_{k=1}^{n_i}\sum_{\ell=1}^{n_j}\left|a_k^{(i)}\overline{\alpha}_\ell^{(j)}\right|\cdot\left|\langle(\gamma_\ell^{(j)})^{-1}\gamma\gamma_k^{(i)}\cdot\xi,\xi\rangle-\langle(\gamma_\ell^{(j)})^{-1}\gamma\gamma_k^{(i)}\cdot\eta,\eta\right|<\epsilon$$

for all $\gamma\in F, 1\leq i,j\leq n$. Put

$$\eta_i=\sum_{k=1}^{n_i}\alpha_k^{(i)}(\gamma_k^{(i)}\cdot\eta)\in H_{\infty\cdot\lambda_\Gamma}.$$

Then

$$\begin{aligned}&|\langle\gamma\cdot\xi_i,\xi_j\rangle-\langle\gamma\cdot\eta_i,\eta_j\rangle|\\
&\leq 2\epsilon(\epsilon+M)+\Big|\langle\gamma\cdot\sum_{k=1}^{n_i}\alpha_k^{(i)}(\gamma_k^{(i)}\cdot\xi),\sum_{k=1}^{n_j}\alpha_k^{(j)}(\gamma_k^{(j)}\cdot\xi)\rangle\\
&\qquad-\langle\gamma\cdot\sum_{k=1}^{n_i}\alpha_k^{(i)}(\gamma_k^{(i)}\cdot\eta),\sum_{k=1}^{n_j}\alpha_k^{(j)}(\gamma_k^{(j)}\cdot\eta\rangle\Big|\\
&=2\epsilon(\epsilon+M)+\Big|\sum_{k=1}^{n_i}\sum_{\ell=1}^{n_j}\alpha_k^{(i)}\overline{\alpha}_\ell^{(i)}\langle(\gamma_\ell^{(j)})^{-1}\gamma\gamma_k^{(i)}\cdot\xi,\xi\rangle\\
&\qquad-\sum_{k=1}^{n_i}\sum_{\ell=1}^{n_j}\alpha_k^{(i)}\overline{\alpha}_\ell^{(j)}\langle(\gamma_\ell^{(j)})^{-1}\gamma\gamma_k^{(i)}\cdot\eta,\eta\rangle\Big|\\
&<2\epsilon(\epsilon+M)+\epsilon.\end{aligned}$$

This completes the proof in the case where π has a cyclic vector. In the general case, we can write

$$H_\pi=\bigoplus_k H_{\pi_k},\ \pi=\bigoplus_k\pi_k,$$

where each π_k has a cyclic vector. Given now $\xi_1, \ldots, \xi_n \in H_\pi$, write

$$\xi_i = \bigoplus_k \xi_i^{(k)}, \xi_i^{(k)} \in H_{\pi_k},$$

so that

$$\langle \gamma \cdot \xi_i, \xi_j \rangle = \sum_k \langle \gamma \cdot \xi_i^{(k)}, \xi_j^{(k)} \rangle.$$

Given $\epsilon > 0$, $F \subseteq \Gamma$ finite, find N large enough so that

$$\left| \langle \gamma \cdot \xi_i, \xi_j \rangle - \sum_{k=0}^{N} \langle \gamma \cdot \xi_i^{(k)}, \xi_j^{(k)} \rangle \right| < \epsilon$$

for all $\gamma \in F$. Since $\xi_1^{(k)}, \ldots, \xi_n^{(k)} \in H_{\pi_k}$, by the special case above, we can find $\eta_1^{(k)}, \ldots, \eta_n^{(k)} \in H_{\infty \cdot \lambda_\Gamma}$ so that

$$\left| \langle \gamma \cdot \xi_i^{(k)}, \xi_j^{(k)} \rangle - \langle \gamma \cdot \eta_i^{(k)}, \eta_j^{(k)} \rangle \right| < \epsilon/(N+1)$$

for all $\gamma \in F, 0 \leq k \leq N$. Put

$$\eta_i = \eta_i^{(0)} \oplus \eta_i^{(1)} \oplus \cdots \oplus \eta_i^{(N)},$$

$\eta_i \in H_{(N+1) \cdot (\infty \cdot \lambda_\Gamma)} \cong H_{\infty \cdot \lambda_\Gamma}$. Then

$$\langle \gamma \cdot \eta_i, \eta_j \rangle = \sum_{k=0}^{N} \langle \gamma \cdot \eta_i^{(k)}, \eta_i^{(k)} \rangle,$$

so for each $\gamma \in F$:

$$\begin{aligned}
&\left| \langle \gamma \cdot \xi_i, \xi_j \rangle - \langle \gamma \cdot \eta_i, \eta_j \rangle \right| \\
\leq &\left| \langle \gamma \cdot \xi_i, \xi_j \rangle - \sum_{k=0}^{N} \langle \gamma \cdot \xi_i^{(k)}, \xi_j^{(k)} \rangle \right| \\
+ &\left| \sum_{k=0}^{N} \langle \gamma \cdot \xi_i^{(k)}, \xi_j^{(k)} \rangle - \sum_{k=0}^{N} \langle \gamma \cdot \eta_i^{(k)}, \eta_j^{(k)} \rangle \right| \\
< &\epsilon + (N+1)\epsilon/(N+1) = 2\epsilon.
\end{aligned}$$

(iv) $\Rightarrow$ (v): Let $\xi_1, \ldots, \xi_n$ be orthonormal in H_π, and fix $\epsilon, F \subseteq \Gamma$ finite. Then, by (iv), we can find $\eta'_1, \ldots, \eta'_n \in H_{\infty \cdot \lambda_\Gamma}$ such that for some $\delta < \epsilon$, to be determined later,

$$\left| \langle \gamma \cdot \xi_i, \xi_j \rangle - \langle \gamma \cdot \eta'_i, \eta'_j \rangle \right| < \delta$$

for all $\gamma \in F \cup \{1\}$, so that, in particular,

$$\left|\langle \eta_i', \eta_j' \rangle - \delta_{ij}\right| < \delta,$$

where δ_{ij} is the Kronecker delta. Clearly, by choosing δ small enough, this implies that $\eta_1', \dots, \eta_n'$ are linearly independent, so applying the Gram–Schmidt process, we obtain orthonormal $\eta_1, \dots, \eta_n$ such that, if again δ is chosen small enough,

$$\left\|\eta_i' - \eta_i\right\| < \epsilon, \ i = 1, \dots, n.$$

Then for any $\gamma \in \Gamma, 1 \leq i, j \leq n$,

$$\left|\langle \gamma \cdot \eta_i', \eta_j' \rangle - \langle \gamma \cdot \eta_i, \eta_j \rangle\right| < \epsilon(1+\epsilon) + \epsilon$$

and the proof is complete. □

Remark. Condition (iii) above is equivalent to its strengthening that replaces $\infty \cdot \lambda_\Gamma$ by λ_Γ, see [7, 18.3.5], but we will not need this fact.

Suppose now that a countable group Γ acts in a Borel way on a standard Borel space X with invariant (Borel probability) measure μ. Then this action gives rise to a unitary representation π^X on $H = L^2(X, \mu) \equiv L^2(X)$, given by

$$(\pi(\gamma)f)(x) = (\gamma \cdot f)(x) = f(\gamma^{-1} \cdot x).$$

If 1 denotes the constant 1 function on X, then $\mathbb{C}1$ (= the set of constant functions on X) is a one-dimensional invariant subspace under π^X. Consider its orthogonal complement

$$(\mathbb{C}1)^\perp = L^2_0(X) = \{f \in L^2(X) : \int f d\mu = 0\},$$

which is also invariant under π^X. We denote by π^X_0 the restriction of π^X to $L^2_0(X)$.

Definition 2.3. *We call the action of* Γ *on* X tempered *if* $\pi^X_0 \prec \lambda_\Gamma$.

Recall that the action of Γ on X is *ergodic* if and only if π^X_0 has no invariant vectors if and only if $1_\Gamma \not\leq \pi^X_0$, where 1_Γ is the *trivial* 1-*dimensional* representation of Γ ($\gamma \cdot \xi = \xi$ for $\xi \in \mathbb{C}$). Since $1_\Gamma \not\prec \lambda_\Gamma$, when Γ is non-amenable (see the second paragraph below), it follows that if the action of a non-amenable Γ on X is tempered, then it is also ergodic.

However, it should be pointed out that the converse fails, in fact there are even (strongly) mixing actions of groups which are not tempered. One way to see this is as follows.

It is a general fact that if ρ is a unitary representation of a countable group Γ, then there is a Borel action of Γ on a standard Borel space X with invariant measure μ such that $\rho \leq \pi_0^X$ (see [25, pp. 110–111] and [5, pp. 22–23]). Moreover, if ρ is a c_0-*representation* (i.e., the matrix coefficients $\gamma \mapsto \langle \gamma \cdot \xi, \eta \rangle, \xi, \eta \in H_\rho$, vanish at infinity, $\langle \gamma \cdot \xi, \eta \rangle \to 0$ as $\gamma \to \infty$), so is π_0^X. Finally, recall that the action of Γ on (X, μ) is (strongly) mixing if and only if π_0^X is a c_0-representation (see [2, Chap. I] and [20, 22]). Also, by [5, 2.1.1], the group Γ has the *Haagerup Approximation Property* (HAP) if and only if it admits a unitary representation ρ with $1_\Gamma \prec \rho$ such that ρ is a c_0-representation. Examples of HAP groups include the free groups F_n. It follows that every HAP group Γ admits a Borel action on X with invariant measure μ, such that π_0^X is a c_0-representation and $1_\Gamma \prec \pi_0^X$. Then the action of Γ on X is mixing but not tempered if Γ is not amenable. Indeed, otherwise $\pi_0^X \prec \lambda_\Gamma$, so $1_\Gamma \prec \lambda_\Gamma$. But by a standard result of Hulanicki–Reiter (see [1, 7.3.2]), a countable group Γ is amenable if and onlt if $1_\Gamma \prec \lambda_\Gamma$ (if and only if for every unitary representation σ of $\Gamma, \sigma \prec \lambda_\Gamma$).

Despite this, it turns out that in the case of actions of countable groups by (topological group) automorphisms on compact Polish groups, there is a nice relationship between the property of being tempered and ergodicity properties of the action. This will be discussed in §4 below.

3. Antimodular actions

Let a countable group Γ act in a Borel way on a standard Borel space X. We call the action *antimodular* if for any countable group Δ and any modular action of Δ on a space Y, there is no $\aleph_0$-to-1 Borel homomorphism of E_Γ^X to E_Δ^Y.

It follows immediately from the definition that if $\Gamma_0 \leq \Gamma$ and the action of Γ_0 on X is antimodular, then the action of Γ on X is antimodular.

We now have the following main result.

Theorem 3.1. *Let Γ be a countable group with $F_2 \leq \Gamma$. Suppose Γ acts in a Borel way on a standard Borel space X with non-atomic invariant measure μ. If this action is tempered, then it is antimodular.*

Proof. First we claim that it is enough to consider the case $\Gamma = F_2$. Indeed, assume this result was proved for F_2. In the general case, by the preceding remark, it is enough to show that the action of F_2 on X is antimodular, therefore, by the special case, it is enough to show that it is tempered. From the definition, it follows that if $\pi_0^X|F_2$ is the restriction of the representation π_0^X to F_2, then $\pi_0^X|F_2 \prec \lambda_\Gamma|F_2$, so it is enough to check that $\lambda_\Gamma|F_2 \leq \infty \cdot \lambda_{F_2}$. Consider the orthonormal basis $\{e_\delta\}_{\delta\in\Gamma}$ of $\ell^2(\Gamma)$ given by $e_\delta(\gamma) = 1$ if $\delta = \gamma$, 0 otherwise. Then if $C_1(= F_2), C_2, \cdots$ (finite or infinite) are the right cosets of F_2 in Γ, and $H_i = \langle e_\delta : \delta \in C_i\rangle$, then H_i is F_2-invariant, $H = \bigoplus_i H_i$, and the representation of F_2 on H_i is clearly isomorphic to λ_{F_2}.

So from now on we assume that

$$\Gamma = F_2.$$

(A) We denote by $\mathcal{F}(F_2)$ the set of finite subsets of F_2 and by $\mathcal{F}_0(F_2)$ the set of nonempty finite subsets of F_2. The group F_2 acts on $\mathcal{F}(F_2), \mathcal{F}_0(F_2)$ by left-translation: $\gamma \cdot S = \gamma S$. Thus it acts unitarily on $\ell^2(\mathcal{F}(F_2))$ by

$$\gamma \cdot p(S) = p(\gamma^{-1}S).$$

Denoting by $e_S \in \ell^2(\mathcal{F}(F_2))$ the function

$$e_S(T) = \begin{cases} 1 & \text{if } S = T, \\ 0 & \text{otherwise,} \end{cases}$$

we have that $\{e_S\}_{S\in\mathcal{F}(F_2)}$ is an orthonormal basis for $\ell^2(\mathcal{F}(F_2))$, invariant under F_2 (as $\gamma \cdot e_S = e_{\gamma S}$). The one-dimensional subspace $\langle e_\emptyset\rangle$ is F_2-invariant with orthogonal complement

$$\langle e_\emptyset\rangle^\perp = \langle e_S : S \neq \emptyset\rangle.$$

Clearly, if we also define $f_S \in \ell^2(\mathcal{F}_0(F_2))$ similarly, $\{f_S\}_{S\in\mathcal{F}_0(F_2)}$ is an F_2-invariant orthonormal basis for $\ell^2(\mathcal{F}_0(F_2))$ and the map $e_S \leftrightarrow f_S$ is an isomorphism of the F_2-action on $\langle e_\emptyset\rangle^\perp$ and that on $\ell^2(\mathcal{F}_0(F_2))$.

The action of F_2 on $\mathcal{F}_0(F_2)$ has infinitely many infinite orbits $\mathcal{O}_1, \mathcal{O}_2, \ldots$. Thus

$$\ell^2(\mathcal{F}_0(F_2)) = \bigoplus_n \langle f_S : S \in \mathcal{O}_n\rangle.$$

Now F_2 acts on $\{f_S : S \in \mathcal{O}_n\}$ transitively and freely, from which it immediately follows that the representation of F_2 on $\langle f_S : S \in \mathcal{O}_n\rangle$ is isomorphic to the regular representation of F_2. So if we denote by $\mu_{F_2}^0$ the representation of F_2 on $\ell^2(\mathcal{F}_0(F_2))$, then

$$\mu_{F_2}^0 \cong \infty \cdot \lambda_{F_2},$$

and, if we denote by μ_{F_2} the representation of F_2 on $\ell^2(\mathcal{F}(F_2))$, then, via $\ell^2(\mathcal{F}(F_2)) = \langle e_\emptyset\rangle \oplus \bigoplus_n \langle e_S : S \in \mathcal{O}_n\rangle$,

$$\mu_{F_2} \cong 1_{F_2} \oplus \infty \cdot \lambda_{F_2}.$$

We consider also the Banach spaces $\ell^1(F_2), \ell^1(\mathcal{F}(F_2)), \ell^1(\mathcal{F}_0(F_2))$ on which again F_2 acts (by linear isometries) via $\gamma \cdot f(u) = f(\gamma^{-1}u)$. We will need the following version of a lemma in Hjorth [12]. We will postpone the proof for the end.

Lemma 3.2. *There is a map $\Theta : \ell^2(\mathcal{F}_0(F_2)) \to \ell^1(F_2)$ with the following properties:*
(i) $\|\Theta(f)\|_1 = \|f\|_2, \forall f \in \ell^2(\mathcal{F}_0(F_2))$,
(ii) $\|\Theta(f_1) - \Theta(f_2)\|_1 \leq 3 \cdot \|f_1 - f_2\|_2, \forall f_1, f_2 \in \ell^2(\mathcal{F}_0(F_2))$.
(iii) $\Theta(f) \geq 0, \forall f \in \ell^2(\mathcal{F}_0(F_2))$.
(iv) Θ *is an F_2-map,* $\Theta(\gamma \cdot f) = \gamma \cdot \Theta(f), \forall f \in \ell^2(\mathcal{F}(F_2)), \gamma \in F_2$.

(B) We now discuss weights on F_2 and their centers. All the concepts and facts here come from Hjorth [12].

Definition 3.3. *A* weight *on F_2 is any $w \in \ell^1(F_2), w \geq 0, w \neq 0$. An element $\gamma \in F_2$ is called a* center *of a weight w if each one of the 4 regions around γ in the Cayley graph of F_2 has $(w-)$weight $\leq \frac{1}{2}\|w\|_1$. Formally, this means that if $F_2 = \langle a, b\rangle$, then for every $g \in \{a^{\pm 1}, b^{\pm 1}\}$,*

$$\sum\{w(\gamma g\delta) : g\delta \text{ reduced}\} \leq \tfrac{1}{2}\|w\|_1.$$

We call γ a weak center *of w if each one of the above regions has weight $< \frac{2}{3}\|w\|_1$.*

Fact 3.4. Every weight has at least one center.

Proof. If not, then $1 \in F_2$ is not a center, so let $g_0 \in \{a^{\pm 1}, b^{\pm 1}\}$ be such that $\sum\{w(g_0\delta) : g_0\delta \text{ reduced}\} > \frac{1}{2}\|w\|_1$. Then g_0 is not a center, so find $g_1 \in \{a^{\pm 1}, b^{\pm 1}\}$ such that $\sum\{w(g_0g_1\delta) : g_1\delta \text{ reduced}\} > \frac{1}{2}\|w\|_1$, and note that $g_0g_1 \neq 1$, so g_0g_1 is reduced. Continuing this way, we can find $g_0, g_1, \ldots, g_n, \cdots \in \{a^{\pm 1}, b^{\pm 1}\}$ with $g_0g_1\ldots g_n$ reduced, for each n, such that $\sum\{w(g_0g_1\ldots g_n\delta) : g_n\delta \text{ reduced}\} > \frac{1}{2}\|w\|_1$, contradicting the fact that at least half of the weight of w is concentrated in a finite radius ball around 1 in the Cayley graph of F_2. $\square$

Fact 3.5. The weak centers of a weight form a finite convex (in the sense of the Cayley graph) subset of F_2. Similarly for the centers.

Proof. If γ is in the line (geodesic) connecting a weak center γ_1 to a weak center γ_2, notice that each of the four regions associated to γ is contained in one of the four regions associated with γ_1 or in one of the four regions associated with γ_2.

To see that there are only finitely many weak centers, choose an integer n, so that the sum of the weights of all $\gamma \in F_2$ with distance (in the Cayley graph) $\leq n$ from 1 is $\geq \frac{2}{3} \cdot \|w\|_1$. Then if $\gamma \in F_2$ is at distance $> n+1$, γ is not a weak center, since one of the four regions associated with γ contains the ball of radius n around 1. □

Fact 3.6. The weak centers of a weight w lie on a line (i.e., a geodesic connecting two vertices in the Cayley graph of F_2).

Proof. If not, there is a weak center γ which has three weak centers $\gamma_1, \gamma_2, \gamma_3$ as neighbors in the Cayley graph. Then the three disjoint regions R_1, R_2, R_3 given by $R_i = \{\gamma_i g\delta : g\delta \text{ reduced}, g \in \{a^{\pm 1}, b^{\pm 1}\} \setminus \{g_i\}\}$, with $g_i \in \{a^{\pm 1}, b^{\pm 1}\}$ such that $\gamma g_i^{-1} = \gamma_i$, have weight $> \frac{1}{3} \cdot \|w\|_1$, a contradiction. □

Fact 3.7. Let w_1, w_2 be weights. If

$$\|w_1 - w_2\|_1 < \frac{1}{12} \cdot \|w_2\|_1,$$

then every center of w_1 is a *weak* center of w_2.

Proof. Let γ be a center of w_1, and let R be any of the four regions associated to γ. Then

$$\left| \sum_{\delta \in R} w_1(\delta) - \sum_{\delta \in R} w_2(\delta) \right| \leq \|w_1 - w_2\|_1 < \frac{1}{12} \cdot \|w_2\|_1.$$

Also

$$\|w_1\|_1 \leq \|w_2\|_1 + \|w_1 - w_2\|_1 < \|w_2\|_1 + \frac{1}{12} \cdot \|w_2\|_1 = \frac{13}{12} \cdot \|w_2\|_1.$$

So, since

$$\sum_{\delta \in R} w_1(\delta) \leq \frac{1}{2} \cdot \|w_1\|_1 < \frac{7}{12} \cdot \|w_2\|_1,$$

we have

$$\sum_{\delta\in R} w_2(\delta) < \frac{1}{12}\cdot\|w_2\|_1 + \sum_{\delta\in R} w_1(\delta)$$
$$< \frac{1}{12}\cdot\|w_2\|_1 + \frac{7}{12}\cdot\|w_2\|_1$$
$$= \frac{2}{3}\cdot\|w_2\|_1,$$

so γ is a weak center for w_2. □

Finally we have the fact obvious from the definitions.

Fact 3.8. If C_w is the set of centers for w and WC_w the set of weak centers of w, then

$$\gamma\cdot C_w = C_{\gamma\cdot w},\ \gamma\cdot WC_w = WC_{\gamma\cdot w}.$$

(C) Assume now, towards a contradiction, that $\Gamma = F_2$ acts in a Borel way on X with invariant non-atomic measure μ, a countable group Δ acts in a modular way on Y and $\varphi : X \to Y$ is an $\aleph_0$-to-1 Borel homomorphism of $E^X_{F_2}$ to E^Y_Δ. By 1.2, we can assume that φ is 1-1. Fix a Borel map $\alpha : F_2\times X \to \Delta$ so that

$$\varphi(\gamma\cdot x) = \alpha(\gamma, x)\cdot\varphi(x), \forall x\in X, \gamma\in F_2.$$

Let finally $\mathcal{P}_0 \geq \mathcal{P}_1 \geq \mathcal{P}_2 \geq \dots$ be a decreasing sequence of partitions witnessing the modularity of the Δ-action on Y. For $A\subseteq Y$, let $\widehat{A} = \varphi^{-1}(A)$. The following lemma comes from Hjorth [12].

Lemma 3.9. *For any finite symmetric $F\subseteq F_2$ containing 1, and any $\epsilon > 0$, there is a Borel set $M\subseteq X$ with $\mu(X\setminus M) < \epsilon$, and there is k and $A_1,\dots,A_n\in\mathcal{P}_k$ such that $M\subseteq \widehat{A}_1\cup\dots\cup\widehat{A}_n$, $\frac{1}{2} > \mu(\widehat{A}_i) > 0, i = 1,\dots,n$, and*

$$(*)\ \forall\gamma\in F(x\mapsto\alpha(\gamma,x)\ \text{ is constant on each }\ M\cap\widehat{A}_i, i=1,\dots,n).$$

We will postpone the proof for later.

Take now F = the ball of radius 3 around 1 in the Cayley graph of F_2. Fix also $\delta < \frac{1}{2}$ to be a small enough positive number so that certain inequalities concerning some fixed functions of δ that will come up below are satisfied. (For example, $\delta = 10^{-6}$ is small enough.) Then fix ϵ small enough so that $|F|\cdot\epsilon\cdot(1+\frac{1}{\delta}) < 1/2$. Apply then 3.9 to obtain $k, A_1,\dots,A_n, M$. These will be fixed from now on.

Lemma 3.10. *If* $\gamma \in F, 1 \leq i \leq j \leq n$ *and* $\gamma \cdot (M \cap \widehat{A}_i) \cap (M \cap \widehat{A}_j) \neq \emptyset$, *then*

$$\gamma \cdot (M \cap \widehat{A}_i) \subseteq \widehat{A}_j,$$

$$\gamma^{-1} \cdot (M \cap \widehat{A}_j) \subseteq \widehat{A}_i.$$

Proof. Suppose $x \in M \cap \widehat{A}_i, \gamma \cdot x \in M \cap \widehat{A}_j$. Let $x' \in M \cap \widehat{A}_i$, in order to show that $\gamma \cdot x' \in \widehat{A}_j$ or $\varphi(\gamma \cdot x') = \alpha(\gamma, x') \cdot \varphi(x') \in A_j$. Let $\beta = \alpha(\gamma, x') = \alpha(\gamma, x)$. Then, as $\varphi(x) \in A_i, \varphi(\gamma \cdot x) = \beta \cdot \varphi(x) \in A_j$, and as Δ acts on $\mathcal{P}_k, \beta \cdot A_i = A_j$, so, as $\varphi(x') \in A_i$, we have $\varphi(\gamma \cdot x') = \beta \cdot \varphi(x') \in A_j$ or $\gamma \cdot x' \in \widehat{A}_j$. □

Call $\widehat{A}_i, 1 \leq i \leq n$, *good* if

$$\frac{\mu(\widehat{A}_i \setminus M)}{\mu(\widehat{A}_i)} < \delta.$$

Otherwise $\widehat{A}_i$ is *bad*. Now

$$\epsilon > \mu(X \setminus M) \geq \sum_{\widehat{A}_i \text{ bad}} \mu(\widehat{A}_i \setminus M) \geq \delta \sum_{\widehat{A}_i \text{ bad}} \mu(\widehat{A}_i),$$

so

$$\sum_{\widehat{A}_i \text{ bad}} \mu(\widehat{A}_i) \leq \frac{\epsilon}{\delta}.$$

Put finally

$$N = \left\{ x \in X : \forall \gamma \in F (\gamma \cdot x \in M \;\& \gamma \cdot x \in \bigcup \{\widehat{A}_i : \widehat{A}_i \text{ is good}\} \right\}.$$

Then

$$\mu(X \setminus N) \leq |F| \cdot \left(\epsilon + \frac{\epsilon}{\delta}\right) < 1/2,$$

thus, in particular, there is at least one good $\widehat{A}_i$. Also note that

$$N \subseteq M \cap \bigcup \{\widehat{A}_i : \widehat{A}_i \text{ is good}\}.$$

Lemma 3.11. *If $x \in N, x \in \widehat{A}_i$, $\gamma \in F$ and $\gamma \cdot x \in \widehat{A}_j$, then $(1-\delta) \leq \frac{\mu(\widehat{A}_i)}{\mu(\widehat{A}_j)} \leq \frac{1}{(1-\delta)}$ and*

$$\mu(\gamma \cdot \widehat{A}_i \Delta \widehat{A}_j) \leq 3\delta \cdot \mu(\widehat{A}_j).$$

Proof. We have $\gamma \cdot x \in \gamma \cdot (M \cap \widehat{A}_i) \cap (M \cap \widehat{A}_j)$, so, by 3.10, $\gamma \cdot (\widehat{A}_i \cap M) \subseteq \widehat{A}_j$, thus $\mu(\widehat{A}_i \cap M) \leq \mu(\widehat{A}_j)$. So, since $\widehat{A}_i$ is good,

$$\mu(\widehat{A}_i) \leq \frac{1}{(1-\delta)} \mu(\widehat{A}_i \cap M) \leq \frac{1}{(1-\delta)} \mu(\widehat{A}_j) \leq 2\mu(\widehat{A}_j),$$

since $\delta < \frac{1}{2}$. Thus

$$\mu(\gamma \cdot \widehat{A}_i \setminus \widehat{A}_j) = \mu(\widehat{A}_i \setminus \gamma^{-1} \cdot \widehat{A}_j) \leq \mu(\widehat{A}_i \setminus M) \leq \delta \cdot \mu(\widehat{A}_i) \leq 2\delta \cdot \mu(\widehat{A}_j),$$

and

$$\mu(\widehat{A}_j \setminus \gamma \cdot \widehat{A}_i) \leq \mu(\widehat{A}_j \setminus M) \leq \delta \cdot \mu(\widehat{A}_j),$$

so $\mu(\gamma \cdot \widehat{A}_i \Delta \widehat{A}_j) \leq 3\delta \cdot \mu(A_j)$.

We have already seen that

$$\frac{\mu(\widehat{A}_i)}{\mu(\widehat{A}_j)} \leq \frac{1}{(1-\delta)}.$$

But, by 3.10 again, $\gamma^{-1} \cdot (M \cap \widehat{A}_j) \subseteq \widehat{A}_i$, so $\mu(M \cap \widehat{A}_j) \leq \mu(\widehat{A}_i)$, so

$$\mu(\widehat{A}_j) \leq \frac{1}{(1-\delta)} \mu(\widehat{A}_j \cap M) \leq \frac{1}{(1-\delta)} \mu(\widehat{A}_i),$$

so $\mu(\widehat{A}_i)/\mu(\widehat{A}_j) \geq (1-\delta)$ as well. □

(D) We summarize what we have proved so far:

There is a finite nonempty collection $\mathcal{G}$ (the good sets) of pairwise disjoint Borel subsets of X, with $0 < \mu(C) < \frac{1}{2}$ for $C \in \mathcal{G}$, and a Borel set $N \subseteq \bigcup \mathcal{G}$ with $\mu(N) > 1/2$ such that if $x \in N, x \in C \in \mathcal{G}, \gamma \in F$, then $\gamma \cdot x \in \bigcup \mathcal{G}$, say $\gamma \cdot x \in D \in \mathcal{G}$,

$$(1-\delta) \leq \frac{\mu(C)}{\mu(D)} \leq \frac{1}{(1-\delta)}$$

and

$$\mu(\gamma \cdot C \Delta D) \leq 3\delta \cdot \mu(D).$$

We will derive a contradiction from this. Since the action of F_2 on X is tempered, $\pi_0^X \prec \lambda_{F_2}$, and thus, since $\mu_{F_2}^0 \cong \infty \cdot \lambda_{F_2}$, we have, from 2.2, that for every $\epsilon_1 > 0$ and $\xi_1, \dots, \xi_m \in L_0^2(X)$ there are $\eta_1, \dots, \eta_m \in \ell^2(\mathcal{F}_0(F_2))$ with

$$(*)\ |\langle \gamma \cdot \xi_i, \xi_j \rangle - \langle \gamma \cdot \eta_i, \eta_j \rangle| < \epsilon_1, \forall \gamma \in F.$$

Note that if $\epsilon_1 < \|\xi_i\|_2^2, i = 1, \dots, m$, then $\eta_1, \dots, \eta_m \neq 0$.

Below 1_C is the characteristic function of C. Letting $\mathcal{G} = \{C_1, \dots, C_m\}$, we apply this to

$$\xi_i = 1_{C_i} - \mu(C_i)1, \quad i = 1, \dots, m,$$

(note that $\xi_i \in L_0^2(X)$ and $\|\xi_i\|_2^2 = \mu(C_i)(1 - \mu(C_i)) > 0$), with $\epsilon_1 < \frac{1}{102^2}\mu(C_i)(1 - \mu(C_i)) = \frac{1}{102^2} \cdot \|\xi_i\|_2^2, i = 1, \dots, m$, to find $\eta_1, \dots, \eta_m \in \ell^2(\mathcal{F}_0(F_2))$ satisfying $(*)$, so in particular $\eta_i \neq 0, i = 1, \dots, m$.

Now consider the map Θ given by 3.2. For any $x \in \bigcup \mathcal{G}$, put

$$w(x) = \Theta(\eta_i),$$

where C_i is the unique element of $\mathcal{G}$ such that $x \in C_i$. Thus $w(x)$ is a weight on F_2. Note that if $x \in N, x \in C_i \in \mathcal{G}, \gamma \in F$, then $\gamma \cdot x \in C_j \in \mathcal{G}$ and

$$w(\gamma \cdot x) = \Theta(\eta_j).$$

Lemma 3.12. *Assuming δ has been chosen small enough, we have*

$$\|\gamma \cdot w(x) - w(\gamma \cdot x)\|_1 < \frac{1}{12} \cdot \|w(\gamma \cdot x)\|_1,$$

for any $x \in N, \gamma \in F$.

We will postpone the proof of this for later.

Granting this lemma, we can complete the proof as in Hjorth [12]. For $g \in \{a^{\pm 1}, b^{\pm 1}\}$, put

$$\begin{aligned} N_g = \{x \in N : &\ \text{there is a center of } \ w(x) \ \text{ in the set} \\ &\{g\gamma : g\gamma \ \ \text{reduced}\}\}. \end{aligned}$$

Also let

$$N_1 = \{x \in N : 1 \text{ is a center of } w(x)\}.$$

Then $N = N_1 \cup N_a \cup N_{a^{-1}} \cup N_b \cup N_{b^{-1}}$, so one of these sets has measure $\geq \frac{1}{5}\mu(N)$. Say it is N_a, the other cases being similar.

Let $K \subseteq F$ be the set of 27 elements of F of length exactly 3 that do not end in a^{-1}, and for $\gamma \in K$ put

$$B_\gamma = \{x \in \bigcup \mathcal{G} : \text{ there is a weak center of } w(x) \\ \text{in the set } \{\gamma\gamma' : \gamma\gamma' \text{ reduced}\}\}.$$

Then if $x \in N_a, \gamma \in K$, there is a center θ for $w(x)$ in the set $\{a\delta : a\delta$ is reduced$\}$, so $\gamma\theta$ is a weak center of $w(\gamma \cdot x)$, by 3.12, 3.8, and 3.7. So $\gamma \cdot x \in B_\gamma$, thus $\mu(B_\gamma) \geq \mu(N_a) \geq \frac{1}{5}\mu(N)$.

Now, by the inclusion-exclusion principle,

$$\mu(\bigcup_{\gamma \in K} B_\gamma) = \sum_{\gamma \in K} \mu(B_\gamma) - \sum_{\{\gamma_1,\gamma_2\} \in [K]^2} \mu(B_{\gamma_1} \cap B_{\gamma_2}) \\ + \sum_{\{\gamma_1,\gamma_2,\gamma_3\} \in [K]^3} \mu(B_{\gamma_1} \cap B_{\gamma_2} \cap B_{\gamma_3}) - \ldots,$$

where $[K]^n = \{a \subseteq K : |a| = n\}$.

If for some $\{\gamma_1, \gamma_2, \gamma_3\} \in [K]^3, B_{\gamma_1} \cap B_{\gamma_2} \cap B_{\gamma_3} \neq \emptyset$, with say $x \in B_{\gamma_1} \cap B_{\gamma_2} \cap B_{\gamma_3}$, then clearly $w(x)$ violates 3.6. So for all $\{\gamma_1, \gamma_2, \gamma_3\} \in [K]^3, B_{\gamma_1} \cap B_{\gamma_2} \cap B_{\gamma_3} = \emptyset$, thus if $\{\gamma_1, \gamma_2\} \neq \{\gamma_1', \gamma_2'\}, (B_{\gamma_1} \cap B_{\gamma_2}) \cap (B_{\gamma_1'} \cap B_{\gamma_2'}) = \emptyset$. So

$$\mu(\bigcup \mathcal{G}) \geq \sum_{\{\gamma_1,\gamma_2\}} \mu(B_{\gamma_1} \cap B_{\gamma_2}),$$

thus

$$\mu(\bigcup \mathcal{G}) \geq \mu(\bigcup_{\gamma \in K} B_\gamma) \geq |K| \cdot \frac{\mu(N)}{5} - \mu(\bigcup \mathcal{G})$$

or (since $\mu(N) > 1/2$)

$$\mu(\bigcup \mathcal{G}) \geq \frac{27}{10} \cdot \mu(N) > \frac{27}{20},$$

a contradiction.

So the only thing that remains to be proved is Lemma 3.12 and Lemmas 3.2, 3.9.

(E) **Proof of Lemma 3.12.** .

Let $x \in N, x \in C_i \in \mathcal{G}, \gamma \in F$ and $\gamma \cdot x \in C_j \in \mathcal{G}$. Then $w(x) = \Theta(\eta_i), w(\gamma \cdot x) = \Theta(\eta_j)$. We need to verify that

$$\|\gamma \cdot w(x) - w(\gamma \cdot x)\|_1 < \frac{1}{12} \cdot \|w(\gamma \cdot x)\|_1$$

or

$$\|\gamma \cdot \Theta(\eta_i) - \Theta(\eta_j)\|_1 < \frac{1}{12} \cdot \|\Theta(\eta_j)\|_1.$$

But, by 3.2, we have $\|\Theta(\eta_j)\|_1 = \|\eta_j\|_2$ and

$$\|\Theta(\gamma \cdot \eta_i) - \Theta(\eta_j)\|_1 \leq 3 \cdot \|\gamma \cdot \eta_i - \eta_j\|_2,$$

so it is enough to check that

$$\|\gamma \cdot \eta_i - \eta_j\|_2^2 < \frac{1}{36^2} \cdot \|\eta_j\|_2^2.$$

Now

$$\|\gamma \cdot \eta_i - \eta_j\|_2^2 = \|\eta_i\|_2^2 + \|\eta_j\|_2^2 - 2\mathrm{Re}\langle \gamma \cdot \eta_i, \eta_j \rangle,$$

$$\|\gamma \cdot \xi_i - \xi_j\|_2^2 = \|\xi_i\|_2^2 + \|\xi_j\|_2^2 - 2\mathrm{Re}\langle \gamma \cdot \xi_i, \xi_j \rangle,$$

so, as $|\|\xi_i\|_2^2 - \|\eta_i\|_2^2| < \epsilon_1$, and $|\|\xi_j\|^2 - \|\eta_j\|^2| < \epsilon_1$, and $(*)$ holds,

$$\|\gamma \cdot \eta_i - \eta_j\|_2^2 - \|\gamma \cdot \xi_i - \xi_j\|_2^2 < 4\epsilon_1,$$

thus

$$\|\gamma \cdot \eta_i - \eta_j\|_2^2 < 4\epsilon_1 + \|\gamma \cdot \xi_i - \xi_j\|_2^2 < \frac{4}{102^2} \cdot \|\xi_j\|_2^2 + \|\gamma \cdot \xi_i - \xi_j\|_2^2.$$

Since

$$\|\xi_j\|_2^2 < 2\|\eta_j\|_2^2,$$

so that

$$\|\gamma \cdot \eta_i - \eta_j\|_2^2 < \frac{1}{36^2} \cdot \|\eta_j\|^2 + \|\gamma \cdot \xi_i - \xi_j\|_2^2,$$

it is enough to verify that

$$\|\gamma \cdot \xi_i - \xi_j\|_2^2 < f(\delta) \cdot \|\xi_j\|^2$$

for some fixed function $f : (0, 1/2) \to \mathbb{R}^+$ with $f(a) \to 0$ as $a \to 0$.

Now

$$\|\gamma \cdot \xi_i - \xi_j\|_2^2 = \|(1_{\gamma \cdot C_i} - 1_{C_j}) + (\mu(C_j) - \mu(C_i))1\|_2^2$$

$$= \|1_{\gamma \cdot C_i} - 1_{C_j}\|_2^2 + \|(\mu(C_j) - \mu(C_i))1\|_2^2$$

$$+2\langle 1_{\gamma \cdot C_i} - 1_{C_j}, (\mu(C_j) - \mu(C_i)1\rangle = \mu(\gamma \cdot C_i \Delta C_j) - (\mu(C_i) - \mu(C_j))^2$$

$$\leq 3\delta \cdot \mu(C_j) + \mu(C_j)^2 \cdot \left(\frac{\mu(C_i)}{\mu(C_j)} - 1 \right)^2,$$

and

$$\|\xi_j\|^2 = \mu(C_j) \cdot (1 - \mu(C_j)).$$

It is enough to find $f(a)$ so that $f(a) \to 0$ as $a \to 0$ and

$$3\delta + \mu(C_j) \cdot \left(\frac{\mu(C_i)}{\mu(C_j)} - 1 \right)^2 \leq f(\delta) \cdot (1 - \mu(C_j)).$$

But, as $\mu(C_j) < \frac{1}{2}$, it is enough to have

$$3\delta + (\frac{\mu(C_i)}{\mu(C_j)} - 1)^2 \leq \frac{1}{2} \cdot f(\delta).$$

Recalling that

$$(1 - \delta) \leq \frac{\mu(C_i)}{\mu(C_j)} \leq \frac{1}{(1 - \delta)},$$

we can simply take

$$f(a) = 6a + 2g(a)^2,$$

where $g(a) = \max(a, \frac{a}{1-a}) = \frac{a}{1-a}$.

(F) Proof of Lemma 3.9.

Consider the partitions $\widehat{\mathcal{P}}_0 \geq \widehat{\mathcal{P}}_1 \geq \ldots$ of X, where $\widehat{\mathcal{P}}_n = \{\widehat{A} : A \in \mathcal{P}_n\}$. We can assume that $\widehat{\mathcal{P}}_0 = \{X\}$. Let T be the rooted tree corresponding to $\{\widehat{\mathcal{P}}_n\}$, i.e., the root of T is X, the children of the root are the elements of $\widehat{\mathcal{P}}_1$, the children of each $\widehat{A} \in \widehat{\mathcal{P}}_1$ are all the elements $\widehat{B} \in \widehat{\mathcal{P}}_2$ contained in it, etc. Define $\psi : X \to [T]$ by

$$\psi(x) = (\widehat{A}_0, \widehat{A}_1, \widehat{A}_2, \ldots)$$

if $x \in \bigcap_n \widehat{A}_n$. Since $\{\mathcal{P}_n\}$ separates points in Y, and φ is 1-1, $\{\widehat{\mathcal{P}}_n\}$ separates points in X, so ψ is 1-1 and Borel. Let $\nu = \psi_* \mu$.

For $f \in \psi(X), \gamma \in F$, let

$$F_\gamma(f) = \alpha(\gamma, \psi^{-1}(f)) \in \Delta.$$

Since F_γ is a Borel function defined on $\psi(X)$, and $\nu(\psi(X)) = 1$, there is a locally finite subtree $S \subseteq T$ with the same root and the property that $[S] \subseteq \psi(X), \nu([T] \setminus [S]) < \epsilon$ and for each $\gamma \in F, F_\gamma|[S] : [S] \to \Delta$ is continuous, where Δ is given the discrete topology. Thus $F_\gamma([S])$ is finite for each γ, and therefore there is a large enough k, so that each $F_\gamma(f), \gamma \in F$, is constant on each

$$\{f \in [S] : f \supseteq u\},$$

where $u = (u_0, \dots, u_k)$ is a finite path in S starting from the root. This means that if $A_1, \dots, A_n \in \mathcal{P}_k$ are such that $\widehat{A}_1, \dots, \widehat{A}_n$ are the elements of S at the k-th level of S (i.e., at distance k from the root), and we let $M = \psi^{-1}([S])$, then $\mu(X \setminus M) = \nu([T] \setminus [S]) < \epsilon$, $M \subseteq \widehat{A}_1 \cup \dots \cup \widehat{A}_n$ and $x \mapsto \alpha(\gamma, x)$ is constant on each $M \cap \widehat{A}_i$, $i = 1, \dots, n, \gamma \in F$. By increasing k if necessary, and using the fact that μ is non-atomic, it is easy to see that we can also guarantee that $\frac{1}{2} > \mu(\widehat{A}_i)$ for $i = 1, \dots, n$, and we can also assume that $\mu(\widehat{A}_i) > 0$ by dropping those that are null.

(G) **Proof of Lemma 3.2.**

First we define

$$f \in \ell^2(\mathcal{F}_0(F_2)) \to f^* \in \ell^1(\mathcal{F}_0(F_2))$$

by

$$f^*(S) = \begin{cases} \dfrac{|f(S)|^2}{\|f\|_2} & \text{if} \quad f \neq 0, \\ 0 & \text{if} \quad f = 0. \end{cases}$$

Clearly

$$\|f^*\|_1 = \|f\|_2, \qquad f^* \geq 0.$$

Let also $\widetilde{f}(S) = |f(S)|^2$, for $f \in \ell^2(\mathcal{F}_0(F_2))$.

Sublemma 3.13. *If $f_1, f_2 \in \ell^2(\mathcal{F}_0(F_2))$ are real, then*

$$\|\widetilde{f}_1 - \widetilde{f}_2\|_1 \leq (\|f_1\|_2 + \|f_2\|_2) \cdot \|f_1 - f_2\|_2.$$

Proof. We have

$$\begin{aligned} \|\widetilde{f}_1 - \widetilde{f}_2\|_1 &= \sum_{S \in \mathcal{F}_0(F_2)} |f_1(S)^2 - f_2(S)^2| \\ &= \sum_{S \in \mathcal{F}_0(F_2)} |f_1(S) - f_2(S)| \cdot |f_1(S) + f_2(S)| \\ &\leq \|f_1 - f_2\|_2 \cdot \|f_1 + f_2\|_2 \text{ (Cauchy-Schwartz)} \\ &\leq (\|f_1\|_2 + \|f_2\|_2) \cdot \|f_1 - f_2\|_2. \end{aligned}$$

Sublemma 3.14. *If $f_1, f_2 \in \ell^2(\mathcal{F}_0(F_2))$, then*

$$\|f_1^* - f_2^*\|_1 \leq 3 \cdot \|f_1 - f_2\|_2.$$

We postpone the proof for later. Now define $\theta : \ell^1(\mathcal{F}_0(F_2)) \to \ell^1(F_2)$ by

$$\theta(f)(\gamma) = \sum_{\gamma \in S \in \mathcal{F}_0(F_2)} \frac{f(S)}{|S|}.$$

Note that

$$\begin{aligned}
\|\theta(f)\|_1 &= \sum_\gamma |\theta(f)(\gamma)| = \sum_\gamma \left| \sum_{\gamma \in S \in \mathcal{F}_0(F_2)} \frac{f(S)}{|S|} \right| \\
&\leq \sum_\gamma \sum_{\gamma \in S \in \mathcal{F}_0(F_2)} \frac{|f(S)|}{|S|} = \sum_{S \in \mathcal{F}_0(F_2)} \left(\sum_{\gamma \in S} \frac{|f(S)|}{|S|} \right) \\
&= \sum_{S \in \mathcal{F}_0(F_2)} |f(S)| = \|f\|_1.
\end{aligned}$$

Also θ is linear, and

$$f \geq 0 \Rightarrow (\|\theta(f)\|_1 = \|f\|_1, \theta(f) \geq 0).$$

Put for $f \in \ell^2(\mathcal{F}_0(F_2))$,

$$\Theta(f) = \theta(f^*).$$

Then, as $f^* \geq 0$,

$$\|\Theta(f)\|_1 = \|\theta(f^*)\|_1 = \|f^*\|_1 = \|f\|_2.$$

Also

$$\begin{aligned}
\|\Theta(f_1) - \Theta(f_2)\|_1 &= \|\theta(f_1^*) - \theta(f_2^*)\|_2 = \|\theta(f_1^* - f_2^*)\|_1 \\
&\leq \|f_1^* - f_2^*\|_1 \leq 3 \cdot \|f_1 - f_2\|_2,
\end{aligned}$$

and

$$\Theta(f) \geq 0.$$

Moreover, it is easy to check that $f \mapsto f^*$ and θ are F_2-maps, thus Θ is an F_2-map, so it only remains to give the proof of Sublemma 3.14.

Proof of Sublemma 3.14. Assume first that f_1, f_2 are real. Since this inequality is clear when $f_1 = 0$ or $f_2 = 0$, also assume that $f_i \neq 0, i = 1, 2$.

Consider now the case where f_1 and f_2 are collinear, i.e., $f_1 = rf_2$ for some $r \neq 0$. Clearly $r \in \mathbb{R}$. As $(rf_2)^* = \dfrac{(rf_2)^2}{\|rf_2\|_2} = \dfrac{r^2(f_2)^2}{|r| \cdot \|f_2\|_2} = |r| \cdot f_2^*$, we have

$$\|f_1^* - f_2^*\|_1 = \||r|f_2^* - f_2^*\|_1 = |\,|r|-1| \cdot \|f_2^*\|_1 \leq |r-1| \cdot \|f_2\|_2 = \|f_1 - f_2\|_2.$$

So we can assume that f_1, f_2 are not collinear. Now

$$\|f_1^* - f_2^*\|_1 = \sum_{S \in \mathcal{F}_0(F_2)} |f_1^*(S) - f_2^*(S)| = \sum_{S \in \mathcal{F}_0(F_2)} \left| \frac{f_1(S)^2}{\|f_1\|_2} - \frac{f_2(S)^2}{\|f_2\|_2} \right|.$$

Put $a_i = \|f_i\|_2$ and assume, without loss of generality, that $a_1 \leq a_2$. Then the above is equal to

$$\begin{aligned}
&\sum_{S \in \mathcal{F}_0(F_2)} \left| \frac{f_1(S)^2}{a_1} - \frac{f_2(S)^2}{a_2} \right| \\
=&\frac{1}{a_1 a_2} \sum_{S \in \mathcal{F}_0(F_2)} |(\sqrt{a_2} f_1(S))^2 - (\sqrt{a_1} f_2(S))^2| \\
=&\frac{1}{a_1 a_2} \|(\widetilde{\sqrt{a_2} f_1}) - (\widetilde{\sqrt{a_1} f_2})\|_1 \leq
\end{aligned}$$

(by Sublemma 3.13)

$$\begin{aligned}
&\leq \frac{1}{a_1 a_2} (\sqrt{a_2} \cdot a_1 + \sqrt{a_1} \cdot a_2) \cdot \|\sqrt{a_2} f_1 - \sqrt{a_1} f_2\|_2 \\
=&\left\| (\sqrt{a_1} + \sqrt{a_2}) \cdot \left(\frac{f_1}{\sqrt{a_1}} - \frac{f_2}{\sqrt{a_2}} \right) \right\|_2 \\
\leq&\|f_1 - f_2\|_2 + \left\| \sqrt{\frac{a_2}{a_1}} f_1 - \sqrt{\frac{a_1}{a_2}} f_2 \right\|_2 \\
\leq&\|f_1 - f_2\|_2 + \sqrt{\frac{a_1}{a_2}} \left\| \frac{a_2}{a_1} f_1 - f_2 \right\|_2 \\
\leq&\|f_1 - f_2\|_2 + \left\| \frac{a_2}{a_1} f_1 - f_2 \right\|_2, \quad \text{as} \quad a_1 \leq a_2.
\end{aligned}$$

So it is enough to check that

$$\left\| \frac{a_2}{a_1} f_1 - f_2 \right\|_2 \leq 2 \cdot \|f_1 - f_2\|_2.$$

Note that $\|\frac{a_2}{a_1} f_1\|_2 = a_2 = \|f_2\|_2$. Let ψ be the angle between f_1, f_2.

Case I. $0 < \psi \leq \pi/2$. Let $\rho = \frac{\pi - \psi}{2}$. We have $\pi/4 \leq \rho < \pi/2$, so

$$\|f_1 - f_2\|_2 \geq \left\|\frac{a_2}{a_1}f_1 - f_2\right\|_2 \sin\rho \geq \frac{\sqrt{2}}{2}\left\|\frac{a_2}{a_1}f_1 - f_2\right\|_2 \geq \frac{1}{2}\left\|\frac{a_2}{a_1}f_1 - f_2\right\|_2.$$

Case II. $\pi/2 < \psi < \pi$.

Then

$$\left\|f_2 - \frac{a_2}{a_1}f_1\right\|_2 \leq 2 \cdot \|f_2\|_2 \leq 2 \cdot \|f_1 - f_2\|_2.$$

So the proof is complete in the case f_1, f_2 are real. Consider now the general case.

For $f \in \ell^2(\mathcal{F}_0(F_2))$, let $|f|(S) = |f(S)|$. Given $f_1, f_2 \in \ell^2(\mathcal{F}_0(F_2))$, we have

$$\|(|f_1|)^* - (|f_2|)^*\|_1 \leq 3 \cdot \||f_1| - |f_2|\|_2,$$

by the preceding case. But

$$(|f|)^* = f^*,$$

so $\|f_1^* - f_2^*\|_1 \leq 3 \cdot \||f_1| - |f_2|\|_2$, thus it is enough to show that

$$\||f_1| - |f_2|\|_2 \leq \|f_1 - f_2\|_2$$

or, by taking squares of both sides, that

$$\begin{aligned}\sum_{S \in \mathcal{F}_0(F_2)} \Big||f_1|(S) - |f_2|(S)\Big|^2 &= \sum_{S \in \mathcal{F}_0(F_2)} \Big||f_1(S)| - |f_2(S)|\Big|^2 \\ &\leq \sum_{S \in \mathcal{F}_0(F_2)} \Big|f_1(S) - f_2(S)\Big|^2.\end{aligned}$$

This completes the proof of 3.14 and thus of 3.1.

Remark. As pointed out by B. Bekka, the conclusion of 3.1 is still valid under the weaker assumption that the action of F_2 on X is tempered. This follows immediately from the remarks preceding 3.1.

We do not know if 3.1 still holds under the weaker assumption that Γ is non-amenable. We also do not know if 3.1 is true under another weaker assumption, namely that π_0^X does not contain weakly a nonzero finite-dimensional representation. Recall here that if Γ is non-amenable, then λ_Γ does not contain weakly a nonzero finite-dimensional representation, see [7, 18.9.5 and 18.3.6].

4. Actions of countable groups by automorphisms on compact Polish groups

It turns out that for actions of countable groups by automorphisms on compact Polish groups, being tempered fits nicely in a hierarchy of ergodicity properties and can be also expressed in terms of several other equivalent conditions. In particular, such actions provide many interesting new examples of antimodular actions.

We will review first the Peter-Weyl theorem in the representation theory of compact Polish groups. We use as a reference Folland [8].

Recall that a unitary representation of a general topological group G on a separable Hilbert space H is a continuous homomorphism π of G into $U(H), \pi : G \to U(H)$. We also write $H = H_\pi$. (As usual $U(H)$ is given the strong or, equivalently, the weak topology. The *strong topology* is the one generated by the functions $f_x(T) = Tx, T \in U(H), x \in H$, and the *weak topology* is the one generated by the functions $f_{x,y}(T) = \langle Tx, y\rangle, T \in U(H), x, y \in H$. They agree on $U(H)$.)

The dimension of H_π is called the *dimension of* π, $\dim(\pi)$. Clearly $1 \leq \dim(\pi) \leq \infty = \aleph_0$.

Isomorphism of representations π, ρ of G, denoted by $\pi \cong \rho$, is defined as before: $\pi \cong \rho$ if and only if there is a Hilbert space isomorphism $T : H_\pi \to H_\rho$ such that $T(\pi(g)\xi) = \rho(g)(T\xi)$ for $g \in G, \xi \in H_\pi$.

If $\{\pi_i\}_{i\in I}$ is a countable family of representations of G, their direct sum $\pi = \bigoplus_i \pi_i$ is the representation on $\bigoplus_i H_i$ given by $\pi(g)(\oplus_i \xi_i) = \oplus_i \pi_i(g)\xi_i$.

We also view a unitary representation π as an action of G on H_π by unitary operators (with appropriate continuity properties) and write $g \cdot \xi = \pi(g)\xi$. A *subrepresentation* of π is the restriction of this action to an invariant closed subspace. We again say that ρ is contained in $\pi, \rho \leq \pi$, if ρ is isomorphic to a subrepresentation of π. Finally, we say that π is *irreducible* if $H_\pi \neq \{0\}$ and π has no invariant closed subspaces except $\{0\}, H_\pi$.

For each irreducible representation π of G, we denote by $\widehat{\pi}$ its isomorphism class, and we put

$$\widehat{G} = \{\widehat{\pi} : \pi \text{ an irreducible unitary representation of } G\}.$$

This is called the *dual* of G.

We now have

Theorem 4.1 (Peter–Weyl). *Let G be a compact Polish group. Then:*

(i) *every irreducible unitary representation of G is finite-dimensional;*

(ii) *$\widehat{G}$ is countable;*

(iii) *every unitary representation of G is a direct sum of irreducible unitary representations.*

Consider now the *right-regular* representation ρ_G of G. Let $\mu = \mu_G$ be the (normalized) Haar measure on G. Then ρ_G is the right-shift action on $H_{\rho_G} = L^2(G, \mu) \equiv L^2(G) : g \cdot f(h) = f(hg)$. We will now see an explicit version of Theorem 4.1 (iii) for ρ_G.

Fix a representative π for each $\widehat{\pi} \in \widehat{G}$. Fix also an orthonormal basis $e_1^\pi, \dots, e_{d_\pi}^\pi$ for H_π, where

$$d_\pi = \dim(\pi).$$

Let

$$\pi_{ij}(g) = \langle \pi(g) e_j^\pi, e_i^\pi \rangle$$

be the matrix coefficients of π in this basis. These are continuous functions on G, so they belong to $L^2(G)$. Denote by $\mathcal{E}_\pi$ their linear span, a finite-dimensional subspace of $L^2(G)$. Clearly this is the same as the subspace generated by the general matrix coefficients

$$\langle \pi(g)\xi, \eta \rangle, \quad \xi, \eta \in H_\pi,$$

so it only depends on the isomorphism class of π, thus we can write $\mathcal{E}_{\widehat{\pi}}$ for this space.

Theorem 4.2 (Peter–Weyl). *Let G be a compact Polish group. Then:*

(i) $L^2(G) = \bigoplus_{\widehat{\pi} \in \widehat{G}} \mathcal{E}_{\widehat{\pi}}$;

(ii) $\{\sqrt{d_\pi}\pi_{ij}\}_{1 \le i,j \le d_\pi}$ *is an orthonormal basis for $\mathcal{E}_{\widehat{\pi}}$, so $\dim(\mathcal{E}_{\widehat{\pi}}) = d_\pi^2$;*

(iii) *for $i = 1, \dots, d_\pi$, the subspace $\mathcal{E}_{\widehat{\pi},i}$ of $\mathcal{E}_{\widehat{\pi}}$ spanned by the ith row of the matrix $(\sqrt{d_\pi}\pi_{ij})$ is invariant under the right-regular representation, and the subrepresentation of ρ_G determined by $\mathcal{E}_{\widehat{\pi},i}$ is isomorphic to π.*

For each irreducible representation π of a compact Polish group G, its *character* χ_π is defined by

$$\chi_\pi(g) = \mathrm{trace}(\pi(g)).$$

Then χ_π depends only on the isomorphism class of π (since similar matrices have the same trace), so we put

$$\chi_{\widehat{\pi}} = \chi_\pi.$$

Then $\{\chi_{\widehat{\pi}} : \widehat{\pi} \in \widehat{G}\}$ is an orthonormal set in $L^2(G)$ (see [8, 5.23]).

Now suppose Γ is an infinite countable group acting by (topological group) automorphisms on a compact Polish group G. Then clearly Γ preserves the Haar measure μ on G. Moreover, Γ acts naturally on $\widehat{G}$ as follows. If π is an irreducible unitary representation of G and $\gamma \in \Gamma$, define

$$(\gamma \cdot \pi)(g) = \pi(\gamma^{-1} \cdot g) \in U(H_\pi).$$

Then clearly $\gamma \cdot \pi$ is an irreducible unitary representation on H_π. Also, $\pi \cong \rho \Rightarrow \gamma \cdot \pi \cong \gamma \cdot \rho$, so we can define

$$\gamma \cdot \widehat{\pi} = \widehat{\gamma \cdot \pi}.$$

Clearly $\dim(\gamma \cdot \pi) = \dim(\pi)$. Also note that

$$\gamma \cdot \chi_{\widehat{\pi}} = \chi_{\gamma \cdot \widehat{\pi}},$$

since $\gamma \cdot \chi_\pi(g) = \chi_\pi(\gamma^{-1} \cdot g) = \mathrm{trace}(\pi(\gamma^{-1} \cdot g)) = \mathrm{trace}((\gamma \cdot \pi)(g)) = \chi_{\gamma \cdot \pi}(g)$. Finally, we have

$$\gamma \cdot \mathcal{E}_{\widehat{\pi}} = \mathcal{E}_{\gamma \cdot \widehat{\pi}},$$

and γ acts as a Hilbert space isomorphism from $\mathcal{E}_{\widehat{\pi}}$ onto $\mathcal{E}_{\gamma \cdot \widehat{\pi}}$.

Let us finally review a hierarchy of progressively stronger ergodicity conditions for a Borel action of a countable group Γ on a standard Borel space X with invariant (probability Borel) measure μ.

(i) The action is *ergodic* if the unitary representation π_0^X of Γ on $L_0^2(X)$ has no nonzero invariant vectors, i.e., for $\xi \in L_0^2(X)$, $\forall \gamma \in \Gamma(\gamma \cdot \xi = \xi) \Leftrightarrow \xi = 0$. Equivalently

$$1_\Gamma \not\leq \pi_0^X.$$

(ii) The action is *weakly mixing* if π_0^X contains no nonzero finite-dimensional representations (see [2, 2.14]). An equivalent formulation, which we will use later, is the following (see [3, 4.1]):

For any $\xi_1, \dots, \xi_n \in L_0^2(X), \epsilon > 0$ there is $\gamma \in \Gamma$ such that

$$|\langle \gamma \cdot \xi_i, \xi_i \rangle| < \epsilon, \quad i = 1, \dots, n.$$

(iii) The action is *mildly mixing* if $\overline{\lim}_{\gamma \to \infty} |\langle \gamma \cdot \xi, \xi \rangle| < ||\xi||^2$ for each $\xi \in L_0^2(X)$, $\xi \neq 0$ (see [9, 22, 20]).

(iv) The action is (*strongly*) *mixing* if for any Borel $A, B \subseteq X$,

$$\lim_{\gamma \to \infty} \mu(\gamma \cdot A \cap B) = \mu(A)\mu(B).$$

This is equivalent to saying that π_0^X is a c_0-representation, i.e., the matrix coefficients $\langle \gamma \cdot \xi, \eta \rangle$ tend to 0 as $\gamma \to \infty$ for ξ, $\eta \in L_0^2(X)$.

Now we consider these conditions in the case where Γ acts by automorphisms on a compact Polish group $X = G$ and $\mu = \mu_G$. After completing a first draft of this paper, we found out that the next two results, 4.3 and 4.5, were known. See Kitchens–Schmidt [16] and Schmidt [20], where the equivalence of (i), (iii), (iv) in 4.3 and (i), (iii), (iv) in 4.5 is proved, and references to earlier work of Halmos, Kaplansky, and Berens, who dealt with special cases, are also given. The proofs given below seem however to be somewhat different.

Theorem 4.3. (Kitchens–Schmidt [16] and Schmidt [20]). *Let Γ be an infinite countable group and suppose Γ acts by automorphisms on a compact Polish group G (and thus leaves the Haar measure μ invariant). Then the following are equivalent:*

(i) *the action of Γ on G is ergodic;*

(ii) *the action of Γ on G is weakly mixing;*

(iii) *if 1_G denotes the trivial one-dimensional representation of G, then for every $\widehat{\pi} \neq \widehat{1}_G$, $\widehat{\pi} \in \widehat{G}$, $\Gamma/\Gamma_{\widehat{\pi}}$ is infinite, where $\Gamma_{\widehat{\pi}}$ is the stabilizer of the Γ-action on $\widehat{G}$;*

(iv) *for some $g \in G, \Gamma \cdot g$ is dense in G;*

(v) *for μ-almost all $g \in G, \Gamma \cdot g$ is dense in G.*

Proof. (i) $\Rightarrow$ (iii): Assume that for some $\widehat{\pi}_0 \neq \widehat{1}_G$, $\Gamma/\Gamma_{\widehat{\pi}_0}$ is finite, towards a contradiction. Then $\Gamma \cdot \widehat{\pi}_0 = \{\widehat{\pi}_1, \ldots, \widehat{\pi}_n\}$, a finite set. Then $f = \chi_{\widehat{\pi}_1} + \cdots + \chi_{\widehat{\pi}_n} \in L_0^2(G)$, since $\widehat{\pi}_i \neq \widehat{1}_G$, for $i = 1, \ldots, n$ (as $\gamma \cdot \widehat{1}_G = \widehat{1}_G, \forall \gamma \in \Gamma$), so $\chi_{\widehat{\pi}_i} \perp \chi_{\widehat{1}_G} = 1$ (the constant 1 function). However, f is Γ-invariant, so $1_\Gamma \leq \pi_0^G$ (the representation of Γ on $L_0^2(G)$), contradicting ergodicity.

(iii) $\Rightarrow$ (i): Suppose that $0 \neq f \in L_0^2(G)$ is Γ-invariant, towards a contradiction. Note that $\mathbb{C}1 = \mathcal{E}_{\widehat{1}_G}$, so $L_0^2(G) = \bigoplus_{\widehat{\pi} \neq \widehat{1}_G} \mathcal{E}_{\widehat{\pi}}$. Let $\widehat{\pi} \neq \widehat{1}_G$ and denote by $p_{\widehat{\pi}}(f)$ the projection of f on $\mathcal{E}_{\widehat{\pi}}$. We will show that $p_{\widehat{\pi}}(f) = 0$, which will complete the proof.

By formula 5.19 in Folland [8] (the Fourier analysis of $L^2(G)$),

$$p_{\widehat{\pi}}(f)(g) = d_{\widehat{\pi}} \int f(h) \chi_{\widehat{\pi}}(h^{-1} g) dh,$$

so

$$\begin{aligned}p_{\gamma\cdot\widehat{\pi}}(f)(g) &= d_{\gamma\cdot\widehat{\pi}}\int f(h)\chi_{\gamma\cdot\widehat{\pi}}(h^{-1}g)dh\\ &= d_{\widehat{\pi}}\int f(h)\mathrm{trace}(\gamma\cdot\widehat{\pi}(h^{-1}g))dh\\ &= d_{\widehat{\pi}}\int f(h)\mathrm{trace}(\pi((\gamma^{-1}\cdot h)^{-1}(\gamma^{-1}\cdot g)))dh\end{aligned}$$

(as $f(h)=f(\gamma^{-1}\cdot h)$ μ-a.e.)

$$\begin{aligned}&= d_{\widehat{\pi}}\int f(\gamma^{-1}\cdot h)\mathrm{trace}(\pi((\gamma^{-1}\cdot h)^{-1}(\gamma^{-1}\cdot g)))dh\\ &= d_{\widehat{\pi}}\int f(h)\mathrm{trace}(\pi(h^{-1}(\gamma^{-1}g)))dh\\ &= p_{\widehat{\pi}}(f)(\gamma^{-1}\cdot g) = (\gamma\cdot p_{\widehat{\pi}}(f))(g),\end{aligned}$$

i.e., $p_{\gamma\cdot\widehat{\pi}}(f)=\gamma\cdot p_{\widehat{\pi}}(f)$. So, as γ acts isomorphically from $\mathcal{E}_{\widehat{\pi}}$ to $\mathcal{E}_{\gamma\cdot\widehat{\pi}}$, we have $\|p_{\gamma\cdot\widehat{\pi}}(f)\|_2=\|p_{\widehat{\pi}}(f)\|_2$, i.e., $\|p_{\widehat{\rho}}(f)\|_2$ is constant for $\widehat{\rho}\in\Gamma\cdot\widehat{\pi}$. Since $\Gamma\cdot\widehat{\pi}$ is infinite, and $\sum_{\widehat{\rho}\in\Gamma\cdot\widehat{\pi}}\|p_{\widehat{\rho}}(f)\|_2^2\leq\|f\|_2^2<\infty$, this means that $p_{\widehat{\pi}}(f)=0$.

(i) $\Rightarrow$ (v): Define the following equivalence relation on G:

$$g\sim h\Leftrightarrow\overline{\Gamma\cdot g}=\overline{\Gamma\cdot h}.$$

Then $\sim$ is smooth (via $g\mapsto\overline{\Gamma\cdot g}$) and $gE_\Gamma^G h\Rightarrow g\sim h$, thus, by ergodicity, there is $g_0\in G$, with $\overline{\Gamma\cdot g}=\overline{\Gamma\cdot g_0}$ a.e. (g). If $F=\overline{\Gamma\cdot g_0}$, then $\mu(F)=1$. So, as F is closed, $F=G$, thus $\Gamma\cdot g$ is dense in G a.e. (g).

(v) $\Rightarrow$ (iv): Obvious.

(iv) $\Rightarrow$ (iii): If (iii) fails, find $\widehat{\pi}\neq\widehat{1}_G$ such that $\Gamma\cdot\widehat{\pi}$ is finite, say $\Gamma\cdot\widehat{\pi}=\{\widehat{\pi}_1,\dots,\widehat{\pi}_n\}$. Thus

$$f=\chi_{\widehat{\pi}_1}+\cdots+\chi_{\widehat{\pi}_n}$$

is continuous and Γ-invariant. If $g\in G$ is such that $\Gamma\cdot g$ is dense in G, then $f(\gamma\cdot g)=f(g)$ for all $\gamma\in\Gamma$, so f is constant, thus $f\perp f$, a contradiction.

(ii) $\Rightarrow$ (i): Obvious.

(iii) $\Rightarrow$ (ii): We need to verify that if $\xi_1,\dots,\xi_n\in L_0^2(G),\epsilon>0$, there is $\gamma\in\Gamma$ such that $|\langle\gamma\cdot\xi_i,\xi_i\rangle|<\epsilon, i=1,\dots,n$. Assume $D\subseteq L_0^2(G)$ is norm-dense and we can verify this condition for $\xi_1',\dots,\xi_n'\in D$. Then, given $\xi_1,\dots,\xi_n\in L_0^2(G)$, find $\xi_1',\dots,\xi_n'\in D$ with

$$\|\xi_i-\xi_i'\|_2\cdot(\|\xi_i\|_2+\|\xi_i'\|_2)<\frac{\epsilon}{2},$$

and find $\gamma \in \Gamma$ with $|\langle \gamma \cdot \xi'_i, \xi'_i \rangle| < \epsilon/2$ for $i = 1, \ldots, n$. Then

$$\begin{aligned} &|\langle \gamma \cdot \xi_i, \xi_i \rangle - \langle \gamma \cdot \xi'_i, \xi'_i \rangle| = |\langle \gamma \cdot (\xi_i - \xi'_i), \xi_i \rangle + \langle \gamma \cdot \xi'_i, \xi_i - \xi'_i \rangle| \\ \leq &\|\xi_i - \xi'_i\|_2 \cdot \|\xi_i\|_2 + \|\xi'_i\|_2 \cdot \|\xi_i - \xi'_i\|_2 \leq \frac{\epsilon}{2}, \end{aligned}$$

so $|\langle \gamma \cdot \xi_i, \xi_i \rangle| < \epsilon$. Thus, since

$$L^2_0(G) = \bigoplus_{\widehat{\pi} \neq \widehat{1}_G} \mathcal{E}_{\widehat{\pi}},$$

we can assume that each ξ_i is of the form

$$\xi_i = \sum_{j=1}^{N} \xi_j^{(i)}$$

for some $\{\widehat{\pi}_1, \ldots, \widehat{\pi}_N\} \subseteq \widehat{G} \setminus \{\widehat{1}_G\}$ and $\xi_j^{(i)}$ = the projection of ξ_i to $\mathcal{E}_{\widehat{\pi}_j}$. Then

$$\langle \gamma \cdot \xi_i, \xi_i \rangle = \sum_{j,k=1}^{N} \langle \gamma \cdot \xi_j^{(i)}, \xi_k^{(i)} \rangle.$$

Now if γ is such that

$$\gamma \cdot \widehat{\pi}_j \neq \widehat{\pi}_k$$

for all $1 \leq j,\ k \leq N$, then

$$\langle \gamma \cdot \xi_i, \xi_i \rangle = 0, \quad i = 1, \ldots, n,$$

and we are done. So it is enough to find such γ, i.e., to show:

$$\forall \widehat{\pi}_1, \ldots, \widehat{\pi}_N \in \widehat{G} \setminus \{\widehat{1}_G\} \exists \gamma \in \Gamma(\gamma \cdot \{\widehat{\pi}_1, \ldots, \widehat{\pi}_N\} \cap \{\widehat{\pi}_1, \ldots, \widehat{\pi}_N\} = \emptyset).$$

This follows from the following well-known result in group theory (taking $X = \widehat{G} \setminus \{\widehat{1}_G\}$).

Lemma 4.4. . *Let a group Γ act on a set X. Then the following are equivalent:*

(i) *all orbits are infinite (i.e., all stabilizers have infinite index);*

(ii) *$\forall F \subseteq X, F$ finite $\exists \gamma \in \Gamma(\gamma \cdot F \cap F = \emptyset)$.*

Proof. (ii) $\Rightarrow$ (i): If $\Gamma \cdot x = F$ is finite, clearly for any $\gamma \in \Gamma, \gamma \cdot F = F$.

(i) $\Rightarrow$ (ii): We will use the following fact:

If Γ is a group and $\Gamma_1, \ldots, \Gamma_k \leq \Gamma$ are subgroups of infinite index, then Γ is not the union of finitely many (left) cosets of $\Gamma_1, \ldots, \Gamma_k$.

Granting this, say $F = \{x_1, \dots, x_k\}$. Let Γ_i = stabilizer of x_i. Then $\Gamma_{ij} = \{\gamma \in \Gamma : \gamma \cdot x_i = x_j\}$ is a coset of Γ_i. So by the fact, and our hypothesis (i), there is $\gamma \in \Gamma$ with $\gamma \notin \Gamma_{ij}$ for all i, j, so $\gamma \cdot x_i \neq x_j$, i.e., $\gamma \cdot F \cap F = \emptyset$.

We now verify this fact, by induction on k. It is obvious for $k = 1$. Assume now that $\Gamma_1, \dots, \Gamma_k, \Gamma_{k+1}$ have infinite index but

$$\Gamma = \bigcup_{\gamma \in Y_1} \gamma\Gamma_1 \cup \cdots \cup \bigcup_{\gamma \in Y_k} \gamma\Gamma_k \cup \bigcup_{\gamma \in Y_{k+1}} \gamma\Gamma_{k+1},$$

where $Y_1, \dots, Y_k, Y_{k+1} \subseteq \Gamma$ are finite. Since Γ_{k+1} has infinite index, let $\delta \in \Gamma$ be such that

$$\delta\Gamma_{k+1} \cap \bigcup_{\gamma \in Y_{k+1}} \gamma\Gamma_{k+1} = \emptyset,$$

so $\delta\Gamma_{k+1} \subseteq \bigcup_{i=1}^k \bigcup_{\gamma \in Y_i} \gamma\Gamma_i$. Thus

$$\Gamma_{k+1} \subseteq \bigcup_{i=1}^{k} \bigcup_{\gamma \in Y_i} \delta^{-1}\gamma\Gamma_i,$$

and

$$g\Gamma_{k+1} \subseteq \bigcup_{i=1}^{k} \bigcup_{\gamma \in Y_i} g\delta^{-1}\gamma\Gamma_i$$

for all $g \in Y_{k+1}$. Consequently $\bigcup_{g \in Y_{k+1}} g\Gamma_{k+1}$ is contained in the union of finitely many cosets of $\Gamma_1, \dots, \Gamma_k$ and thus so does Γ, contradicting the induction hypothesis. □

The following characterizes mixing actions.

Theorem 4.5 (Kitchens–Schmidt [16] and Schmidt [21]). *Let an infinite countable group Γ act by automorphisms on a compact Polish group G. Then the following are equivalent:*

(i) *the action of Γ on G is mixing;*

(ii) *the action of Γ is mildly mixing;*

(iii) *for any $\widehat{\pi} \neq 1_G, \Gamma_{\widehat{\pi}}$ is finite;*

(iv) *for any infinite $\Gamma_0 \leq \Gamma$, the Γ_0-action on G is ergodic.*

If, moreover, Γ is torsion-free, the above are equivalent to:

(v) *$\pi_0^G \cong n \cdot \lambda_\Gamma$ for some $1 \leq n \leq \infty$.*

Proof. (i) ⇒ (iv): Obvious.

(iv) $\Rightarrow$ (iii): Assume that there is $\widehat{\pi} \neq \widehat{1}_\Gamma$ with $\Gamma_{\widehat{\pi}}$ infinite. Since $\chi_{\widehat{\pi}} \in L^2_0(G)$ is $\Gamma_{\widehat{\pi}}$-invariant, the action of $\Gamma_{\widehat{\pi}}$ on G is not ergodic, contradicting (iv).

(iii) $\Rightarrow$ (i): To verify that a unitary representation ρ of Γ on H_ρ is a c_0-representation, i.e, $\langle \gamma \cdot \xi \eta \rangle \to 0$ as $\gamma \to \infty$ for all ξ, $\eta \in H_\rho$, it is enough to check this for all $\xi, \eta \in D$, which is norm-dense in H_ρ, since if $\xi_n \to \xi, \eta_n \to \eta$ in H_ρ, then $\langle \gamma \cdot \xi_n, \eta_n \rangle \to \langle \gamma \cdot \xi, \eta \rangle$ uniformly in γ. Let $\mathcal{O}_1, \mathcal{O}_2, \dots$ (finite or infinite list) be the orbits of Γ on $\widehat{G}$. Then $L^2_0(G) = \bigoplus_n (\bigoplus_{\widehat{\rho} \in \widehat{\mathcal{O}}_n} \mathcal{E}_{\widehat{\rho}})$, so it is enough to check that the representation of Γ on each $\bigoplus_{\widehat{\rho} \in \widehat{\mathcal{O}}_n} \mathcal{E}_{\widehat{\rho}}$ is a c_0-representation. Say $\mathcal{O} = \mathcal{O}_n = \Gamma \cdot \widehat{\pi}, \widehat{\pi} \neq \widehat{1}_G$. Since $\bigoplus_{\widehat{\rho} \in \mathcal{O}} \mathcal{E}_{\widehat{\rho}}$ has as dense set $\bigcup_k \mathcal{E}_{\widehat{\rho}_1} \oplus \cdots \oplus \mathcal{E}_{\widehat{\rho}_k}$, where $\mathcal{O} = \{\widehat{\rho}_1, \widehat{\rho}_2, \dots\}$, it is enough to check that if $\gamma_1 \cdot \widehat{\pi}, \gamma_2 \cdot \widehat{\pi} \in \mathcal{O}$ and $\xi \in \mathcal{E}_{\gamma_1 \cdot \widehat{\pi}}, \eta \in \mathcal{E}_{\gamma_2 \cdot \widehat{\pi}}$, we have

$$\langle \gamma \cdot \xi, \eta \rangle \to 0 \quad \text{as } \gamma \to \infty.$$

But $\gamma \cdot \xi \in \mathcal{E}_{\gamma\gamma_1 \cdot \widehat{\pi}} \perp \mathcal{E}_{\gamma_2 \cdot \widehat{\pi}}$, if $\gamma\gamma_1 \cdot \widehat{\pi} \neq \gamma_2 \cdot \widehat{\pi}$, i.e., if $\gamma_2^{-1}\gamma\gamma_1 \notin \Gamma_{\widehat{\pi}}$ or $\gamma \notin \gamma_2 \Gamma_\pi \gamma_1^{-1}$. So $\langle \gamma \cdot \xi, \eta \rangle = 0$ off a finite set, and we are done.

(i) $\Rightarrow$ (ii): Obvious.

(ii) $\Rightarrow$ (iii): Fix $\widehat{\pi} \neq \widehat{1}_G$, and assume $\Gamma_{\widehat{\pi}}$ is infinite. Let $\xi = \chi_{\widehat{\pi}}$. Then, clearly, for $\gamma \in \Gamma_{\widehat{\pi}}, \gamma \cdot \xi = \xi$, so $|\langle \gamma \cdot \xi, \xi \rangle| = \|\xi\|^2$ for $\gamma \in \Gamma_{\widehat{\pi}}$, thus $\overline{\lim}_{\gamma \to \infty} |\langle \gamma \cdot \xi, \xi \rangle| = \|\xi\|^2$, contradicting mild mixing.

Assume now that Γ is also torsion-free.

(iii) $\Rightarrow$ (v): Denote by $\mathcal{O}_1, \mathcal{O}_2, \dots$ the orbits of Γ on $\widehat{G} \setminus \{\widehat{1}_G\}$, a finite or infinite list. By (iii), Γ acts freely on each $\mathcal{O}_n$. Fix $\mathcal{O} = \mathcal{O}_n$, and fix π with $\widehat{\pi} \in \mathcal{O}$. Then $\{\sqrt{d_\pi}\pi_{ij}\}_{1 \leq i,j \leq d_\pi}$ is an orthonormal basis for $\mathcal{E}_{\widehat{\pi}}$ and thus $\{\sqrt{d_\pi}\gamma \cdot \pi_{ij}\}_{1 \leq i,j \leq d_\pi}$ is an orthonormal basis for $\mathcal{E}_{\gamma \cdot \widehat{\pi}} \perp \mathcal{E}_{\widehat{\pi}}$, if $\gamma \neq 1$. So for each $1 \leq i, j \leq n$, $\{\sqrt{d_\pi}\gamma \cdot \pi_{ij}\}_{\gamma \in \Gamma}$ is an orthonormal set on which Γ acts freely and transitively, thus the subrepresentation (of the action of Γ on $L^2_0(X)$) on the closed subspace $\langle \sqrt{d_\pi}\gamma \cdot \pi_{ij} \rangle_{\gamma \in \Gamma}$ is isomorphic to λ_Γ. Then the representation on $\bigoplus_{\widehat{\pi} \in \mathcal{O}} \mathcal{E}_{\widehat{\pi}}$ is isomorphic to $d^2_\pi \cdot \lambda_\Gamma$ and so $\pi^X_0 \cong n \cdot \lambda_\Gamma$, for some $1 \leq n \leq \infty$.

(v) $\Rightarrow$ (i): It is enough to check that $n \cdot \lambda_\Gamma$ is a c_0-representation for $n \leq \infty$. Since the eventually zero functions are dense in $\ell^2(\Gamma)$, this is clear. $\square$

We will finally see that when a countable group Γ acts by automorphisms on a compact Polish group G, the property of being tempered for this action is sandwiched between mixing ($\equiv$ mildly mixing, in this case) and ergodicity ($\equiv$ weak mixing, in this case).

For the following statement recall that a Borel action of a countable group Γ on a standard Borel space X with invariant measure μ is called *E_0-ergodic* if and only if for every hyperfinite equivalence relation E and Borel homomorphism φ from E_Γ^X to E, there is a μ-conull set mapped by φ to a single E-class.

Theorem 4.6. *Let an infinite countable group Γ act by automorphisms on a compact Polish group G. Then the following are equivalent:*

(i) *the action of Γ on G is tempered;*

(ii) *for any $\widehat{\pi} \neq \widehat{1}_G, \Gamma_{\widehat{\pi}}$ is amenable;*

(iii) *for any non-amenable $\Gamma_0 \leq \Gamma$, the Γ_0-action on G is ergodic;*

(iv) *for any non-amenable $\Gamma_0 \leq \Gamma$, the Γ_0-action on G is E_0-ergodic.*

In particular, if the action of Γ on G is tempered, so is the action of $\Gamma_0 \leq \Gamma$ on G, for every infinite $\Gamma_0 \leq \Gamma$.

Proof. (i) $\Rightarrow$ (ii). Assume that the action of Γ on G is tempered, i.e., $\pi_0^G \prec \lambda_\Gamma$, where π_0^G is the representation of Γ on $L_0^2(G)$. Fix $\widehat{\pi} \neq \widehat{1}_G$ and denote by $\lambda_{\Gamma/\Gamma_{\widehat{\pi}}}$ the representation of Γ on $\ell^2(\Gamma/\Gamma_{\widehat{\pi}})$ induced by the canonical action of $\Gamma/\Gamma_{\widehat{\pi}}$ (here $\Gamma/\Gamma_{\widehat{\pi}}$ is the set of left-cosets of $\Gamma_{\widehat{\pi}}$). Then it is clear that the representation $\lambda_{\Gamma/\Gamma_{\widehat{\pi}}}$ is isomorphic to the subrepresentation of Γ on $\langle \Gamma \cdot \chi_{\widehat{\pi}} \rangle$, so $\lambda_{\Gamma/\Gamma_{\widehat{\pi}}} \leq \pi_0^G$, thus $\lambda_{\Gamma/\Gamma_{\widehat{\pi}}} \prec \lambda_\Gamma$. But then, by Burger–de la Harpe [4, 2.3], $\Gamma_{\widehat{\pi}}$ is amenable.

(ii) $\Rightarrow$ (i). Assume now that for any $\widehat{\pi} \neq \widehat{1}_G, \Gamma_{\widehat{\pi}}$ is amenable. Let $\mathcal{O}_1, \mathcal{O}_2, \dots$ be the Γ-orbits on $\widehat{G} \setminus \{\widehat{1}_G\}$. As $L_0^2(G) = \bigoplus_n (\bigoplus_{\widehat{\rho} \in \mathcal{O}_n} \mathcal{E}_{\widehat{\rho}})$, it is enough to show that (the representation of Γ on each $\bigoplus_{\widehat{\rho} \in \mathcal{O}_n} \mathcal{E}_{\widehat{\rho}}) \prec \lambda_\Gamma$. So fix an orbit $\mathcal{O} = \mathcal{O}_n$, and $\widehat{\pi} \in \mathcal{O}$ (so that $\mathcal{O} = \Gamma \cdot \widehat{\pi}$). Clearly $\Gamma_{\widehat{\pi}}$ acts unitarily on $\mathcal{E}_{\widehat{\pi}}$ and we denote this representation of $\Gamma_{\widehat{\pi}}$ by $\rho_{\widehat{\pi}}$. Then it is easy to check that the representation of Γ on $\bigoplus_{\widehat{\rho} \in \mathcal{O}} \mathcal{E}_{\widehat{\rho}}$ is contained in the induced representation $\mathrm{Ind}_{\widehat{\pi}}^{\Gamma} \rho_{\widehat{\pi}}$. (For the definition of induced representations, see, e.g., [8, 6.1].) To verify this, simply send $\xi \in \mathcal{E}_{\widehat{\rho}}$, where $\widehat{\rho} = \gamma \cdot \widehat{\pi}$, to the following vector u_ξ in the space of $\mathrm{Ind}_{\Gamma_{\widehat{\pi}}}^{\Gamma} \rho_{\widehat{\pi}}$: u_ξ maps Γ into $H_{\widehat{\pi}}, u_\xi = 0$ except on $\gamma \Gamma_{\widehat{\pi}}$, and for $\delta = \gamma\theta$, where $\theta \in \Gamma_{\widehat{\pi}}$, let $u_\xi(\theta) = \theta^{-1} \cdot \gamma^{-1} \cdot \xi$ (note that $\gamma^{-1} \cdot \xi \in H_{\widehat{\pi}}$, so $\theta^{-1} \cdot \gamma^{-1} \cdot \xi \in H_{\widehat{\pi}}$ as well). This gives an isomorphic copy of $\bigoplus_{\widehat{\rho} \in \mathcal{O}} \mathcal{E}_{\widehat{\rho}}$ inside the space of $\mathrm{Ind}_{\Gamma_{\widehat{\pi}}}^{\Gamma} \rho_{\widehat{\pi}}$, which preserves the Γ-action.

Now, since $\Gamma_{\widehat{\pi}}$ is amenable, by the Hulanicki–Reiter theorem (see [1, 7.3.2]) $\rho \prec \lambda_{\Gamma_{\widehat{\pi}}}$ for any unitary representation ρ of $\Gamma_{\widehat{\pi}}$, So, as inducing preserves weak containment (see [1, 6.3.5]),

$$\mathrm{Ind}_{\Gamma_{\widehat{\pi}}}^{\Gamma} \rho_{\widehat{\pi}} \prec \mathrm{Ind}_{\Gamma_{\widehat{\pi}}}^{\Gamma} \lambda_{\Gamma_{\widehat{\pi}}} \cong \lambda_\Gamma.$$

We are using here that if $\Gamma_0 \leq \Gamma$, then $\mathrm{Ind}_{\Gamma_0}^{\Gamma}\lambda_{\Gamma_0} \cong \lambda_\Gamma$. (To see this, consider the chain $\{1\} \leq \Gamma_0 \leq \Gamma$. Then by induction in stages (see [1, 5.2.4]), $\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\mathrm{Ind}_{\{1\}}^{\Gamma_0} 1_{\{1\}}) \cong \mathrm{Ind}_{\{1\}}^{\Gamma} 1_{\{1\}}$. But $\mathrm{Ind}_{\{1\}}^{\Gamma_0} 1_{\{1\}} \cong \lambda_{\Gamma_0}$ (see [1, 5.1.7]), so

$$\mathrm{Ind}_{\Gamma_0}^{\Gamma}\lambda_{\Gamma_0} \cong \lambda_\Gamma.)$$

Thus the representation of Γ on $\bigoplus_{\rho\in\widehat{\mathcal{O}}}\mathcal{E}_{\widehat{\rho}}$ is contained in $\mathrm{Ind}_{\Gamma_{\widehat{\pi}}}^{\Gamma}\rho_{\widehat{\pi}} \prec \lambda_\Gamma$, and we are done.

(ii) $\Rightarrow$ (iv): Let $\Gamma_0 \leq \Gamma$ be non-amenable. For each $\widehat{\pi} \neq \widehat{1}_G, \Gamma_{\widehat{\pi}}$ is amenable, so $\Gamma_0 \cap \Gamma_{\widehat{\pi}}$ is amenable. Consider now the Γ_0 action on G. Then for $\widehat{\pi} \neq \widehat{1}_G, (\Gamma_0)_{\widehat{\pi}} = \Gamma_0 \cap \Gamma_{\widehat{\pi}}$, so applying (ii) $\Rightarrow$ (i) before to Γ_0 instead of Γ, we have that $\pi_0^X|\Gamma_0 \prec \lambda_{\Gamma_0}$. Since Γ_0 is not amenable, $1_{\Gamma_0} \not\prec \lambda_{\Gamma_0}$, so $1_{\Gamma_0} \not\prec \pi_0^X|\Gamma_0$. This is equivalent to saying that the action of Γ_0 on $L_0^2(G)$ does not have nonzero almost invariant vectors, thus the action of Γ_0 on G has no I-sequence, thus it does not have nontrivial almost invariant sets (see, e.g., [13, A3]). Then by the result of Jones–Schmidt (see [15] or [13, A2.2]), the action of Γ_0 on X is E_0-ergodic. Note that, by 4.3, (ii) implies that this action is ergodic since $\Gamma_0/(\Gamma_0)_{\widehat{\pi}}$ is infinite.

(iv) $\Rightarrow$ (iii): Obvious.

(iii) $\Rightarrow$ (ii): If, for some $\widehat{\pi} \neq \widehat{1}_G$, $\Gamma_{\widehat{\pi}}$ is not amenable, then, by (iii), $\Gamma_{\widehat{\pi}}$ acts ergodically on G. But clearly $\Gamma_{\widehat{\pi}}$ leaves invariant $\chi_{\widehat{\pi}} \in L_0^2(G)$, a contradiction. □

Corollary 4.7. *Let a non-amenable countable group Γ act by automorphisms on a compact Polish group G. Then* (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Leftrightarrow$ (v), *where*

(i) *the action is mixing;*
(ii) *the action is mildly mixing;*
(iii) *the action is tempered;*
(iv) *the action is weakly mixing;*
(v) *the action is ergodic.*

In particular, if $F_2 \leq \Gamma$ and the action is mixing, then it is antimodular.

Remark. It should be pointed out that in the general case of a Borel action of a countable infinite group Γ on a standard Borel space X with invariant measure μ, tempered does not imply mixing (or even mild mixing) and mixing does not imply tempered. An example of a tempered action which is not mildly mixing, even by compact group automorphisms, is given below in 5 (B). To see an example of a mixing action that is not tempered, let Γ be a non-amenable group with the HAP (e.g., F_2).

Then, by [5, 2.2.2], Γ has a mixing action on X with $1_\Gamma \prec \pi_0^X$, and so $\pi_0^X \not\prec \lambda_\Gamma$. However, for every non-amenable group Γ, being tempered implies weak mixing. This is because, by a result of Fell (see [7, 18.9.5, 18.3.6]), if Γ is non-amenable, then λ_Γ does not weakly contain a nonzero finite-dimensional representation.

In case G is Abelian, every irreducible unitary representation of G is one-dimensional, thus for each $\widehat{\pi} \in \widehat{G}, \mathcal{E}_{\widehat{\pi}} = \langle \chi_{\widehat{\pi}} \rangle$. Now $\chi_{\widehat{\pi}} : G \to \mathbb{T}$ is a continuous homomorphism, and conversely for every continuous homomorphism $\chi : G \to \mathbb{T}$ there is a unique $\widehat{\pi} \in \widehat{G}$ with $\chi_{\widehat{\pi}} = \chi$. Thus we can identify, in this case, $\widehat{G}$ with the usual dual of G, i.e., the group of its characters, which is a countable (discrete) group.

If a countable group Γ acts by automorphisms on G, then the induced action of Γ on $\widehat{G}$ is an action of Γ by (algebraic group) automorphisms on $\widehat{G}$. Conversely, an action of Γ by automorphisms on $\widehat{G}$ induces canonically an action by automorphisms on $G = \widehat{\widehat{G}}$.

5. Examples and questions

(A) Let Γ be an infinite countable group, K an Abelian compact Polish group with Haar measure ν. Consider the compact Polish group $G = K^\Gamma = \{\{k_\gamma\}_\gamma : k_\gamma \in K\}$ (with pointwise multiplication). Then ν^Γ(= the product measure of ν) is the Haar measure of G. The dual group $\widehat{G}$ can be identified with

$$\widehat{K}^{<\Gamma} = \{\{\chi_\gamma\}_\gamma : \chi_\gamma \in \widehat{K}, \chi_\gamma = 1 \text{ for all but finitely many } \gamma\},$$

where $\{\chi_\gamma\}_\gamma = \chi : G \to \mathbb{T}$ is defined by

$$\{\chi_\gamma\}_\gamma(\{k_\gamma\}_\gamma) = \prod_\gamma \chi_\gamma(k_\gamma).$$

Now Γ acts on K^Γ by shift

$$\delta \cdot \{k_\gamma\} = \{k_{\gamma^{-1}\delta}\},$$

and this is an action by automorphisms on K^Γ. Its corresponding action on $\widehat{G} = \widehat{K}^{<\Gamma}$ is also the shift action: $\delta \cdot \{\chi_\gamma\} = \{\chi_{\gamma^{-1}\delta}\}$.

If $K \neq \{1\}$, so that $\widehat{K} \neq \{1\}$, clearly the action of Γ on $\widehat{G} \setminus \{1\} = \widehat{K}^{<\Gamma} \setminus \{1\}$ has finite stabilizers, so it is mixing, thus it is tempered, and so, if $\Gamma \geq F_2$, it is antimodular. So the action of $\Gamma \geq F_2$ by shift on $2^\Gamma (= \mathbb{Z}_2^\Gamma)$ with the usual product measure is antimodular, and in fact so is the action on X^Γ for any finite non-singleton set X, since there are Abelian groups of any finite cardinality. Similarly, the action of $\Gamma \geq F_2$ by shift on $\mathbb{T}^\Gamma$ with the product measure is antimodular (and of course $\mathbb{T}$ can be replaced here by any Polish space with non-atomic measure). These were the original results proved in Hjorth [12].

(B) Consider now the group $\mathrm{SL}_n(\mathbb{Z}), n = 2, 3, \ldots$, and its usual action on $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$ by matrix multiplication, which is clearly an action by automorphisms. The dual of $\mathbb{T}^n$ is $\mathbb{Z}^n$, where $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ is identified with the character

$$\chi(z_1, \ldots, z_n) = z_1^{k_1} z_2^{k_2} \ldots z_n^{k_n}.$$

It follows that the induced action of $\mathrm{SL}_n(\mathbb{Z})$ on $\mathbb{Z}^n$ is given by

$$A \cdot (k_1, \ldots, k_n) = (A^{-1})^t \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix}.$$

Since the map $A \mapsto (A^{-1})^t$ is an automorphism on $\mathrm{SL}_n(\mathbb{Z})$, the structure of the stabilizers of this action on $\mathbb{Z}^n$ is the same as the structure of the stabilizers of the usual action by matrix multiplication on $\mathbb{Z}^n$.

If $n \geq 3$, then it is easy to see that not all stabilizers of the elements of $\mathbb{Z}^n \backslash \{0\}$ are amenable (e.g., the stabilizer of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ consists of all matrices of the form $\begin{pmatrix} 1 & b & c \\ 0 & d & e \\ 0 & f & g \end{pmatrix}$, thus contains a copy of $\mathrm{SL}_2(\mathbb{Z})$). It follows that the action of $\mathrm{SL}_n(\mathbb{Z})$ on $\mathbb{T}^n$ for $n \geq 3$, which is well-known to be weak mixing but not mixing, is also not tempered.

Consider now the case $n = 2$. Then it is easy to see that the stabilizers of the elements of $\mathbb{Z}^2 \backslash \{0\}$ are in fact infinite cyclic. This is clear for $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, as its stabilizer consists of the matrices of the form $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, $a \in \mathbb{Z}$. Consider then any other point $\begin{pmatrix} p \\ q \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $p, q \in \mathbb{Z}$. We can assume that $\gcd(p, q) = 1$ (where $\gcd(n, 0) = n$), so find $a, b \in \mathbb{Z}$ with $ap + bq = 1$. Then $A = \begin{pmatrix} a & b \\ -q & p \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $A \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, so the stabilizer of

$\begin{pmatrix} p \\ q \end{pmatrix}$ is conjugate to that of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, so cyclic. It follows that the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{T}^2$ is not only weak mixing but actually tempered (but of course is not mixing). So it is also antimodular. Moreover, the last two examples show that the arrows in 4.7 cannot be reversed.

(C) The group F_2 can be embedded in $\mathrm{SL}_2(\mathbb{Z})$ in various ways. First, there is a copy of F_2 contained in $\mathrm{SL}_2(\mathbb{Z})$ that has finite index on $\mathrm{SL}_2(\mathbb{Z})$. Identifying this copy with F_2, we have an action of F_2 on $\mathbb{T}^2$ by automorphisms. The stabilizers of the induced action of F_2 on $\mathbb{Z}^2$ are amenable but infinite, so this action of F_2 on $\mathbb{T}^2$ is tempered, so also antimodular, but not mixing.

On the other hand, F_2 can be embedded in $\mathrm{SL}_2(\mathbb{Z})$, so all the matrices in this copy of F_2 are *hyperbolic*, i.e., have trace of absolute value > 2 (see [24, p. 86, pp. 97–98]). Every hyperbolic matrix clearly has no fixed points on $\mathbb{Z}^2 \setminus \{0\}$, from which it follows that if we identify F_2 with *this* copy, and we consider its action on $\mathbb{T}^2$, then the induced action on $\mathbb{Z}^2 \setminus \{0\}$ has trivial stabilizers, thus this action of F_2 on $\mathbb{T}^2$ is mixing, thus also antimodular.

(D) We can also use (C) to see that the action of $\mathrm{SL}_n(\mathbb{Z})$ on $\mathbb{T}^n$ is antimodular, for any $n \geq 2$, even when we restrict it to an invariant conull subset of $\mathbb{T}^n$. The reason is that there is a copy of F_2 in $\mathrm{SL}_n(\mathbb{Z})$, so that its action on $\mathbb{T}^n$ is tempered (although the action of $\mathrm{SL}_n(\mathbb{Z})$ itself on $\mathbb{T}^n$ is not, if $n \geq 3$). To verify this, it is enough to show that we can find a copy of F_2 in $\mathrm{SL}_n(\mathbb{Z})$ with the property that the induced action of F_2 on $\mathbb{Z}^n \setminus \{0\}$ has cyclic stabilizers. We have already seen this for $n = 2$, and by diagonal embedding it follows for all even n. To verify it then for all $n \geq 2$, it is enough to check it for $n = 3$ (using diagonal embedding again), and this is true by an unpublished result of G. Soifer (private communication).

We sketch an elementary proof of this below. It is based on a reformulation of Soifer's idea, due to S. Adams. We have replaced Adams' argument for the cyclicity of the stabilizers by an argument using only elementary linear algebra.

Consider the group $\mathrm{SL}_2(\mathbb{R})$ and its action on the set of 2×2 real matrices with trace 0 by conjugation (this is the so-called adjoint representation of $\mathrm{SL}_2(\mathbb{R})$). Identifying such a traceless matrix $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ with the vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$, this action assigns to each $A \in \mathrm{SL}_2(\mathbb{R})$ a matrix $\varphi(A) \in \mathrm{GL}_3(\mathbb{R})$ and one can check easily that actually $\varphi(A) \in \mathrm{SL}_3(\mathbb{R})$, so φ: $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_3(\mathbb{R})$ is a homomorphism. One can easily also check that the kernel of φ consists of the matrices $\{I, -I\}$, so φ induces an embedding, also denoted by φ, of $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm I\}$ into $\mathrm{SL}_3(\mathbb{R})$. Since $\mathrm{PSL}_2(\mathbb{R})$ contains a copy of F_2, it is enough to check that the action of $\varphi(\mathrm{PSL}_2(\mathbb{R}))$ on $\mathbb{R}^3 \setminus \{0\}$ has Abelian stabilizers. For this, it is enough to show that if M is a nonzero 2×2 matrix with trace 0, then the set of all elements in $\mathrm{SL}_2(\mathbb{R})$ that commute with M is Abelian. Since every nonzero 2×2 matrix with trace 0 is conjugate by a matrix in $\mathrm{GL}_2(\mathbb{C})$ to a matrix of the form $\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$ or $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$, for some $a \in \mathbb{C} \setminus \{0\}$, it is enough to show that the set of matrices in $\mathrm{GL}_2(\mathbb{C})$ that commute with a given matrix of one of the above two types forms an Abelian group. This can be easily checked by direct calculation.

(E) A. Valette and B. Bekka brought to my attention the following example of a tempered action. Let G be a discrete infinite subgroup of $\mathrm{SL}_2(\mathbb{R})$ and consider its canonical action on $X = \mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$. Then this action is tempered. For this and various results and questions related to extensions of this result, one can consult Lubotzky [18].

(F) B. Bekka also communicated to me the following result. Suppose G is a simple real Lie group with finite center and with $\mathbb{Q}$-rank ≥ 2. Let G act in a Borel way on a standard Borel space with ergodic invariant measure, and let $\Gamma \leq G$ be an arithmetic lattice in G. Then Γ contains a copy of F_2 such that the action of F_2 on X is tempered, and thus the action of Γ on X is antimodular.

(G) Let Γ be a residually finite group and $\Gamma = \Gamma_0 \geq \Gamma_1 \geq \dots$ a decreasing sequence of finite index normal subgroups of Γ with $\bigcap_n \Gamma_n = \{1\}$. Let $\pi_n : \Gamma/\Gamma_{n+1} \to \Gamma/\Gamma_n$ be the canonical projection, $\pi_n(\gamma\Gamma_{n+1}) = \gamma\Gamma_n$, and let K be the inverse limit of (Γ_n, π_n), a closed subgroup of the compact group $\prod_n \Gamma/\Gamma_n$. Thus $K = \{(\gamma_n\Gamma_n) : \gamma_0\Gamma_0 \supseteq \gamma_1\Gamma_1 \supseteq \gamma_2\Gamma_2 \dots\}$. We can embed Γ into K via $\varphi : \gamma \mapsto (\gamma\Gamma_n)_{n\in\mathbb{N}}$. Then $\varphi(\Gamma)$ is a dense subgroup of K. Moreover, if $H_n = \overline{\varphi(\Gamma_n)}$, then $[K : H_n] < \infty$ and $\bigcap_n H_n = \{1\}$, since

$$\begin{aligned} H_n = &\{(\Gamma_0, \dots, \Gamma_n, \gamma_{n+1}\Gamma_{n+1}, \gamma_{n+2}\Gamma_{n+2}, \dots) : \Gamma_n \supseteq \gamma_{n+1}\Gamma_{n+1} \supseteq \\ &\gamma_{n+2}\Gamma_{n+2} \supseteq \dots\}. \end{aligned}$$

It follows from Example (E) of §1, that the (left) translation action of Γ on K is (finitely) modular.

So any residually finite countable group Γ (like, e.g., F_n) can be densely embedded in a compact Polish group K, so that the induced (left) translation action of Γ on K is (finitely) modular.

On the other hand, if we embed F_2 into $\mathrm{SO}_3(\mathbb{R})$ (the group of rotations in $\mathbb{R}^3$), then is the translation action of F_2 on $\mathrm{SO}_3(\mathbb{R})$ modular? antimodular? (**Addendum.** Orestis Raptis has recently used the fact that any countable non-Abelian torsion-free subgroup of $\mathrm{SO}_3(\mathbb{R})$ is dense in $\mathrm{SO}_3(\mathbb{R})$, to show that the translation action of any copy of F_2 in $\mathrm{SO}_3(\mathbb{R})$, as well as its conjugation action on $\mathrm{SO}_3(\mathbb{R})$ and its natural action on S^2, are not modular.) Notice that if Γ is a non-amenable countable dense subgroup of a compact group K, then the left-translation of Γ on K is not weakly mixing, so not tempered, since by the Peter–Weyl theorem, the corresponding representation on $L^2_0(K)$ admits nonzero finite-dimensional subrepresentations.

(H) Let a countable group Γ act in a Borel way on a standard Borel space X with non-atomic invariant measure μ. If the action is weakly mixing, then it cannot be modular. Indeed, if it is modular with a witnessing partition $\{\mathcal{P}_n\}$, then we can find n large enough so that at least one $A \in \mathcal{P}_n$ has measure $0 < \mu(A) < 1$. Then the span of $\{1_A - \mu(A) \cdot 1 : A \in \{\mathcal{P}_n\}\}$ is a nonzero finite-dimensional subspace of $L^2_0(X)$ invariant under Γ, a contradiction.

It is an open question however whether a weakly mixing action or even a (strongly) mixing action must be always antimodular.

Acknowledgments. I would like to thank Alain Valette and Bachir Bekka for many valuable comments on an earlier version of this paper, and for pointers to references and suggestions of several additional examples of tempered actions, that I was not aware of. Moreover, Bachir Bekka suggested the terminology "tempered action," used here, instead of "regular action" that I used in an earlier version. This is motivated by terminology from group representation theory, where unitary representations of semisimple Lie groups that are weakly contained in the regular representation are usually called "tempered representations," because of the decay behavior of the corresponding matrix coefficients. See, for example, [17, 18, 6] (as well as Haagerup [10] for the case of free groups). Finally, I would like to thank S. Adams, G. Hjorth, J. Melleray, G. Soifer, and S. Thomas for their comments at various stages of the preparation of this paper.

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Поступило 30 марта 2005 г.

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