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REPRESENTATIONS OF BORNOLOGICAL ALGEBRAS

Abstract. Bounded representations of bornological algebras are considered. The left and right bornological radicals in bornological algebras are introduced. It is shown that the left (right) bornological radical of a bornological algebra A is equal to the intersection of all bornologically closed maximal regular left (respectively, right) ideals of A and these both radicals of A and the Jacobson radical of A coincide when A is an advertive and simplicial bornological algebra (in particular, a bornological Q-algebra).

Properties of bornological algebras have been studied in several papers and books (see, for example, [1, 2, 6, 9, 11–13, 15] and [16]), but mainly in the case of convex bornological algebras with jointly bounded multiplication. In the present paper we consider bounded representations and radicals of a bornological algebra with separately bounded multiplication.

1. Introduction

1. Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ of real numbers or $\mathbb{C}$ of complex numbers and X a set. A collection $\mathcal{B}$ of subsets of X is called a *bornology* on X if

a) $\mathcal{B}$ covers X,

b) if $n \in \mathbb{N}$ and $B_1, \dots, B_n \in \mathcal{B}$, then $\bigcup\limits_{k=1}^{n} B_k \in \mathcal{B}$,

and

c) if $B \in \mathcal{B}$ and $B' \subset B$, then $B' \in \mathcal{B}$.

Herewith, the pair $(X, \mathcal{B})$ is called a *bornological space* and every $B \in \mathcal{B}$ a *bounded subset* in X. In particular, let X be a linear space over $\mathbb{K}$ and the bornology $\mathcal{B}$ on X satisfy the conditions

d) if $B_1, B_2 \in \mathcal{B}$, then $B_1 + B_2 \in \mathcal{B}$,

e) if $\lambda \in \mathbb{K}$ and $B \in \mathcal{B}$, then $\lambda B \in \mathcal{B}$,

and

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f) if $B \in \mathcal{B}$, then $\bigcup\limits_{|\lambda| \leqslant 1} \lambda B \in \mathcal{B}$.

Then the pair $(X, \mathcal{B})$ is called a *bornological linear space* over $\mathbb{K}$ (in short, a bornological linear space). It is easy to see that the addition and the multiplication over $\mathbb{K}$ in every bornological linear space $(X, \mathcal{B})$ are bounded[1] maps by conditions d)–f). A bornological linear space $(X, \mathcal{B})$ is called a *separated bornological linear space* if $\{\theta_X\}$ is the only bounded linear subspace in X (here θ_X denotes the zero element of X).

2. Let now X be an algebra over $\mathbb{K}$ and let a bornology $\mathcal{B}$ on X satisfy, in addition to conditions d)–f), the condition

g) if $x \in X$ and $B \in \mathcal{B}$, then xB, $Bx \in \mathcal{B}$.

Then $(X, \mathcal{B})$ is called a *bornological algebra* over $\mathbb{K}$ (in short, a *bornological algebra*). In particular, let the bornology $\mathcal{B}$ on X satisfy conditions d)–f) and

h) if $B_1, B_2 \in \mathcal{B}$, then $B_1 B_2 \in \mathcal{B}$.

Then $(X, \mathcal{B})$ is called a *bornological algebra with jointly bounded multiplication*. The multiplication in $(X, \mathcal{B})$ is *separately bounded*, by condition g), and *jointly bounded*, by condition h).

3. Let $(X, \mathcal{B})$ be a bornological linear space. It is said that the net $(x_\lambda)_{\lambda \in \Lambda}$ in X *bornologically converges* (or *Mackey converges*) to x_0 if there is a balanced[2] $B \in \mathcal{B}$ and for any $\epsilon > 0$ an index $\lambda_\epsilon \in \Lambda$ such that $x_\lambda - x_0 \in \epsilon B$ whenever $\lambda > \lambda_\epsilon$. It is known (see [10, p. 28, Proposition 4]) that a bornological linear space $(X, \mathcal{B})$ is separated if and only if every bornologically convergent net in X has a unique limit.

A subset U of X is called a *bornologically closed set* if from conditions $(x_\lambda)_{\lambda \in \Lambda}$ is a net in U and $(x_\lambda)_{\lambda \in \Lambda}$ converges bornologically to x_0 follows that $x_0 \in U$.

A map f from a bornological linear space $(X, \mathcal{B}_X)$ to another bornological linear space $(Y, \mathcal{B}_Y)$ is a *bornologically continuous map* if for any net $(x_\lambda)_{\lambda \in \Lambda}$ of elements of X such that $(x_\lambda)_{\lambda \in \Lambda}$ bornologically converges to x_0 in X follows that the net $(f(x_\lambda))_{\lambda \in \Lambda}$ bornologically converges to $f(x_0)$ in Y. In particular, when f is a linear map, then f is bounded if and only if f is bornologically continuous (see [8, p. 10, Proposition 1]).

4. Let $(X, \mathcal{B}_X)$ be a bornological linear space over $\mathbb{K}$ and $\mathcal{L}(X)$ be the set of all bounded linear maps from $(X, \mathcal{B}_X)$ to $(X, \mathcal{B}_X)$. We define the

[1] A map f from $(X, \mathcal{B}_X)$ to $(Y, \mathcal{B}_Y)$ is called *bounded* if $f(B) \in \mathcal{B}_Y$ for all $B \in \mathcal{B}_X$.

[2] A set U is called *balanced*, or *circled*, if $\lambda U \subset U$ whenever $\lambda \in \mathbb{K}$ and $|\lambda| \leqslant 1$.

addition of elements and the scalar multiplication in $\mathcal{L}(X)$ point-wise and the multiplication of elements in $\mathcal{L}(X)$ by the composition (that is, $(f+g)(x) = f(x)+g(x)$, $(\lambda f)(x) = \lambda f(x)$ and $(fg)(x) = f(g(x))$ for each $f, g \in \mathcal{L}(X)$ and $x \in X$). Then $\mathcal{L}(X)$ is an algebra over $\mathbb{K}$. We endowe $\mathcal{L}(X)$ with the bornology $\mathcal{B}_{\mathcal{L}(X)}$ which consists of the sets

$$\Big\{H \subset \mathcal{L}(X) : \bigcup_{L \in H} L(B) \in \mathcal{B}_X \quad \text{for each} \quad B \in S_X\Big\},$$

where $\mathcal{S}_X$ is a set of all finite subsets in X. It is easy to see that $(\mathcal{L}(X), \mathcal{B}_{\mathcal{L}(X)})$ is a bornological algebra.

Let now $(X, \mathcal{B}_X)$ be a separated bornological linear space, H a bounded linear subspace of $(\mathcal{L}(X), \mathcal{B}_{\mathcal{L}(X)})$ and x_0 an arbitrary element of X. Then

$$D = \bigcup_{L \in H} L(\{x_0\}) \in \mathcal{B}_X.$$

Since D is a linear subspace of X and $(X, \mathcal{B}_X)$ is separated, we have $D = \{\theta_X\}$. Hence, $H = \{\theta_{\mathcal{L}(X)}\}$. Consequently, $(\mathcal{L}(X), \mathcal{B}_{\mathcal{L}(X)})$ is a separated bornological algebra.

5. Let A be an algebra over $\mathbb{K}$ and X a linear space over $\mathbb{K}$. Any homomorphism (antihomomorphism)[3] π of A into $\mathcal{L}(X)$ is called a *representation* (respectively, *antirepresentation*) of A on X. Every representation (antirepresentation) π of A on X defines on X a left (respectively, right) module multiplication ${}_\pi\cdot$ (respectively, $\cdot_\pi$) if we put $a\,{}_\pi\!\cdot x = \pi(a)(x)$ (respectively, $x \cdot_\pi a = \pi(a)(x)$) for each $a \in A$ and $x \in X$. In this case X becomes a left (right) A-module that we denote by ${}_\pi X$ (respectively, by X_π).

6. A left (right) A-module X is *nontrivial* if $AX \neq \{\theta_X\}$ (respectively, $XA \neq \{\theta_X\}$), and is called *irreducible* if it is a nontrivial left (right) A-module in which X and $\{\theta_X\}$ are the only A-submodules of X. A representation (antirepresentation) π of A on X is called *irreducible* if ${}_\pi X$ is an irreducible left (respectively, if X_π is an irreducible right) A-module.

7. Let A be an algebra. The intersection $\operatorname{Rad} A$ of kernels of all irreducible representations of A on linear spaces is called a *Jacobson radical* of A. If there are no irreducible representations, then A is called a *radical algebra* (in this case we write $\operatorname{Rad} A = A$) and if $\operatorname{Rad} A = \{\theta_A\}$, then A is called a *semisimple algebra*.

[3] That is, a linear map π such that $\pi(ab) = \pi(b)\pi(a)$ for all $a, b \in A$.

8. Let $(A, \mathcal{B}_A)$ be a bornological algebra, M a bornologically closed maximal regular or modular left (right) ideal of $(A, \mathcal{B}_A)$, $A - M$ the quotient space of A modulo M (it consists of M-cosets which contain such elements $a, b \in A$ for which $a - b \in M$) and κ the canonical homomorphism from A onto $A - M$. We define the addition and the multiplication over $\mathbb{K}$ in $A - M$ as usual (that is, $x_1 + x_2 = \kappa(a_1 + a_2)$ and $\mu x_1 = \kappa(\mu a_1)$ for each $x_1, x_2 \in A - M$ and $\mu \in \mathbb{K}$, where a_1 and a_2 are arbitrary elements of the M-cosets x_1 and x_2, respectively) and endowe $A - M$ with the *quotient bornology*

$$\mathcal{B}_{A-M} = \{\kappa(B) : B \in \mathcal{B}_A\}.$$

Then $(A - M, \mathcal{B}_{A-M})$ is a separated[4] bornological linear space. If we define the module multiplication in $A - M$ over A by $ax = \kappa(aa')$ ($xa = \kappa(a'a)$) for each $a \in A$ and $x \in A - M$, where a' is an arbitrary element of the M-coset x, then we can consider $A - M$ as a left (respectively, right) A-module.

9. Let $(A, \mathcal{B})$ be a bornological algebra. For each $a \in A$ and each bornologically closed maximal regular left (right) ideal M of $(A, \mathcal{B})$, let ${}_aL^M$ (respectively, L_a^M) be a map from $A - M$ into $A - M$ defined by ${}_aL^M(x) = ax$ (respectively, by $L_a^M(x) = xa$) for each $x \in A - M$ and let ${}_ML$ (respectively, L_M) be a map from A into $\mathcal{L}(A - M)$ defined by ${}_ML(a) = {}_aL^M$ (respectively, by $L_M(a) = L_a^M$) for each $a \in A$. Then ${}_ML$ is a representation and L_M is an antirepresentation of A on $A - M$, ${}_aL^M(x) = ax$, $L_a^M(x) = xa$, $a\,{}_{L_M}\!\cdot x = ax$, $a \cdot_{L_M} x = xa$ for each $a \in A$ and $x \in A - M$ and the left A-module ${}_ML(A - M)$ (the right A-module $(A - M)_{L_M}$) is irreducible (see [4, Proposition 5, p. 120]). Hence, ${}_ML$ is an irreducible representation of A on $A - M$ and L_M is an irreducible antirepresentation of A on $A - M$. Here with

$$\ker {}_ML = \{a \in A : aA \subset M\} \quad (\ker L_M = \{a \in A : Aa \subset M\}).$$

Both kernels $\ker {}_ML$ and $\ker L_M$ are called *primitive ideals* in A for every maximal regular left (respectively, right) ideal of A.

To show that ${}_ML$ is bounded,[5] let $B \in \mathcal{B}_A$ and $B' \in \mathcal{S}_{A-M}$. Then there are $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in A - M$ such that $B' = \{x_1, \ldots, x_n\}$

[4] The quotient space $(A - M, \mathcal{B}_{A-M})$ is separated if and only if M is bornolodically closed (see [10, p. 37, Proposition 2]).

[5] The boundedness of L_M is shown similarily.

and for every v an element $a_v \in A$ such that $x_v = \kappa(a_v)$. Since ${}_M L(B) = \{{}_b L^M : b \in B\}$ and

$${}_b L^M(x_v) = \kappa(ba_v) \in \kappa(Ba_v) \subset \bigcup_{v=1}^{n} \kappa(Ba_v) = \kappa\Big[\bigcup_{v=1}^{n} Ba_v\Big] \in \mathcal{B}_{A-M}$$

for each $b \in B$ and each v, we have ${}_M L(B) \in \mathcal{B}_{\mathcal{L}(A-M)}$ for each $B \in \mathcal{B}_A$. Therefore, ${}_M L$ is bounded.

Let $(a_\lambda)_{\lambda\in\Lambda}$ be a net in ker ${}_M L$ and let $(a_\lambda)_{\lambda\in\Lambda}$ bornologically converge to a_0 in A. Since ${}_M L$ is bounded, ${}_M L$ is bornologically continuous. Therfore, the net $({}_M L(a_\lambda))_{\lambda\in\Lambda}$ bornologically converges to ${}_M L(a_0))$. Since ${}_M L(a_\lambda) = \theta_{\mathcal{L}(A-M)}$ for each $\lambda \in \Lambda$ and $(\mathcal{L}(A-M), \mathcal{B}_{\mathcal{L}(A-M)})$ is separated, ${}_M L(a_0) = \theta_{\mathcal{L}(A-M)}$. Therefore, from

$${}_{a_0} L^M(\kappa(a)) = a_0\kappa(a) = \kappa(a_0 a) = \theta_{A-M}$$

$(a \in A)$ it follows that $a_0 A \subset M$. Hence, $a_0 \in \ker_M L$.

It means that ${}_M L$ is a bounded irreducible representation, L_M is a bounded irreducible antirepresentation of $(A, \mathcal{B}_A)$ on $(A - M, \mathcal{B}_{A-B})$, and ker ${}_M L$ and ker L_M are bornologically closed subsets of A.

2. Bornological radicals

Let $(A, \mathcal{B})$ be a bornological algebra. We say that the intersection $\mathrm{rad}_l\, A$ of kernels of all bounded irreducible representations of $(A, \mathcal{B})$ on separated bornological linear spaces is a *bornological left radical* of $(A, \mathcal{B})$ and the intersection $\mathrm{rad}_r\, A$ of kernels of all bounded irreducible antirepresentations of $(A, \mathcal{B})$ on separated bornological linear spaces is a *bornological right radical* of A. If $(A, \mathcal{B})$ has no bounded irreducible representations of $(A, \mathcal{B})$, then we say that $(A, \mathcal{B})$ is a *bornologically left radical algebra*, and if $(A, \mathcal{B})$ has no bounded irreducible antirepresentations of $(A, \mathcal{B})$, then we say that $(A, \mathcal{B})$ is a *bornologically right radical algebra*. In the first case, we write $\mathrm{rad}_l\, A = A$ and in the second case, $\mathrm{rad}_r\, A = A$. In particular, if $\mathrm{rad}_l\, A = \{\theta_A\}$, then $(A, \mathcal{B})$ is a *bornologically left semisimple algebra*, and if $\mathrm{rad}_r\, A = \{\theta_A\}$, then $(A, \mathcal{B})$ is a *bornologically right semi-simple algebra*.

Theorem 1. *Let $(A, \mathcal{B})$ be a bornological algebra. If $\mathrm{rad}_l\, A \neq A$, then $\mathrm{rad}_l\, A$ is equal to*

a) *the intersection of all bornologically closed maximal regular left ideals of $(A, \mathcal{B})$,*

b) *the intersection of all primitive ideals of* $(A, \mathcal{B})$ *that are defined by bornologically closed maximal regular left ideals of* $(A, \mathcal{B})$.

If $\operatorname{rad}_r A \neq A$, *then* $\operatorname{rad}_r A$ *is equal to*

a′) *the intersection of all bornologically closed maximal regular right ideals of* $(A, \mathcal{B})$,

b′) *the intersection of all primitive ideals of* $(A, \mathcal{B})$ *that are defined by bornologically closed maximal regular right ideals of* $(A, \mathcal{B})$.

Proof. For each bornological algebra $(A, \mathcal{B})$, let $m_l(A)$ $(m_r(A))$ denote the set of all bornologically closed maximal regular left (respectively, right) ideals of $(A, \mathcal{B})$. If[6] $\operatorname{rad}_l A \neq A$, then

$$\operatorname{rad}_l A \subset \bigcap_{M \,\in\, m_l(A)} \ker{}_M L \subset \bigcap_{M \in m_l(A)} M,$$

because $\ker{}_M L \subset M$ for each $M \in m_l(A)$. To show the converse inclusion, let $a \in \cap\{M : M \in m_l(A)\}$. If $a \notin \operatorname{rad}_l A$, then (by assumption that $\operatorname{rad}_l A \neq A$) there is a bounded irreducible representation π of $(A, \mathcal{B})$ on some separated bornological linear space $(X, \mathcal{B}_X)$ such that $a \notin \ker \pi$. Therefore, there exists $x_0 \in X$ such that $\pi(a)(x_0) \neq \theta_X$. Hence $a \notin I = \ker \rho_{\pi,x_0}$ where $\rho_{\pi,x_0}(a) = \pi(a)(x_0)$ for each $a \in A$. It is known (see [4, p. 120]) that $\ker\rho_{\pi,x_0}$ is a maximal regular left ideal in A. To show that $\ker \rho_{\pi,x_0}$ is bornologically closed, we first show that the map ρ_{π,x_0} is bounded. To this end, let B be an arbitrary element in $\mathcal{B}_A$. Then $\pi(B) \in \mathcal{B}_{\mathcal{L}(X)}$. Therefore, from

$$\rho_{\pi,x_0}(B) \subset \bigcup_{L \in \pi(B)} L(\{x_0\})$$

it follows that $\rho_{\pi,x_0}(B) \in \mathcal{B}_X$. Hence ρ_{π,x_0} is a bounded map. Thus, ρ_{π,x_0} is bornologically continuous. Since $(X, \mathcal{B}_X)$ is separated, I is a bornologically closed maximal regular left ideal in $(A, \mathcal{B})$ which does not contain a. But this is impossible. Consequently, $a \in \operatorname{rad}_l A$. Hence, statement a) holds. Since $\ker{}_M L \subset M$ for each $M \in m_l(A)$, statement b) holds too.

Corollary 1. *Let* $(A, \mathcal{B})$ *be a bornological algebra. If* $\operatorname{rad}_l A \neq A$ $(\operatorname{rad}_r A \neq A)$, *then* $\operatorname{rad}_l A$ *(respectively,* $\operatorname{rad}_r A$*) is a bornologically closed two-sided ideal in* $(A, \mathcal{B})$.

[6] The proof for $\operatorname{rad}_r A$ is similar.

Corollary 2. *Let $(A, \mathcal{B})$ be a bornological algebra. Then* $\operatorname{Rad} A \subset \operatorname{rad}_l A \cap \operatorname{rad}_r A$.

Proof. It is well known (see, for example, [7, Theorem 24.10.1]) that in any algebra A the intersection of all maximal regular left ideals of A is equal to the intersection of all maximal regular right ideals of A. Therefore, $\operatorname{Rad} A \subset \operatorname{rad}_l A \cap \operatorname{rad}_r A$ by Theorem 1.

Theorem 2. *Let A be a bornological algebra. If* $\operatorname{rad}_l A \neq A$, *then* $A/\operatorname{rad}_l A$ *is a bornologically left semi-simple algebra, and if* $\operatorname{rad}_r A \neq A$, *then* $A/\operatorname{rad}_r A$ *is a bornologically right semi-simple algebra.*

Proof. Let $(A, \mathcal{B}_A)$ be a bornological algebra, κ the canonical homomorphism from A onto $A/\operatorname{rad}_l A$ and π a bounded irreducible representation of $(A, \mathcal{B}_A)$ on a separated bornological linear space $(X, \mathcal{B}_X)$. If $\operatorname{rad}_l A \neq A$, then there exists a representation ω of $A/\operatorname{rad}_l A$ on X such that $\pi = \omega \circ \kappa$.

Let $a, a' \in A$ be such that $k(a) = k(a')$. Then $a - a' \in \operatorname{rad}_l A \subset \ker \pi$. Therefore, $\pi(a) = \pi(a')$. That is, ω is well defined. We define the module multiplication ${}_\omega\cdot$ on X by $z\,{}_\omega\!\cdot x = a\,{}_\pi\!\cdot x$ for each $z \in A/\operatorname{rad}_l A$ and $x \in X$, where $a \in A$ is such that $z = k(a)$. Since

$$a\,{}_\pi\!\cdot x = \pi(a)(x) = \pi(a')(x) = a'\,{}_\pi\!\cdot x$$

for each such $a, a' \in A$ for which $k(a) = k(a')$ and $x \in X$, the product $z\,{}_\omega\!\cdot x$ does not depend on the representatives of $\operatorname{rad}_l A$-coset z. Thus, X_ω is a left $A/\operatorname{rad}_l A$-module. To show that X_ω is irreducible, let S be a left $A/\operatorname{rad}_l A$-submodule in ${}_\omega X$. Since S is also a left A-submodule in ${}_\pi X$ and ${}_\pi X$ is irreducible, $S = X$ or $S = \{\theta_X\}$. Hence ${}_\omega X$ is also irreducible. To show that ω is bounded, let C be an element of the quotient bornology $\mathcal{B}_{A/\operatorname{rad}_l A}$ on $A/\operatorname{rad}_l A$. Then there is $B \in \mathcal{B}_A$ such that $C = \kappa(B)$, and $\pi(B) \in \mathcal{B}_{\mathcal{L}(X)}$ because π is bounded. Therefore, $\omega(C) = \pi(B) \in \mathcal{B}_{\mathcal{L}(X)}$. Hence, ω is a bounded irreducible representation of $(A/\operatorname{rad}_l A, \mathcal{B}_{A/\operatorname{rad}_l A})$ on $(X, \mathcal{B}_X)$.

Let now z be an arbitrary element of $\operatorname{rad}_l(A/\operatorname{rad}_l A)$. Then $z = \kappa(a)$ for some $a \in A$ and $z_\omega\!\cdot X = \{\theta_X\}$. Therefore, $a_\pi\!\cdot X = \{\theta_X\}$. It means that $a \in \ker \pi$. Since π is an arbitrary bounded irreducible representation of $(A, \mathcal{B}_A)$, $a \in \operatorname{rad}_l A$. It means that z is the zero element of $A/\operatorname{rad}_l A$. Hence $(A/\operatorname{rad}_l A, \mathcal{B}_{A/\operatorname{rad}_l A})$ is a bornologically left semi-simple algebra.

The proof for $(A/\operatorname{rad}_r A, \mathcal{B}_{A/\operatorname{rad}_r A})$ is similar.

3. Bornologically quasi-ivertible elements

1. Let $(A, \mathcal{B})$ be a bornological algebra. We will say that $a \in A$ is *bornologically left quasi-invertible* or *bornologically left advertible* (*bornologically right quasi-invertible* or *bornologically right advertible*) in $(A, \mathcal{B})$ if there exists a net $(a_\lambda)_{\lambda \in \Lambda}$ of elements of A such that the net $(a_\lambda \circ a)_{\lambda \in \Lambda}$ (respectively, $(a \circ a_\lambda)_{\lambda \in \Lambda}$) bornologically converges to θ_A (here $a \circ b = a + b - ab$ for all $a, b \in A$). The bornological left and right invertibility of elements in $(A, \mathcal{B})$ is defined similarily. It is easy to see that every left (right) quasi-invertible element[7] $a \in A$ is bornologically left (respectively, right) quasi-invertible. Denoting by $\operatorname{Qinv}_l A$, $\operatorname{Qinv}_r A$, $Qinv_l\, A$, and $Qinv_r\, A$, respectively, the sets of all left quasi-invertible, right quasi-invertible, bornologically left quasi-invertible and bornologically right quasi-invertible elements in A, it is clear that $\operatorname{Qinv}_l A \subset Qinv_l\, A$ and $\operatorname{Qinv}_r A \subset Qinv_r\, A$.

Let $\operatorname{Qinv} A = \operatorname{Qinv}_l A \cap \operatorname{Qinv}_r A$ and $Qinv\, A = Qinv_l A \cap Qinv_r\, A$. The following example shows that there exists a bornological algebra $(A, \mathcal{B})$ for which $\operatorname{Qinv} A \neq Qinv\, A$.

Example. Let $C_b(\mathbb{R})$ be the algebra of all $\mathbb{R}$-valued bounded continuos functions on $\mathbb{R}$ and $\mathcal{P} = \{p_n : n \in \mathbb{N}\}$ the countable family of seminorms on $C_b(\mathbb{R})$ defined by $p_n(f) = \sup_{|x| \leqslant n} |f(x)|$ for each $n \in \mathbb{N}$ and $f \in C_b(\mathbb{R})$, τ the topology on $C_b(\mathbb{R})$ defined by the family $\mathcal{P}$, and $\mathcal{B}_{\mathcal{N}}$ the von-Neumann bornology on $C_b(\mathbb{R})$ defined by the topology τ; that is, $\mathcal{B}_{\mathcal{N}}$ consists of bounded subsets of $(C_b(\mathbb{R}), \tau)$. Then $(C_b(\mathbb{R}), \mathcal{B}_{\mathcal{N}})$ is a commutative bornological algebra.

Let $f(x) = 1 - \dfrac{1}{\pi}\operatorname{arccot} x$ and $g(x) = 1 - \dfrac{\pi}{\operatorname{arccot} x}$ for each $x \in \mathbb{R}$. Then $f \in C_b(\mathbb{R})$ and $g \circ f = f \circ g = 0$. Since $g \notin C_b(\mathbb{R})$, then $f \notin \operatorname{Qinv} C_b(\mathbb{R})$. Let

$$g_n(x) = \begin{cases} 1 - \dfrac{\pi}{\operatorname{arccot} x}, & |x| \leqslant n \\ 1 - \dfrac{\pi}{\operatorname{arccot} n}, & x > n \\ 1 - \dfrac{\pi}{\operatorname{arccot}(-n)}, & x < -n \end{cases}$$

for each $n \in \mathbb{N}$. Then $(g \circ g_n)(x) = (g_n \circ g)(x) = 0$ for each $|x| \leqslant n$.

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[7] An element $a \in A$ is *left quasi-invertible* or *left advertible* (*right quasi-invertible* or *right advertible*) if there is an element $b \in A$ such that $b \circ a = \theta_A$ (respectively, $a \circ b = \theta_A$).

Let O be a neighbourhood of zero in $(C_b(\mathbb{R}), \tau)$. Now there are $\delta > 0$ and an index $N \in \mathbb{N}$ such that $\{f \in C_b(\mathbb{R}) : p_N(f) < \delta\} \subset O$. Let $\epsilon \in (0, \delta)$. Then $p_N(g \circ g_n) < \epsilon$ whenever $n > N$. Therefore, the sequence $(g \circ g_n)$ converges to $\theta_{C_b(\mathbb{R})}$ in the topology τ. Since $(C_b(\mathbb{R}), \tau)$ is a metrizable space, then (see [10, p. 27, Proposition 3]) the sequence $(g \circ g_n)$ bornologically converges to $\theta_{C_b(\mathbb{R})}$ in the bornology $\mathcal{B}_{\mathcal{N}}$. Hence, $f \in Qinv\,(C_b(\mathbb{R}), \mathcal{B}_{\mathcal{N}}) \setminus \mathrm{Qinv}\,(C_b(\mathbb{R}), \mathcal{B}_{\mathcal{N}})$.

Proposition 1. *Let $(A, \mathcal{B})$ be a bornological algebra and $a, b \in A$. If $ab \in Qinv_l\,A$ ($ab \in Qinv_r\,A$), then $ba \in Qinv_l\,A$ (respectively, $ba \in Qinv_r\,A$).*

Proof. Let $ab \in Qinv_l A$ ($ab \in Qinv_r A$). Then there exists a net $(c_\lambda)_{\lambda\in\Lambda}$ in A such that the net $(c_\lambda \circ ab)_{\lambda\in\Lambda}$ (respectively, $(ab \circ c_\lambda)_{\lambda\in\Lambda}$) bornologically converges to θ_A. It means that there is $B \in \mathcal{B}$ and, for every $\epsilon > 0$, an index $\lambda_0 \in \Lambda$ such that $c_\lambda \circ ab \in \epsilon B$ (respectively, $ab \circ c_\lambda \in \epsilon B$) whenever $\lambda > \lambda_0$. It is easy to see that $ba \circ (bc_\lambda a - ba) = b(ab \circ c_\lambda)a$ (respectively, $(bc_\lambda a - ba) \circ ba = b(c_\lambda \circ ab)a$). Therefore, $ba \circ (bc_\lambda a - ba) \in \epsilon(bBa)$ (respectively, $(bc_\lambda a - ba) \circ ba \in \epsilon(bBa)$) whenever $\lambda > \lambda_0$. Since $(A, \mathcal{B})$ is a bornological algebra, $bBa \in \mathcal{B}$. Hence, $ba \in Qinv_l A$ (respectively, $ba \in Qinv_r A$). □

2. We say that a bornological algebra $(A, \mathcal{B})$ is

a) *left (right) advertive bornological algebra* if $\mathrm{Qinv}_l A = Qinv_l A$ (respectively, $\mathrm{Qinv}_r A = Qinv_r A$) and an *advertive bornological algebra* if $\mathrm{Qinv}\,A = Qinv\,A$;

b) a *bornological Q_l-algebra* (*bornological Q_r-algebra*) if $\mathrm{Qinv}\,l_l A - a$ is bornivore[8] for each $a \in \mathrm{Qinv}_l A$ (respectively, $\mathrm{Qinv}\,l_r A - a$ is bornivore for each $a \in \mathrm{Qinv}_r A$).

Proposition 2. *Let $(A, \mathcal{B})$ be a bornological algebra. If $\mathrm{Qinv}_l A$ ($\mathrm{Qinv}_r A$) is a bornivore in $(A, \mathcal{B})$, then $(A, \mathcal{B})$ is a left (respectively, right) advertive bornological algebra.*

Proof. Let $(A, \mathcal{B})$ be a bornological algebra and $a \in Qinv_l A$. Then there exists a net $(a_\lambda)_{\lambda\in\Lambda}$ such that the net $(a_\lambda \circ a)_{\lambda\in\Lambda}$ bornologically converges to θ_A. It means that there are $B \in \mathcal{B}$ and for any $\epsilon > 0$ an index $\lambda_\epsilon \in \Lambda$ such that $a_\lambda \circ a \in \epsilon B$ whenever $\lambda > \lambda_\epsilon$. Since $\mathrm{Qinv}_l A$ is a bornivore, then there is a number $\mu_B > 0$ such that $B \subset \mu_B \mathrm{Qinv}_l A$. If $\epsilon < \mu_B^{-1}$, then

[8] A set O in a bornological algebra $(A, \mathcal{B})$ is a *bornivore* if for each $B \in \mathcal{B}$ there is a number $\mu_B > 0$ such that $B \subset \mu_B O$.

$a_\lambda \circ a \in \mathrm{Qinv}_l A$ whenever $\lambda > \lambda_\epsilon$. Hence, for any fixed $\lambda_0 > \lambda_\epsilon$ holds $a_{\lambda_0} \circ a \in \mathrm{Qinv}_l A$. Consequently, $a \in \mathrm{Qinv}_l$.

The proof for $\mathrm{Qinv}_r A$ is similar. □

Corollary 3. *Every bornological Q_l-algebra*[9] *is a left advertive bornological algebra and every bornological Q_r-algebra is a right advertive bornological algebra.*

4. Descriptions of bornological radicals

Let $(A, \mathcal{B})$ be a bornological algebra and I a bornologically closed regular left (right or two-sided) ideal of $(A, \mathcal{B})$. Then there exists a maximal regular left (right or two-sided) ideal J of A such that $I \subset J$. But J is not necessarily bornologically closed. Therefore, we will say that a bornological algebra $(A, \mathcal{B})$ is a *simplicial algebra* if every bornologically closed regular left (right or two-sided) ideal of $(A, \mathcal{B})$ is contained in some bornologically closed maximal regular left (right or two-sided) ideal of $(A, \mathcal{B})$. Hence, if $(A, \mathcal{B})$ is a simplicial bornological algebra, then $\mathrm{rad}_l\, A \neq A$ and $\mathrm{rad}_r\, A \neq A$.

It is easy to show that every maximal regular left ideal is bornologically closed in a bornological Q_l-algebra and every maximal regular right ideal is bornologically closed in a bornological Q_r-algebra. Therefore, every bornological Q-algebra is a simplicial bornological algebra.

Proposition 3. *Let $(A, \mathcal{B})$ be a simplicial bornological algebra. Then $\mathrm{rad}_l\, A \subset Qinv_l A$ and $\mathrm{rad}_r\, A \subset Qinv_r A$.*

Proof. Let $a_0 \in A \backslash Qinv_l A$ and I be the intersection of all bornologically closed sets containing $\{a - aa_0 : a \in A\}$. Then I is a bornologically closed regular left ideal of $(A, \mathcal{B})$ and a_0 is a unit of A modulo I. Since $(A, \mathcal{B})$ is a simplicial bornological algebra, then there is a bornologically closed maximal regular left ideal M such that $I \subset M$. Hence $a_0 \notin M$. Consequently, $a_0 \notin \mathrm{rad}_l A$ by Theorem 1. This means that $\mathrm{rad}_l A \subset Qinv_l A$.

The proof for $\mathrm{rad}_r\, A$ is similar. □

We say that an ideal I of $(A, \mathcal{B})$ is a *bornologically left (right) quasi-regular ideal* if $I \subset Qinv_l A$ (respectively, $I \subset Qinv_r A$).

[9] This result for unital bornological Q-algebra has been proved in [3, Proposition 4.1], and for metrizable case in [5, p. 75.]

Proposition 4. *Let $(A,\mathcal{B})$ be a simplicial bornological algebra. Then* $\operatorname{rad}_l A$ $(\operatorname{rad}_r A)$ *is a bornologically left (respectively, right) quasi-regular ideal that contains all other bornologically left (respectively, right) quasi-regular ideals of* $(A,\mathcal{B})$.

Proof. It is clear that by Proposition 3, $\operatorname{rad}_l A$ is a bornologically left quasi-regular ideal in $(A,\mathcal{B})$. Let now I be an arbitrary bornologically left quasi-regular ideal in $(A,\mathcal{B})$. If $I \setminus \operatorname{rad}_l A$ is nonempty, then there is an element $a_0 \in I \setminus \operatorname{rad}_l A$ and a bounded irreducible representation π of $(A,\mathcal{B})$ on some separated bornological linear space $(X,\mathcal{B}_X)$ such that $\pi(a_0) \neq \theta_{\mathcal{L}(X)}$ or $y_0 = \pi(a_0)(x_0) \neq \theta_X$ for some nonzero $x_0 \in X$. Since $Y = \{\pi(a)(y_0) : a \in A\}$ is an A-submoddule of ${}_\pi X$ and ${}_\pi X$ is irreducible, we have $Y =_\pi X$ and, consequently, $x_0 = \pi(a_1)(y_0)$ for an element $a_1 \in A$. Hence, from $\pi(a_1a_0)(x_0) = x_0$ it follows that $\pi(a \circ a_1a_0)(x_0) = x_0$ for each $a \in A$. If now $a_1a_0 \in \mathrm{Qinv}_l A$, then there is a net $(a_\lambda)_{\lambda\in\Lambda}$ in A such that the net $(a_\lambda \circ (a_1a_0))_{\lambda\in\Lambda}$ bornologically converges to θ_A. As π is bornologically continuous, $(\pi(a_\lambda \circ (a_1a_0)))_{\lambda\in\Lambda}$ bornologically converges to $\theta_{\mathcal{L}_\tau(X)}$. Therefore, there is a set $B \in \mathcal{B}_{\mathcal{L}(X)}$ and, for any $\epsilon > 0$, an index $\lambda_0 \in \Lambda$ such that $\pi(a_\lambda \circ (a_1a_0)) \in \epsilon(B \cap \pi(A))$ whenever $\lambda > \lambda_0$. Hence

$$\pi(a_\lambda \circ (a_1a_0))(x_0) \in \epsilon\Big(\bigcup_{L\in B\cap\pi(A)} L(\{x_0\})\Big)$$

whenever $\lambda > \lambda_0$. Since

$$\bigcup_{L\in B\cap\pi(A)} L(\{x_0\}) \in \mathcal{B}_X,$$

$((\pi(a_\lambda \circ (a_1a_0)))(x_0))_{\lambda\in\Lambda}$ bornologically converges to θ_X. As $(X,\mathcal{B}_X)$ is separated, $x_0 = \theta_X$. But this is impossible. Consequently, $I \subset \operatorname{rad}_l A$. The proof for $\operatorname{rad}_r A$ is similar. □

Theorem 3. *Let $(A,\mathcal{B})$ be a simplicial bornological algebra. Then*

$$\begin{aligned}\operatorname{rad}_l A &= \{a \in A : \lambda a + ba \in Qinv_l A \quad \text{for each} \quad \lambda \in \mathbb{K} \quad \text{and} \quad b \in A\}\\ &= \{a \in A : \lambda a + ab \in Qinv_l A \quad \text{for each} \quad \lambda \in \mathbb{K} \quad \text{and} \quad b \in A\}\end{aligned}$$

and

$$\begin{aligned}\operatorname{rad}_r A &= \{a \in A : \lambda a + ba \in Qinv_r A \quad \text{for each} \quad \lambda \in \mathbb{K} \quad \text{and} \quad b \in A\}\\ &= \{a \in A : \lambda a + ab \in Qinv_r A \quad \text{for each} \quad \lambda \in \mathbb{K} \quad \text{and} \quad b \in A\}.\end{aligned}$$

Proof. Let $a \in \mathrm{rad}_l\, A$. Since $\mathrm{rad}_l\, A$ is a two-sided ideal in A, $\lambda a + ba \in \mathrm{rad}_l\, A$ for all $\lambda \in \mathbb{K}$ and $b \in A$. Hence

$$\mathrm{rad}_l\, A \subset \{a \in A : \lambda a + ba \in Qinv_l A \quad \text{for each} \quad \lambda \in \mathbb{K} \quad \text{and} \quad b \in A\},$$

by Proposition 3. To show the converse inclusions, let $a \in A \backslash \mathrm{rad}_l\, A$. Then there exists a bounded irreducible representation π of $(A, \mathcal{B})$ on some separated bornological linear space $(X, \mathcal{B}_X)$ such that $a \notin \ker\pi$. Therefore, there exists an element $x_0 \in X \backslash \{\theta_X\}$ such that $y_0 = \pi(a)(x_0) \neq \{\theta_X\}$. It is easy to see that $Y = A_{\pi} \cdot y_0$ is an A-submodule of ${}_{\pi}X$. Since ${}_{\pi}X$ is irreducible, $Y = \{\theta_X\}$ or $Y = X$. To show that the first case is impossible, let $Z = \{x \in X : A \cdot_{\pi} x = \theta_X\}$. It is easy to see that Z is also an A-submodule of ${}_{\pi}X$. If $Y = \{\theta_X\}$, then $y_0 \in Z$. Hence $Z = X$ or $A \cdot_{\pi} X = \{\theta_X\}$. But this is impossible since ${}_{\pi}X$ is nontrivial. Consequently, $Y = X$. It means that

$$x_0 = \pi(a_0)(y_0) = \pi(a_0)[\pi(a)(x_0)] = (\pi(a_0) \circ \pi(a))(x_0) = \pi(a_0 a)(x_0)$$

for some $a_0 \in A$.

Suppose now that $a_0 a \in Qinv_l A$. Then there is a net $(a_\lambda)_{\lambda \in \Lambda}$ in A such that $(a_\lambda \circ (a_0 a))_{\lambda \in \Lambda}$ bornologically converges to θ_A. Now, like in the proof of Proposition 4, it is easy to show that $(\pi(a_\lambda \circ a_0 a)(x_0))_{\lambda \in \Lambda}$ bornologically converges to $\theta_{\mathcal{L}(X)}$. As

$$\begin{aligned}\pi(a_\lambda \circ (a_0 a))(x_0) &= \pi(a_\lambda)(x_0) + \pi(a_0 a)(x_0) - \pi(a_\lambda)[\pi(a_0 a)(x_0)] = \\ &= \pi(a_\lambda)(x_0) + x_0 - \pi(a_\lambda)(x_0) = x_0\end{aligned}$$

for each $\lambda \in \Lambda$, $x_0 = \theta_X$, which is impossible. Therefore, $0a + a_0 a \notin Qinv_l A$ and, by Proposition 1, also $0a + a a_0 \notin Qinv_l A$. Consequently,

$$\{a \in A : \lambda a + ba \in Qinv_l A \quad \text{for each} \quad \lambda \in \mathbb{K} \quad \text{and} \quad b \in A\} \subset \mathrm{rad}_l\, A$$

and

$$\{a \in A : \lambda a + ba \in Qinv_l A \quad \text{for each} \quad \lambda \in \mathbb{K} \quad \text{and} \quad b \in A\} \subset \mathrm{rad}_l\, A.$$

The proof for $\mathrm{rad}_r\, A$ is similar. □

Proposition 5. *Let $(A, \mathcal{B})$ be an advertive and simplicial bornological algebra. Then* $\mathrm{Rad}\, A = \mathrm{rad}_l\, A = \mathrm{rad}_r\, A$.

Proof. Let $a \in \mathrm{rad}_l\, A$. Then $a \in Qinv_l A$, by Proposition 3. Since $(A, \mathcal{B})$ is an advertive bornological algebra, $\mathrm{Qinv}_l A = Qinv_l A$. Therefore, there is an element $b \in A$ such that $b \circ a = \theta_A$. As $b = ba - a \in \mathrm{rad}_l\, A$, by

Corollary 1, also $b \in \mathrm{Qinv}_l A$. Hence, there is an element $c \in A$ such that $c \circ b = \theta_A$. Since

$$a = \theta_A \circ a = (c \circ b) \circ a = c \circ (b \circ a) = c \circ \theta_A = c,$$

$a \in \mathrm{Qinv}\, A \subset Qinv_r A$. Consequently, $\mathrm{rad}_l\, A \subset \mathrm{rad}_r\, A$ by Proposition 3. Similarily, $\mathrm{rad}_r\, A \subset \mathrm{rad}_l\, A$. Hence, $\mathrm{rad}_l\, A = \mathrm{rad}_r\, A$. It is known (see [14, p. 55]) that

$$\mathrm{Rad}\, A = \{a \in A : \lambda a + ba \in \mathrm{Qinv}\, A \quad \text{for each} \quad \lambda \in \mathbb{K} \quad \text{and} \quad b \in A\}.$$

Since $(A, \mathcal{B})$ is an advertive bornological algebra, $\mathrm{Qinv} A = Qinv\, A$. Therefore, $\mathrm{Rad}\, A = \mathrm{rad}_l\, A = \mathrm{rad}_r\, A$ by Theorem 1.

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