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ON INSTABILITY OF AXIALLY SYMMETRIC EQUILIBRIUM FIGURES OF ROTATING VISCOUS INCOMPRESSIBLE LIQUID

ABSTRACT. We prove that axially symmetric equilibrium figures of uniformly rotating viscous incompressible liquid are unstable when the second variation of the energy functional can take negative values.

**Dedicated to the memory of
Olga Aleksandrovna Ladyzhenskaya**

1. INTRODUCTION

Equilibrium figure $\mathcal{F}$ of an incompressible liquid subjected to the capillary and self-gravitation forces and rotating as a rigid body with the angular velocity ω about the x_3-axis is defined by the equation

$$\sigma\mathcal{H}(x) + \frac{\omega^2}{2}|x'|^2 + \kappa\mathcal{U}(x) + p_0 = 0, \qquad x \in \mathcal{G} = \partial\mathcal{F}, \qquad (1.1)$$

where $\sigma = \text{const} > 0$ is the coefficient of the surface tension, $\mathcal{H}(x)$ is twice the mean curvature of the surface $\mathcal{G}$ at the point x negative for convex domains, $p_0 = \text{const}$, $\mathcal{U}(x) = \int_{\mathcal{F}} |x-y|^{-1}dy$ is the Newtonian potential, $x' = (x_1, x_2, 0)$ and κ is the gravitational constant. The case of the absence of self -gravitation ($\kappa = 0$) is not excluded. The density of the liquid equals one. The velocity vector field and the pressure of the rotating liquid are given by

$$\mathcal{V}(x) = \omega(e_3 \times x), \qquad \mathcal{P}(x) = \frac{\omega^2}{2}|x'|^2 + p_0, \qquad (1.2)$$

where $e_3 = (0,0,1)$ is a unit vector in the direction of the x_3-axis. We assume that the equilibrium figure $\mathcal{F}$ is an axially symmetric bounded domain with a smooth boundary $\mathcal{G}$ and with the barycenter located at the origin which means that

$$\int_{\mathcal{F}} x_i dx = 0, \qquad i = 1, 2, 3. \qquad (1.3)$$

The angular momentum of the rotating liquid, $\int_{\mathcal{F}} x\times\mathcal{V}(x)dx$, is parallel to the axis of rotation, i.e.

$$\int\limits_{\mathcal{F}} x \times \mathcal{V}(x)dx = \beta e_3,$$

$$\int\limits_{\mathcal{F}} x_j x_3 dx = 0, \quad j = 1,2, \qquad \beta = \omega \int\limits_{\mathcal{F}} |x'|^2 dx. \tag{1.4}$$

In the present paper, we continue the analysis of stability of equilibrium figures carried out in [1–6]. We consider evolution free boundary problem for the perturbations of velocity, pressure and of the figure $\mathcal{F}$. This problem written in the coordinate system rotating with the angular velocity ω about the x_3 -axis consists in the determination of a bounded domain $\Omega_t \in R^3$, $t > 0$, of a vector field $v(x,t) = (v_1, v_2, v_3)$ and of a function $p(x,t)$, $x \in \Omega_t$, satisfying the relations

$$v_t + (v\cdot\nabla)v + 2\omega(e_3 \times v) - \nu\nabla^2 v + \nabla p = 0,$$

$$\nabla\cdot v = 0, \qquad x\in\Omega_t, \qquad t>0, \tag{1.5}$$

$$T(v,p)n = (\sigma H(x) + \frac{\omega^2}{2}|x'|^2 + p_0 + \kappa U(x,t))n, \qquad V_n = v\cdot n, \qquad x\in\Gamma_t$$

$$v(x,0) = v_0(x), \qquad x\in\Omega_0.$$

Here ν is a constant positive viscosity coefficient, n is the exterior normal to the free surface $\Gamma_t = \partial\Omega_t$, V_n is the velocity of evolution of Γ_t in the normal direction, H is twice the mean curvature of Γ_t,

$$U(x,t) = \int\limits_{\Omega_t} \frac{dy}{|x-y|}$$

is the Newtonian potential computed for the unknown domain Ω_t and, finally,

$$T(v,p) = -pI + \nu S(v)$$

and

$$S(v) = \nabla v + (\nabla v)^T = \Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\Big)_{i,j=1,2,3}$$

are the stress and the doubled rate-of-strain tensors, respectively. The domain Ω_0 is given.

Concerning the initial data it should be assumed that v_0 is a small divergence free vector field satisfying the compatibility conditions

$$S(v_0)n - n(n \cdot S(v_0)n)\Big|_{\Gamma_0} = 0, \tag{1.6}$$

and Ω_0 is close to $\mathcal{F}$ which means that the surface Γ_0 can be given by the equation

$$x = y + N(y)\rho_0(y), \qquad y \in \mathcal{G} \tag{1.7}$$

with a certain small function $\rho_0(y)$; moreover, the total and angular momenta corresponding to $v_0 + \mathcal{V}$ should be the same as for $\mathcal{V}$, i.e.

$$\int_{\Omega_0} v_0(x)dx = 0, \qquad \int_{\Omega_0} x \times (v_0(x) + \mathcal{V}(x))dx = \beta e_3.$$

It can be verified that this implies

$$\int_{\Omega_t} v(x,t)dx = 0, \qquad \int_{\Omega_t} x \times (v(x,t) + \mathcal{V}(x))dx = \beta e_3, \qquad \forall t > 0,$$

i.e.

$$\int_{\Omega_t} v(x,t)dx = 0,$$

$$\int_{\Omega_t} v(x,t)\cdot\eta_i(x)dx + \omega\int_{\Omega_t} \eta_3(x)\cdot\eta_i(x)dx = \omega\int_{\mathcal{F}} \eta_3(x)\eta_i(x)dx, \qquad i = 1,2,3, \tag{1.8}$$

where $\eta_i(x) = e_i \times x$ is a vector of rigid rotation about the x_i-axis. Finally, for arbitrary $t \geq 0$ there holds

$$|\Omega_t| = |\mathcal{F}|, \qquad \int_{\Omega_t} x_i dx = 0, \qquad i = 1,2,3. \tag{1.9}$$

Problem (1.5) has a stationary solution $v(x,t) = 0,\ p(x,t) = 0,\ x \in \mathcal{F}$. It is stable, if the second variation of the energy functional,

$$\delta_0^2 R[\rho] = \int_{\mathcal{G}} (\sigma|\nabla_{\mathcal{G}}\rho(y)|^2 - b(y)\rho^2(y))dS$$

$$+\frac{\omega^2}{\int_{\mathcal{F}}|x'|^2dx}\Big(\int_{\mathcal{G}} \rho(y)|y'|^2 dS\Big)^2 - \kappa\int_{\mathcal{G}}\int_{\mathcal{G}} \rho(y)\rho(z)\frac{dS_y\,dS_z}{|y-z|},$$

where

$$b(y) = \sigma(\mathcal{H}^2(y) - 2\mathcal{K}(y)) + \frac{\omega^2}{2}\frac{\partial}{\partial N}|y'|^2 + \kappa\frac{\partial \mathcal{U}(y)}{\partial N},$$

and $\mathcal{K}$ is the Gaussian curvature of $\mathcal{G}$, is a positive definite quadratic form for every $\rho(y)$ satisfying the constraints

$$\int\limits_{\mathcal{G}} \rho(y)dS = 0, \qquad \int\limits_{\mathcal{G}} \rho(y)y_i dS = 0, \qquad i = 1,2,3. \tag{1.10}$$

This criterium of stability generally accepted in the theory of equilibrium figures (see, for instance, [7, 8]) was justified in [5] where it was shown that in the case $\delta_0^2 R[\rho] > 0$ problem (1.5) with initial data described above has a solution defined for all $t \geq 0$ and that $v(x,t) \to 0$, $p(x,t) \to 0$ as $t \to \infty$ exponentially. In the present article the instability of this solution is proved in the case when $\delta_0^2 R$ can take negative values for some ρ satisfying (1.10). The precise formulation of this result will be given in Sec. 3. Assuming that also Γ_t can be prescribed by the equation of the type (1.7), namely,

$$x = y + N(y)\rho(y,t), \qquad y \in \mathcal{G},$$

we show that problem (1.5) can be written as a nonlinear problem in a fixed domain $\mathcal{F}$:

$$\begin{gathered}
w_t + 2\omega(e_3 \times w) - \nu\nabla^2 w + \nabla s = f(w,s,\rho), \\
\nabla \cdot w = 0, \qquad x \in \mathcal{F}, \\
T(w,s)N(x) + NB_0\rho = \nu b(w,\rho) + Nd(w,\rho), \\
\rho_t(x,t) = w(x,t)\cdot N(x) + g(w,\rho), \\
\rho(x,0) = \rho_0(x), \qquad x \in \mathcal{G}, \\
w(x,0) = w_0(x), \qquad x \in \mathcal{F},
\end{gathered} \tag{1.11}$$

where unknown are $w(x,t)$, $s(x,t)$, $x \in \mathcal{F}$, $\rho(x,t)$, $x \in \mathcal{G}$, B_0 is an elliptic integro-differential operator on $\mathcal{G}$ of the form

$$B_0\rho = -\sigma\Delta_{\mathcal{G}}\rho - b(x)\rho(x) - \kappa\int\limits_{\mathcal{F}} \frac{\rho(z)dz}{|x-z|},$$

$\Delta_{\mathcal{G}}$ is the Laplace–Beltrami operator on $\mathcal{G}$ and f, b, d, g are nonlinear functions of w, s, ρ and of their derivatives computed below in Sec. 2. To the stationary solution $v(x,t)=0, p(x,t)=0,\ x\in\mathcal{F}$ of (1.5) corresponds the zero solution $w=0,\ s=0,\ \rho=0$ of (1.11).

In Sec. 2 existence theorems are obtained for problem (1.11) and for the corresponding linear problem (when f,b,d,g are given functions of x,t). To simplify estimates of nonlinear terms, we work in the Hölder spaces $C^l(\mathcal{F})$ and $C^l(\mathcal{G})$. Finally, in Sec. 3 the instability of the zero solution of (1.11) is proved.

2. Some existence theorems and auxiliary proposition

Let us consider a non-homogeneous linear problem

$$v_t+2\omega(e_3\times v)-\nu\nabla^2 v+\nabla p=f(x,t),\quad \nabla\cdot v=0,\qquad x\in\mathcal{F},\quad (2.1)$$

$$\Pi_0 S(v)N=b(x,t), \qquad (2.2)$$

$$N\cdot T(v,p)N+B_0\rho=d(x,t), \qquad (2.3)$$

$$\rho_t-v\cdot N=g(x,t),\quad \rho(x,0)=\rho_0(x),\qquad x\in\mathcal{G}, \qquad (2.4)$$

$$v(x,0)=v_0(x),\qquad x\in\mathcal{F}, \qquad (2.5)$$

where $\Pi_0 h=h-N(N\cdot h)$ is a tangential component of the vector field $h(x),\ x\in\mathcal{G}$. Boundary conditions (2.2), (2.3) are tangential and normal projections of the equation $T(v,p)N+NB_0\rho=\nu b+Nd$. The substitution

$$\rho(x,t)=\rho_0(x)+\int_0^t v(x,\tau)\cdot N(x)d\tau+\int_0^t g(x,\tau)d\tau \qquad (2.6)$$

excludes ρ and reduces (2.1)–(2.5) to the following problem for v and p:

$$v_t+2\omega(e_3\times v)-\nu\nabla^2 v+\nabla p=f(x,t),\quad \nabla\cdot v=0,\qquad x\in\mathcal{F},$$

$$\Pi_0 S(v)N=b(x,t), \qquad (2.7)$$

$$N\cdot T(v,p)N+\int_0^t B_0(v\cdot N)d\tau=d(x,t)-B_0\rho_0-\int_0^t B_0 g\,d\tau,\quad x\in\mathcal{G},$$

$$v(x,0)=v_0(x)\qquad x\in\mathcal{F}.$$

It differs from a similar problem

$$v_t-\nu\nabla^2 v+\nabla p=f(x,t),\quad \nabla\cdot v=0,\qquad x\in\mathcal{F},$$

$$\Pi_0 S(v)N = b(x,t), \tag{2.8}$$

$$N\cdot T(v,p)N - \sigma N\cdot \Delta_{\mathcal{G}}\int_0^t v\,d\tau = d(x,t) + \int_0^t D(x,\tau)d\tau, \quad x\in\mathcal{G},$$

$$v(x,0) = v_0(x), \qquad x\in\mathcal{F}$$

studied in [9, 10] only by some lower order terms in the equation (2.1) and in the boundary condition. It was proved in [9.10] that problem (2.8) is uniquely solvable and that the solution satisfies the inequality

$$\begin{aligned}
&\sup_{\tau\le t}|v_\tau(\cdot,\tau)|_{C^\alpha(\mathcal{F})} + \sup_{\tau\le t}|v(\cdot,\tau)|_{C^{2+\alpha}(\mathcal{F})} + \sup_{\tau\le t}|p(\cdot,\tau)|_{C^{1+\alpha}(\mathcal{F})}\\
&\le c(t)\Big(\sup_{\tau\le t}|f(\cdot,\tau)|_{C^\alpha(\mathcal{F})} + |b|_{C^{1+\alpha,(1+\alpha)/2}(G_t)}\\
&+\sup_{\tau\le t}|d(\cdot,\tau)|_{C^{1+\alpha}(\mathcal{G})} + \sup_{\tau\le t}|D(\cdot,\tau)|_{C^\alpha(\mathcal{G})} + |v_0|_{C^{2+\alpha}(\mathcal{F})}\Big).
\end{aligned} \tag{2.9}$$

The following theorem is a consequence of this result.

Theorem 2.1. *Let the data of problem (2.1)–(2.5) satisfy the following hypotheses:* $f(\cdot,t)\in C^\alpha(\mathcal{F})$, $\alpha\in(0,1)$, $\forall t\in[0,T]$, $b\in C^{1+\alpha,(1+\alpha)/2}(G_T)$, $G_T = \mathcal{G}\times[0,T]$, $d(\cdot,t)\in C^{1+\alpha}(\mathcal{G})$, $g(\cdot,t)\in C^{2+\alpha}(\mathcal{G})$, $\forall t\in[0,T]$, $v_0\in C^{2+\alpha}(\mathcal{F})$, $\rho_0\in C^{3+\alpha}(\mathcal{G})$, *and the compatibility conditions* $\nabla\cdot v_0(x) = 0$,

$$b(x,t)\cdot N(x) = 0, \qquad \Pi_0 S(v_0)N(x) = b(x,0), \qquad x\in\mathcal{G}$$

hold. Then problem (2.1)–(2.5) has a unique solution $v(\cdot,t)\in C^{2+\alpha}(\mathcal{F})$ *with* $v_t(\cdot,t)\in C^\alpha(\mathcal{F})$, $p(\cdot,t)\in C^{1+\alpha}(\mathcal{F})$, $\rho(\cdot,t)\in C^{3+\alpha}(\mathcal{G})$, $\forall t\in[0,T]$, *and the inequality*

$$Y_t(v,p,\rho)\le c(t)M_t \tag{2.10}$$

is satisfied for arbitrary $t\in[0,T]$, $c(t)$ *being a non-decreasing function of* t,

$$\begin{aligned}
&Y_t(v,p,\rho)\\
&=\sup_{\tau\le t}|v_\tau(\cdot,\tau)|_{C^\alpha(\mathcal{F})} + \sup_{\tau\le t}|v(\cdot,\tau)|_{C^{2+\alpha}(\mathcal{F})}\\
&+\sup_{\tau\le t}|p(\cdot,\tau)|_{C^{1+\alpha}(\mathcal{F})} + \sup_{\tau\le t}|\rho(\cdot,\tau)|_{C^{3+\alpha}(\mathcal{G})}
\end{aligned} \tag{2.11}$$

and

$$M_t = \sup_{\tau\le t} |f(\cdot,\tau)|_{C^\alpha(\mathcal{F})} + |b|_{C^{1+\alpha,(1+\alpha)/2}(G_t)}$$

$$+\sup_{\tau\le t} |d(\cdot,\tau)|_{C^{1+\alpha}(\mathcal{G})} + \sup_{\tau\le t} |g(\cdot,\tau)|_{C^{2+\alpha}(\mathcal{G})} + |v_0|_{C^{2+\alpha}(\mathcal{F})} + |\rho_0|_{C^{3+\alpha}(\mathcal{G})}.$$

Proof. We consider problem (2.7) that can be written in the form

$$v_t - \nu\nabla^2 v + \nabla p = f(x,t) - 2\omega(e_3\times v), \quad \nabla\cdot v = 0, \quad x\in\mathcal{F},$$

$$\Pi_0 S(v)N = b(x,t), \tag{2.12}$$

$$N\cdot T(v,p)N - \sigma N\cdot\Delta_{\mathcal{G}}\int_0^t v d\tau = d(x,t) - B_0\rho_0(x) - \int_0^t B_0 g(x,\tau)d\tau$$

$$-\int_0^t (B_0(v\cdot N) + \sigma N\cdot\Delta_{\mathcal{G}} v)d\tau, \quad x\in\mathcal{G},$$

$$v(x,0) = v_0(x). \qquad x\in\mathcal{F}$$

Making use of the inequalities

$$|B_0(v\cdot N) + \sigma N\cdot\Delta v|_{C^\alpha(\mathcal{G})} \le c|v|_{C^{1+\alpha}(\mathcal{G})}$$

$$\le \epsilon|v|_{C^{2+\alpha}(\mathcal{F})} + c(\epsilon)\sup_{x\in\mathcal{F}}|v(x,t)|,$$

$$|v(\cdot,t)|_{C^\alpha(\mathcal{F})} \le |v(\cdot,0)|_{C^\alpha(\mathcal{F})} + \int_0^t |v_\tau(\cdot,\tau)|_{C^\alpha(\mathcal{F})}d\tau$$

with a small $\epsilon > 0$ and of estimate (2.9) we obtain

$$\sup_{\tau\le t}|v_t|_{C^\alpha(\mathcal{F})} + \sup_{\tau\le t}|v|_{C^{2+\alpha}(\mathcal{F})} + \sup_{\tau\le t}|p|_{C^{1+\alpha}(\mathcal{F})}$$

$$\le c\Big(\sup_{\tau\le t}|f - 2\omega(e_3\times v)|_{C^\alpha(\mathcal{F})} + |b|_{C^{1+\alpha,(1+\alpha)/2}(G_t)}$$

$$+\sup_{\tau\le t}|d - B_0\rho_0|_{C^{1+\alpha}(\mathcal{G})} + \sup_{\tau\le t}|B_0(v\cdot N) + \sigma N\cdot\Delta v|_{C^\alpha(\mathcal{G})} + |B_0\rho_0|_{C^{1+\alpha}(\mathcal{G})}$$

$$\le c(t)M_t + c\int_0^t (|v_\tau|_{C^\alpha(\mathcal{F})} + |v|_{C^{2+\alpha}(\mathcal{G})})d\tau.$$

From this inequality one easily arrives at

$$\sup_{\tau\le t}|v_t|_{C^\alpha(\mathcal{F})}+\sup_{\tau\le t}|v|_{C^{2+\alpha}(\mathcal{F})}+\sup_{\tau\le t}|p|_{C^{1+\alpha}(\mathcal{F})}\le cM_t$$

with the help of Granwall's lemma. The solvability of problem (2.12) can be easily proved, for instance, by successive approximations. Finally, the estimate of the function $\rho(x,t)$ follows from (2.6) and from the Schauder estimates for an elliptic equation

$$B_0\rho=-N\cdot T(v,p)N+d$$

whose second term is already evaluated. The theorem is proved.

We notice that in the case $f=0,\ b=0,\ d=0,\ g=0$ (2.10) implies

$$\begin{aligned}&|v(\cdot,t)|_{C^{2+\alpha}(\mathcal{F})}+|\rho(\cdot,t)|_{C^{3+\alpha}(\mathcal{G})}\\ &\le c\Big(|v_0|_{C^{2+\alpha}(\mathcal{F})}+|\rho_0|_{C^{3+\alpha}(\mathcal{G})}\Big),\qquad \forall t\le T.\end{aligned}\tag{2.13}$$

Let us pass to nonlinear problem and show that (1.5) can be written in the form (1.11). We extend $N(x)$ and $\rho(x,t)$ from $\mathcal{G}$ into $\mathcal{F}$ in such a way that N remains smooth (for our purposes it is sufficient that $N\in C^{3+\alpha}(\mathcal{F})$) and ρ satisfies the condition $\frac{\partial}{\partial N}\rho(x,t)|_{\mathcal{G}}=0$ and the inequalities

$$|\rho(\cdot,t)|_{C^{3+\alpha}(\mathcal{F})}\le c|\rho(\cdot,t)|_{C^{3+\alpha}(\mathcal{G})},$$

$$|\rho_t(\cdot,t)|_{C^{2+\alpha}(\mathcal{F})}\le c|\rho_t(\cdot,t)|_{C^{2+\alpha}(\mathcal{G})},$$

$$|\rho(\cdot,t)|_{C^1(\mathcal{F})}\le\delta\ll 1.$$

Now, we map $\mathcal{F}$ onto Ω_t by the transformation

$$x=y+N(y)\rho(y,t)\equiv e_\rho(y),\qquad y\in\mathcal{F}\tag{2.14}$$

that is invertible if δ is small enough, and we pass in (1.5) to the variables $y\in\mathcal{F}$. We introduce the following notations: $\mathcal{L}=\frac{\partial e_\rho}{\partial y}$ is the Jacobi matrix of this transformation with the elements

$$l_{ij}=\delta_{ij}+\frac{\partial}{\partial y_j}N_i(y)\rho(y,t),\tag{2.15}$$

$L_\rho=\det\mathcal{L}$, l^{ij} are elements of the inverse matrix $\mathcal{L}^{-1}$, $\widehat{L}_{ij}=L_\rho l^{ij}$ are elements of the adjugate matrix $\widehat{\mathcal{L}}$. The change of variables (2.14) transforms ∇_x into $\widetilde{\nabla} = \mathcal{L}^{-T}\nabla_y$ (the superscript T means transposition, $\mathcal{L}^{-T} = (\mathcal{L}^{-1})^T$). Since

$$0 = \nabla_x \cdot v(x,t) = \sum_{j,k=1}^{3} l^{jk} \frac{\partial v_k(e_\rho(y),t)}{\partial y_j} = \frac{1}{L_\rho} \sum_{j,k=1}^{3} \frac{\partial}{\partial y_j} \widehat{L}_{jk} v_k(e_\rho(y),t)$$

it is natural to introduce a new (divergence free) vector field

$$w(y,t) = \widehat{\mathcal{L}} v(e_\rho(y),t).$$

Making use of the relations

$$\begin{aligned}\frac{\partial v(e_\rho(y),t)}{\partial t} &= \frac{\partial v(x,t)}{\partial t} + \sum_{k=1}^{3} \frac{\partial v(x,t)}{\partial x_k} N_k \frac{\partial \rho(y,t)}{\partial t} \\ &= \frac{\partial v(x,t)}{\partial t} + \frac{\partial \rho}{\partial t}(\mathcal{L}^{-1}N \cdot \nabla_y) v(e_\rho(y),t),\end{aligned}$$

$v(e_\rho(y),t) = L_\rho^{-1}\mathcal{L}w$ and $(v \cdot \nabla_x)v = L_\rho^{-1}(w\cdot\nabla)(L_\rho^{-1}\mathcal{L}w)$, we obtain for w and $s(y,t) = p(e_\rho(y),t)$ the system of equations

$$\begin{aligned}&\frac{\partial}{\partial t}(L_\rho^{-1}\mathcal{L}w) - \frac{\partial \rho}{\partial t}(\mathcal{L}^{-1}N\cdot\nabla)(L_\rho^{-1}\mathcal{L}w) + L_\rho^{-1}(w\cdot\nabla)(L_\rho^{-1}\mathcal{L}w) \\ &+2\omega(e_3 \times L_\rho^{-1}\mathcal{L}w) - \nu\widetilde{\nabla}\cdot\widetilde{\nabla}(L_\rho^{-1}\mathcal{L}w) + \widetilde{\nabla}s = 0, \quad \nabla\cdot w = 0\end{aligned}$$

that can be also written in the form

$$w_t + 2\omega(e_3 \times w) - \nu\nabla^2 w + \nabla s = f(w,s,\rho), \qquad \nabla\cdot w = 0$$

with

$$\begin{aligned}f =& (w_t - L_\rho^{-1}\mathcal{L}w_t) - (L_\rho^{-1}\mathcal{L})_t w + \rho_t(\mathcal{L}^{-1}N\cdot\nabla)(L_\rho^{-1}\mathcal{L}w) \\ &-L_\rho^{-1}(w\cdot\nabla)(L_\rho^{-1}\mathcal{L}w) \\ &+2\omega(e_3\times(w - L_\rho^{-1}\mathcal{L}w)) + \nu\Big(\widetilde{\nabla}\cdot\widetilde{\nabla}(L_\rho^{-1}\mathcal{L}w) - \nabla^2 w\Big) + (\nabla - \widetilde{\nabla})s. \quad (2.16)\end{aligned}$$

Now, we turn to dynamic boundary condition

$$T(v,p)n = (\sigma H + \frac{\omega^2}{2}|x'|^2 + \kappa U + p_0)n$$

that can be written in an equivalent form (in the case $n(e_\rho(y))\cdot N(y) > 0$, i.e. for small ρ) as follows:

$$\Pi_0 \Pi S(v(x,t))n = 0,$$

$$-p(x,t)+\nu n\cdot S(v)n$$

$$=\Big(\sigma(H(x)-\mathcal{H}(y))+\frac{\omega^2}{2}(|x'|^2-|y|^2)+\kappa(U(x,t)-\mathcal{U}(y))\Big).$$

Here $y\in\mathcal{G}$, $x=y+N(y)\rho(y,t)\in\Gamma_t$, $\Pi h=h-n(n\cdot h)$. We make the change of variables (2.14) and note that this mapping transforms $S(v(x,t))=\nabla v+(\nabla v)^T$ into

$$\widetilde{S}(v(e_\rho(y),t)=\widetilde{\nabla}v(e_\rho(y),t)+(\widetilde{\nabla}v(e_\rho(y),t))^T. \qquad (2.17)$$

Further. we write the differences $H-\mathcal{H}$ and $U-\mathcal{U}$ as sums of their first variations (with respect to ρ) and of some remainders. We use the formulas

$$H(x)-\mathcal{H}(y)=n\cdot\Delta_\Gamma(y+N\rho)-N\cdot\Delta_\mathcal{G}y$$

$$=N\cdot\Delta_\mathcal{G}(N\rho)+N\cdot\delta_0\Delta_\Gamma y+N\cdot(\Delta_\Gamma-\Delta_\mathcal{G}-\delta_0\Delta_\Gamma)y$$

$$+(n-N)\cdot\Delta_\Gamma(N\rho)+N\cdot(\Delta_\Gamma-\Delta_\mathcal{G})(N\rho)+(n-N)\cdot\Delta_\Gamma y,$$

$$U(x,t)-\mathcal{U}(y)=\int\limits_\mathcal{G}\frac{L_\rho(z)dz}{|e_\rho(y)-e_\rho(z)|}-\int\limits_\mathcal{G}\frac{dz}{|y-z|}$$

$$=\int\limits_0^1 d\mu\int\limits_\mathcal{G}\frac{d}{d\mu}\frac{L_{\mu\rho}(z)}{|e_{\mu\rho}(y)-e_{\mu\rho}(z)|}dz$$

$$=\delta_0U(x,t)+\int\limits_0^1(1-\mu)d\mu\int\limits_\mathcal{G}\frac{d^2}{d\mu^2}\frac{L_{\mu\rho}(z)}{|e_{\mu\rho}(y)-e_{\mu\rho}(z)|}dz$$

where $\Delta_\mathcal{G}$ and Δ_Γ are the Laplace–Beltrami operators on $\mathcal{G}$ and Γ_t at the corresponding points y and $x=y+N\rho$, respectively. It can be verified that

$$N\cdot\Delta_\mathcal{G}(N\rho)+N\cdot\delta_0\Delta_\Gamma y=\delta_0(H-\mathcal{H})=\Delta_\mathcal{G}\rho+(\mathcal{H}^2(y)-2\mathcal{K})\rho$$

and

$$\delta_0U(x,t)=\int\limits_\mathcal{G}\frac{d}{d\mu}\frac{L_{\mu\rho}(z)dz}{|e_{\mu\rho}(y,t)-e_{\mu\rho}(z,t)|}\Big|_{\mu=0}=\rho\frac{\partial\mathcal{U}(y)}{\partial N}+\int\limits_\mathcal{G}\frac{\rho(z,t)dS}{|y-z|}$$

(see [4]), hence, the dynamic boundary condition can be written in the form

$$\Pi_0S(w)N=b(w,\rho),$$

$$-s+\nu N\cdot S(w)N+B_0\rho=d(w,\rho),$$

where

$$b(w,\rho)=\Pi_0(\Pi_0 S(w)N-\Pi\widetilde{S}(L_\rho^{-1}\mathcal{L}w)n), \tag{2.18}$$

$$d(w,\rho)=\nu d_1(w,\rho)+\sigma d_2(w,\rho)+\kappa d_3(w,\rho), \tag{2.19}$$

$$d_1(w,\rho)=N\cdot S(w)N-n\cdot\widetilde{S}(L_\rho^{-1}\mathcal{L}w)n,$$

$$d_2(w,\rho)=N\cdot(\Delta_\Gamma-\Delta_{\mathcal{G}}-\delta_0\Delta_\Gamma)y+(n-N)\cdot\Delta_\Gamma(N\rho)$$

$$+N\cdot(\Delta_\Gamma-\Delta_{\mathcal{G}})(N\rho)+(n-N)\cdot(\Delta_\Gamma-\Delta_{\mathcal{G}})y+(n\cdot N-1)\mathcal{H}+\frac{\omega^2}{2\sigma}(N_1^2+N_2^2)\rho^2,$$

$$d_3(w,\rho)=\int\limits_0^1(1-\mu)d\mu\int\limits_{\mathcal{F}}\frac{d^2}{d\mu^2}\frac{L_{\mu\rho}(z)}{|e_{\mu\rho}(y,t)-e_{\mu\rho}(z,t)|}dz.$$

Next, we consider the kinematic boundary condition $V_n=v\cdot n$ the equivalent form of which is

$$\rho_t(y,t)=\frac{v(e_\rho(y),t)\cdot n(e_\rho(y))}{n(e_\rho(y))\cdot N(y)}=\frac{v(e_\rho(y),t)\cdot\widehat{\mathcal{L}}^T N(y)}{N(y)\cdot\widehat{\mathcal{L}}^T N(y)}=\frac{w(y,t)\cdot N(y)}{N(y)\cdot\widehat{\mathcal{L}}^T N(y)},$$

because

$$n(e_\rho)=\frac{\widehat{\mathcal{L}}^T N(y)}{|\widehat{\mathcal{L}}^T N(y)|}. \tag{2.20}$$

We have

$$\Lambda(y;\rho)\equiv N(y)\cdot\widehat{\mathcal{L}}^T N(y)=1-\rho\mathcal{H}(y)+\rho^2\mathcal{K}(y)$$

(see calculation in [3]), hence, this boundary condition reduces to

$$\frac{\partial}{\partial t}\varphi(y;\rho)=w(y,t)\cdot N(y)$$

or

$$\rho_t=w\cdot N+g(w,\rho)$$

where

$$\varphi(y;\rho)=\rho-\frac{\rho^2}{2}\mathcal{H}(y)+\frac{\rho^3}{3}\mathcal{K}(y),$$

$$g(w,\rho)=(1-\Lambda(y;\rho))w\cdot N=(\rho\mathcal{H}(y)-\rho^2\mathcal{K}(y))w\cdot N. \tag{2.21}$$

Thus, problem (1.5) can be written in the form (1.11) with f, b, d, g defined in (2.16), (2.18), (2.19), (2.21).

Finally, we write orthogonality conditions for w and ρ in new coordinates. The restrictions (1.9) can be expressed in terms of ρ as follows:

$$\int\limits_{\mathcal{G}} \varphi(y;\rho)dS = 0, \quad \int\limits_{\mathcal{G}} \varphi(y;\rho)y_i\,dy = -\int\limits_{\mathcal{G}} N_i(y)\Big(\frac{\rho^2}{2} - \frac{\rho^3}{3}\mathcal{H}(y) + \frac{\rho^4}{4}\mathcal{K}\Big)dS,$$

$i = 1, 2, 3$, and (1.8) reduces to

$$\int\limits_{\mathcal{F}} \mathcal{L}(y;\rho)w(y,t)dy = 0,$$

$$\int\limits_{\mathcal{F}} \mathcal{L}(y;\rho)w(y,t)\cdot\eta_i(e_\rho(y))dy =$$

$$= -\omega\int\limits_{\mathcal{F}} L_\rho\eta_3(e_\rho)\cdot\eta_i(e_\rho)dy + \omega\int\limits_{\mathcal{F}} \eta_3(y)\cdot\eta_i(y)dy.$$

We transform the last two integrals using the formula

$$\int\limits_{\mathcal{F}} L_\rho f(e_\rho)dy - \int\limits_{\mathcal{F}} f(y)dy = \int\limits_0^1 d\mu \int\limits_{\mathcal{G}} f(e_{\mu\rho})\rho\Lambda(y;\mu\rho)dS$$

$$= \int\limits_{\mathcal{G}} f(y)\rho(y,t)dS + \int\limits_0^1 (1-\mu)d\mu \int\limits_{\mathcal{G}} \frac{d}{d\mu} f(e_{\mu\rho})\rho\Lambda(y;\mu\rho)dS$$

(see [3]) and obtain

$$\int\limits_{\mathcal{F}} w\,dy = \int\limits_{\mathcal{F}} (I-\mathcal{L})w\,dy,$$

$$\int\limits_{\mathcal{F}} w\cdot\eta_i dy + \omega\int\limits_{\mathcal{G}} \rho\eta_3(y)\cdot\eta_i(y)dS = \int\limits_{\mathcal{F}} w\cdot\eta_i dy - \int\limits_{\mathcal{F}} \mathcal{L}(y;\rho)w(y,t)\cdot\eta_i(e_\rho(y)dy$$

$$-\omega\int\limits_0^1 (1-\mu)d\mu \int\limits_{\mathcal{G}} \frac{d}{d\mu}\eta_3(e_{\mu\rho})\cdot\eta_i(e_{\mu\rho})\rho\Lambda(y;\mu\rho)dS.$$

Since $\nabla\cdot w = 0$ and $\varphi(y;\rho)_t = w\cdot N$, there should be $\int_{\mathcal{F}} w_i dy = \int_{\mathcal{G}} \varphi(y;\rho)_t y_i dS$, $i = 1, 2, 3$, and, as a consequence,

$$\int\limits_{\mathcal{F}} (I-\mathcal{L})w\,dy = -\frac{\partial}{\partial t}\int\limits_{\mathcal{G}} N(y)\Big(\frac{\rho^2}{2} - \frac{\rho^3}{3}\mathcal{H}(y) + \frac{\rho^4}{4}\mathcal{K}\Big)dS$$

$$= \int_{\mathcal{G}} \Lambda(y,\rho)(w \cdot N + g(w,\rho))dS.$$

For further transformation of the problem (1.5) we need the following auxiliary proposition.

Proposition 2.2. *Given the functions $l_i(t)$, $m_i(t)$, $i = 1,2,3$, and a tangential vector field $h(x,t), x \in \mathcal{G}$, $t \in [0,T]$, there exist a function $r(x,t), x \in \mathcal{G}$ and a divergence free vector field $w(x,t)$, $x \in \mathcal{F}$, such that*

$$r_t = w \cdot N, \qquad \Pi_0 S(w)N = h(x,t), \qquad x \in \mathcal{G},$$

$$\int_{\mathcal{G}} r(x,t)dS = 0, \qquad \int_{\mathcal{G}} r(x,t)x dS = l(t) = (l_1(t), l_2(t), l_3(t)),$$

$$\int_{\mathcal{F}} w(x,t)dx = l'(t), \qquad \int_{\mathcal{F}} w(x,t) \cdot \eta_i(x)dx = m_i(t), \qquad i = 1,2,3,$$

and

$$|r(\cdot,t)|_{C^{3+\alpha}(\mathcal{G})} \le c|l(t)|,$$

$$|w(\cdot,t)|_{C^{2+\alpha}(\mathcal{F})} \le c\Big(|l(t)| + |l'(t)| + |m(t)| + |h|_{C^{1+\alpha}(\mathcal{G})}\Big).$$

Proof. We set $r(x,t) = |\mathcal{F}|^{-1} l(t) \cdot N(x)$ and we construct a divergence free vector field $w_1(x,t)$ such that $w_1 \cdot N|_{\mathcal{G}} = |\mathcal{F}|^{-1} l'(t) \cdot N(x)$ and

$$|w_1(\cdot,t)|_{C^{2+\alpha}(\mathcal{F})} \le c|l'(t)|.$$

It satisfies $\int_{\mathcal{F}} w_1(x,t)dx = l'(t)$. Further, we find another solenoidal vector field w_2 such that

$$\Pi_0 S(w_2)N(x) = h(x,t) - \Pi_0 S(w_1)N(x) \equiv h'(x,t).$$

We take it in the form $w_2(x,t) = \operatorname{rot} \Phi(x,t)$ with Φ satisfying the conditions

$$\Phi(x,t) = \frac{\partial \Phi}{\partial N} = 0, \quad \frac{\partial^2 \Phi}{\partial N^2} = h'(x,t) \times N(x), \qquad x \in \mathcal{G},$$

and the estimate

$$|\Phi(\cdot,t)|_{C^{3+\alpha}(\mathcal{F})} \le c|h'(\cdot,t)|_{C^{1+\alpha}(\mathcal{G})}.$$

It is clear that $w_2(x,t) = 0$ on $\mathcal{G}$ and $\int_{\mathcal{G}} w_2(x,t)dS = \int_{\mathcal{G}} w_2(x,t) \cdot N(x)xdS = 0$, moreover,

$$\frac{\partial w_2(x,t)}{\partial N} = N(x) \times \frac{\partial^2 \Phi(x,t)}{\partial N^2}, \qquad x \in \mathcal{G},$$

which implies $N \cdot \frac{\partial w_2}{\partial N}|_{\mathcal{G}} = 0$ and

$$\Pi_0 S(w_2)N = \frac{\partial w_2}{\partial N} = N \times [h' \times N] = h', \quad x \in \mathcal{G}.$$

Finally, we set

$$w_3(x,t) = \sum_{i=1}^{3} \widehat{m}_i(t) \operatorname{rot} e_i A(x)$$

where $A \in C_0^\infty(\mathcal{F})$, $\int_{\mathcal{F}} A(x,t)dx = 1/2$, and

$$\widehat{m}_i(t) = m_i(t) - \int_{\mathcal{F}} (w_1(x,t) + w_2(x,t)) \cdot \eta_i(x)dx.$$

Since $\operatorname{rot} \eta_i = 2e_i$, we have

$$\int_{\mathcal{F}} w_3(x,t) \cdot \eta_i(x)dx = \sum_{j=1}^{3} \widehat{m}_j(t)e_j \cdot e_i = \widehat{m}_i(t);$$

there also holds the estimate

$$|w_3|_{C^{2+\alpha}(\mathcal{F})} \le c \sum_{j=1}^{3} |\widehat{m}_j(t)|.$$

It is clear that $r(x,t)$ defined above and $w(x,t) = w_1 + w_2 + w_3$ satisfy all the necessary requirements. The proposition is proved.

Proposition 2.2 enables us to make the following last transformation of problem (1.5). We set $r = \varphi(y;\rho)$,

$$l(t) = -\int_{\mathcal{G}} N(y)\Big(\frac{\rho^2}{2} - \frac{\rho^3}{3}\mathcal{H}(y) + \frac{\rho^4}{4}\mathcal{K}\Big)dS,$$

$$m_i(t) = \int_{\mathcal{F}} w \cdot \eta_i dy - \int_{\mathcal{F}} \mathcal{L}(y;\rho)w(y,t) \cdot \eta_i(e_\rho(y))dy$$

$$-\omega\int\limits_0^1(1-\mu)d\mu\int\limits_{\mathcal{G}}\frac{d}{d\mu}\eta_3(e_{\mu\rho})\cdot\eta_i(e_{\mu\rho})\rho\Lambda(y;\mu\rho)dS,$$

$$h(x,t)=b(w,\rho)=\Pi_0(\Pi_0 S(w)N-\Pi\widetilde{S}(L_\rho^{-1}\mathcal{L}w)n)$$

and we compute the functions w'', r'' corresponding to these $l(t)$, $m(t)$, $h(x,t)$, according to Proposition 2.2. Then $w'(x,t)=w-w''$, $r'(x,t)=r-r''$ satisfy the conditions

$$\Pi_0 S(w')N(x)=0,\quad x\in\mathcal{G},$$

$$\int\limits_{\mathcal{G}}r'dS=0,\quad\int\limits_{\mathcal{G}}r'x_i dS=0,\quad i=1,2,3. \tag{2.22}$$

$$\int\limits_{\mathcal{F}}w'dx=0,\quad\int\limits_{\mathcal{F}}w'\cdot\eta_i dx+\omega\int\limits_{\mathcal{G}}r'\eta_3\cdot\eta_i dS=0,\quad i=1,2,3.$$

Now, we define u_1, q_1, r_1 as the solution to a linear problem

$$u_{1t}+2\omega(e_3\times u_1)-\nu\nabla^2u_1+\nabla q_1=0,\quad\nabla\cdot u_1=0,\qquad x\in\mathcal{F},$$

$$T(u_1,q_1)N=-NB_0r_1,\qquad r_{1t}=u_1\cdot N, \tag{2.23}$$

$$u_1(x,0)=w'(x,0),\qquad r_1(x,0)=r'(x,0).$$

It can be verified that (u_1,r_1) also satisfy (2.22) for arbitrary $t\geqslant 0$, and $w-u_1=u_2$, $s-q_1=q_2$, $\rho-r_1=\rho_2$ satisfy the relations

$$u_{2t}+2\omega(e_3\times u_2)-\nu\nabla^2u_2+\nabla q_2=f(u_1+u_2,q_1+q_2,r_1+\rho_2),$$

$$\nabla\cdot u_2=0,\qquad x\in\mathcal{F},$$

$$\Pi_0 S(u_2)N=b(u_1+u_2,r_1+\rho_2), \tag{2.24}$$

$$-q_2+\nu N\cdot S(u_2)N(x)+B_0\rho_2=d(u_1+u_2,r_1+\rho_2),$$

$$\rho_{2t}=u_2\cdot N+g(u_1+u_2,r_1+\rho_2),$$

$$\rho_2(x,0)=\rho_0-r'(x,0)=(\rho_0-\varphi(\rho_0))+r''(x,0),\quad u_2(x,0)=w_0''(x,0).$$

Proposition 2.3. *Given arbitrary $T>0$, there exists a number $\epsilon_1(T)>0$ such that in the case*

$$|w_0|_{C^{2+\alpha}(\mathcal{F})}+|\rho_0|_{C^{3+\alpha}(\mathcal{G})}\le \epsilon_1$$

problems (1.11) *and* (2.24) *are uniquely solvable in the interval of time $t\in[0,T]$, and the solutions satisfy the inequalities*

$$Y_t(w,s,\rho)\le c\Big(|w_0|_{C^{2+\alpha}(\mathcal{F})}+|\rho_0|_{C^{3+\alpha}(\mathcal{G})}\Big), \tag{2.25}$$

$$Y_t(u_2,q_2,\rho_2)\le c\Big(|w_0|_{C^{2+\alpha}(\mathcal{F})}+|\rho_0|_{C^{3+\alpha}(\mathcal{G})}\Big)^2 \tag{2.26}$$

where Y_t is defined in (2.11).

The solvability of the first problem (in the form (1.5)) was already established in [3–6] by the method of Lagrangean coordinates but it can be also proved directly on the basis of proposition 2.1, for instance, by successive approximations. As for the estimates (2.25), (2.26), it should be observed in the first line that nonlinear functions $f(w,s,\rho)$, $b(w,\rho)$, $d(w,\rho)$, $g(w,\rho)$ defined in (2.16), (2.18), (2.19), (2.21) satisfy

$$|f|_{C^{\alpha}(\mathcal{F})}+|d|_{C^{1+\alpha}(\mathcal{G})}+|g|_{C^{2+\alpha}(\mathcal{G})}\le cY_t^2(w,s,\rho),$$
$$|b|_{C^{1+\alpha,(1+\alpha)/2}(G_t)}\le cY_t^2(w,s,\rho)$$

where $G_t=\mathcal{G}\times[0,t]$ and $t\in[0,T]$ is arbitrary (in fact, slightly sharper estimates hold). The proof of these inequalities that are consequences of (2.16), (2.17), (2.20) and of Proposition 3.1 in [4] is transparent but lengthy and is omitted. Moreover, the initial data in (2.24) satisfy the inequality

$$|u_2(\cdot,0)|_{C^{2+\alpha}(\mathcal{F})}+|\rho_2(\cdot,0)|_{C^{3+\alpha}(\mathcal{G})}$$
$$\le\Big(|w''(\cdot,0)|_{C^{2+\alpha}(\mathcal{F})}+|r''(\cdot,0)|_{C^{3+\alpha}(\mathcal{G})}+|\rho_0-\varphi(\rho_0)|_{C^{3+\alpha}(\mathcal{G})}\Big)$$
$$\le c\Big(|l(0)|+|l'(0)|+|m(0)|+|h(\cdot,0)|_{C^{1+\alpha}(\mathcal{G})}+|\rho_0-\varphi(\rho_0)|_{C^{3+\alpha}(\mathcal{G})}\Big)$$
$$\le c\Big(|w_0|_{C^{2+\alpha}(\mathcal{F})}+|\rho_0|_{C^{3+\alpha}(\mathcal{G})}\Big)^2,$$

hence, (2.10) implies

$$Y_t(w,s,\rho)\le c\Big(|w_0|_{C^{2+\alpha}(\mathcal{F})}+|\rho_0|_{C^{3+\alpha}(\mathcal{G})}+Y_t^2(w,s,\rho)\Big),$$

$$Y_t(u_2, q_2, \rho_2) \le c\Big(|w_0|_{C^{2+\alpha}(\mathcal{F})} + |\rho_0|_{C^{3+\alpha}(\mathcal{G})}\Big)^2 + cY_t^2(u_2, q_2, \rho_2)$$

which gives (2.25), (2.26) in the case of small ϵ_1.

3. Proof of instability.

We recall that in the case $f = 0,\ b = 0,\ d = 0,\ g = 0$ problem (2.1)–(2.5) can be written as Cauchy problem in a Banach space (see [13, 11]):

$$\frac{d\phi}{dt} - A\phi = 0, \qquad \phi|_{t=0} = \phi_0,$$

where $\phi = (w, \rho)$, $\phi_0 = (w_0, \rho_0)$ and the operator A is given by

$$A\phi = \Big(A_{11}w + A_{12}\rho, \quad A_{21}w\Big)^T,$$

$$A_{11}w = -2\omega P_J(e_3 \times w) + \nu\nabla^2 w - \nabla s_1, \qquad A_{12}\rho = -\nabla s_2,$$

$$A_{21}w = w \cdot N|_{\mathcal{G}}.$$

By P_J we mean the orthogonal in $L_2(\mathcal{F})$ projector onto the subspace $J(\mathcal{F}) \subset L_2(\mathcal{F})$ of divergence free vector fields, and s_i are harmonic functions in $\mathcal{F}$ satisfying the conditions

$$s_1(x,t) = \nu N(x) \cdot S(w)N(x), \qquad s_2 = B_0\rho(x,t), \qquad x \in \mathcal{G}.$$

Thus, the pressure is excluded. The domain of A is characterized by the conditions

$$\nabla \cdot w = 0, \qquad \Pi_0 S(w)N(x)|_{\mathcal{G}} = 0, \tag{3.1}$$

$$\int_{\mathcal{F}} w dx = 0, \quad \int_{\mathcal{F}} w \cdot \eta_i dx + \omega \int_{\mathcal{G}} \rho\eta_3 \cdot \eta_i dS = 0, \quad i = 1, 2, 3,$$

and conditions (1.10) for ρ. As a basic space we choose $X \equiv C^{2+\alpha}(\mathcal{F}) \times C^{3+\alpha}(\mathcal{G})$: $w \in C^{2+\alpha}(\mathcal{F})$, $\rho \in C^{3+\alpha}(\mathcal{G})$ and we set

$$|\phi|_X = |w|_{C^{2+\alpha}(\mathcal{F})} + |\rho|_{C^{3+\alpha}(\mathcal{G})}.$$

By X_0 we mean the subspace of X whose elements satisfy (3.1), (1.10). We have seen above that $e^{tA}\phi_0 \in X_0$ for every $\phi_0 \in X_0$ and

$$|e^{tA}\phi_0|_X \le c(t)|\phi_0|_X.$$

As shown in [13, 11], the operator A (extended to $W_2^2(\mathcal{F}) \times W_2^{5/2}(\mathcal{G})$) has a countable set of eigenvalues of a finite algebraic multiplicity located in the sector $\mathrm{Re}\,\lambda < L - a|\mathrm{Im}\lambda|$, $a, A > 0$ and having the only limit point at infinity. We make the following hypothesis concerning A:

(H). The operator A has a finite number of eigenvalues with a positive real part.

In this case the equation $\frac{d}{dt}\phi - A\phi = 0$ has a finite number of linearly independent solutions growing exponentially as $t \to \infty$. It follows from the results of [11] that they are regular functions of x and belong to X_0.

We will show that (H) implies instability of the zero solution of problem (1.11). The scheme of the proof will be the same as in [14], Sec. 4.

Let T be a fixed large positive number and $Z = e^{TA}$. The spectrum of Z, $\sigma(Z)$, consists of two parts, $\sigma_1(Z)$ and $\sigma_2(Z)$, where $\sigma_1(Z)$ is a finite set of eigenvalues $\mu \in C$ with $|\mu| > 1$, whereas the eigenvalues $\mu \in \sigma_2(Z)$ satisfy $|\mu| \leq 1$. By the Riesz formula, Z can be represented in the form $Z = Z_1 + Z_2$ where

$$Z_k = \frac{1}{2\pi i} \int\limits_{\gamma_k} \mu(\mu I - Z)^{-1} d\mu, \qquad k = 1, 2,$$

and γ_k are non-intersecting contours enclosing $\sigma_k(Z)$. Replacing Z_1 with

$$Z_1^n = \frac{1}{2\pi i} \int\limits_{\gamma_1} \mu^n (\mu I - Z)^{-1} d\mu,$$

if necessary (i.e. choosing T large enough), it is possible to satisfy the inequalities

$$|Z_1\psi|_X \geq b_1|\psi|_X, \quad b_1 > 1, \quad \forall \psi \in X_1,$$
$$\|Z_2\|_{X\to X} \leq b_2 < b_1. \tag{3.2}$$

To the decomposition of Z defined above there corresponds decomposition of the space X_0 into a direct sum $X_0 = X_1 + X_2$. The operators

$$P_k = \frac{1}{2\pi i} \int\limits_{\gamma_k} (\mu I - Z)^{-1} d\mu, \qquad k = 1, 2,$$

are projectors onto X_k, and the relations

$$P_1P_2 = P_2P_1 = 0, \quad P_k^2 = P_k, \quad Z_k = ZP_k = P_kZ, \quad P_1Z_2 = P_2Z_1 = 0$$

hold.

Let us represent the initial data in (1.11) in the form

$$w_0(x) = w_0'(x) + w_0''(x), \quad \rho_0(x) = r_0'(x) + \rho_0''(x),$$

as it has been done at the end of Sec. 2, and let $(w_0', r_0') \equiv Q(w_0, \rho_0)$. Our goal is to prove that the solution $\phi = (w, \rho)$ of (1.11) with the initial data satisfying

$$|P_1 Q\phi_0|_X \geq 2|P_2 Q\phi_0|_X \tag{3.3}$$

can not satisfy $|\phi|_X \leq \epsilon$ for all $t > 0$ (the number $\epsilon \leq \epsilon_1(T)$ will be fixed later). Let K_ϵ be the ball $|\phi|_X \leq \epsilon$ in X, $K_\epsilon' \subset K_\epsilon$ its subset whose elements satisfy (3.3) and let W and W' be operators making correspond to ϕ_0 the solutions (w, ρ) of problem (1.11) and (2.24), respectively, at the moment $t = T$. As it has been shown in Sec. 2,

$$W\phi_0 = ZQ\phi_0 + W'\phi_0.$$

Since

$$\begin{aligned} |\phi - Q\phi|_X \leq |w''|_{C^{2+\alpha}(\mathcal{F})} + |r''|_{C^{3+\alpha}(\mathcal{G})} \\ + |\rho - \varphi(\rho)|_{C^{3+\alpha}(\mathcal{G})} \leq c|\phi|_X^2, \end{aligned} \tag{3.4}$$

arbitrary $\phi \in K_\epsilon'$ satisfies the inequality

$$\begin{aligned} |\phi|_X &\leq |P_1 Q\phi|_X + |P_2 Q\phi|_X + |\phi - Q\phi|_X \\ &\leq \frac{3}{2}|P_1 Q\phi|_X + c_1|\phi|_X^2 \leq \frac{3}{2}|P_1 Q\phi|_X + c_1\epsilon|\phi|_X; \end{aligned}$$

and if $c_1\epsilon < 1/2$, then

$$|\phi|_X \leq 3|P_1 Q\phi|_X.$$

Now, we show that for arbitrary $\phi \in K_\epsilon'$ the element $W\phi$ also belongs to K_ϵ', if ϵ is sufficiently small. Since

$$P_1 QW\phi = P_1 W\phi + P_1(Q-1)W\phi = Z_1 P_1 Q\phi + P_1 W'\phi + P_1(Q-1)W\phi,$$

$$P_2 QW\phi = Z_2 P_2 Q\phi + P_2 W'\phi + P_2(Q-1)W\phi,$$

we have, by virtue of (2.26) and (3.2)–(3.4):

$$\begin{aligned} |P_1 QW\phi|_X - 2|P_2 QW\phi|_X &\geq (b_1 - b_2)|P_1 Q\phi|_X - c_3|\phi|_X^2 \\ &\geq (b_1 - b_2 - 3c_3\epsilon)|P_1 Q\phi|_X \geq 0, \end{aligned}$$

if

$$b_1 - b_2 - 3c_3\epsilon > 0.$$

Finally, we estimate $P_1 QW\phi$ from below, again with the help of (2.26), (3.2)–(3.4):

$$|P_1 QW\phi|_X \geq |Z_1 P_1 Q\phi|_X - c_3|\phi|_X^2 \geq (b_1 - 3c_3\epsilon)|P_1 Q\phi|_X.$$

We assume that

$$b_1' \equiv b_1 - 3c_3\epsilon > 1;$$

then

$$|P_1QW\phi|_X \geq b_1'|P_1Q\phi|_X. \tag{3.5}$$

If the solution $\phi = (w, \rho)$ of (1.11) remains in K_ϵ for all $t > 0$, and $\phi_0 = (w_0, r_0) \in K_\epsilon'$, then the same arguments show that also $W^n\phi_0 \in K_\epsilon'$ for $n = 1, 2, \dots$, and, by (3.5),

$$|P_1QW^n\phi_0|_X \geq b_1^{'}|P_1QW^{n-1}\phi_0|_X \geq \dots \geq b_1^{'n}|P_1Q\phi_0|_X.$$

For large n, this contradicts to $W^n\phi_0 \in K_\epsilon$.

Hence, the following proposition is proved.

Proposition 3.1. *Under the hypothesis (H), there exists a number $\epsilon > 0$ and non-zero initial data $\phi_0 = (w_0, \rho_0) \in X$ with $|\phi_0|_X$ arbitrarily small, such that the solution of problem* (1.11) *has the norm $|\phi|_X > \epsilon$ at certain $t > 0$.*

It is shown in [12] that the hypothesis (H) holds, if $\delta_0^2 R[\rho]$ can take negative values for some ρ satisfying (1.10).

We observe at the conclusion that the same analysis can be made in the Sobolev spaces W_2^l.

It is also worth noticing that by similar arguments exponential stability of the zero solution of (1.11) can be established under the condition that all the eigenvalues of the operator A have negative real parts. This is the case when $\delta_0^2 R[\rho]$ is positive definite for all ρ satisfying (1.10). In [5] it was done by another method.

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