Записки научных
семинаров ПОМИ
Том 318, 2004 г.

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NON BLOW–UP OF THE 3D IDEAL MAGNETOHYDRODYNAMICS EQUATIONS FOR A CLASS OF THREE–DIMENSIONAL INITIAL DATA IN CYLINDRICAL DOMAINS

Abstract. Non blow-up of the 3D ideal incompressible magnetohydrodynamics (MHD) equations is proven for a class of three-dimensional initial data characterized by both uniformly large vorticity and magnetic field in bounded cylindrical domains. There are no conditional assumptions on the properties of solutions at later times, nor are the global solutions close to some 2D manifold. The approach of proving regularity is based on investigation of fast singular oscillating limits and nonlinear averaging methods in the context of almost periodic functions. We establish the global regularity of the 3D limit resonant MHD equations without any restriction on the size of 3D initial data. After establishing strong convergence to the limit resonant equations, we bootstrap this into the regularity on arbitrary large time intervals of the solutions of 3D MHD Equations with weakly aligned uniformly large vorticity and magnetic field at $t = 0$.

Dedicated to the Memory of O.A. Ladyzhenskaya

1. Introduction and main results

Non blow-up of the 3D incompressible ideal MHD Equations is proven for a class of three-dimensional initial data characterized by uniformly large vorticity and magnetic field in bounded cylindrical domains. There are no conditional assumptions on the properties of solutions at later times, nor are the global solutions close to some 2D manifold. Fully three-dimensional initial conditions with uniformly large vorticity and magnetic field excite fast Alfven-Poincaré magneto-vorticity waves [3], [5]. Since individual Alfven-Poincaré wave modes are related to the eigenfunctions of the curl operator, they are exact time-dependent solutions of the full nonlinear 3D MHD equations. Of course, their linear superposition does not preserve this property. In this article we prove regularity on arbitrary long time intervals for fully 3D initial data which are superposition of 'slow' poloidal fields (two-dimensional velocity and magnetic fields with three components) and an infinite countable spectrum of 'fast' Alfven-Poincaré waves; this is achieved with a detailed mathematical analysis of multiscale phenomena which arise in the fully nonlinear interactions between 3D waves and the mean flow (poloidal). The techniques involve detailed investigations of fast singular oscillating limits [12], [2].

We study initial value problem for the three-dimensional ideal incompressible MHD equations with initial data characterized by uniformly large vorticity and axial magnetic field:

$$\partial_t \mathbf{V} + \operatorname{curl} \mathbf{V} \times \mathbf{V} - \operatorname{curl} \mathbf{B} \times \mathbf{B} = -\nabla p, \tag{1.1}$$

$$\partial_t \mathbf{B} = \operatorname{curl}(\mathbf{V} \times \mathbf{B}), \tag{1.2}$$

$$\nabla \cdot \mathbf{V} = 0, \quad \nabla \cdot \mathbf{B} = 0, \tag{1.3}$$

$$\begin{aligned} \mathbf{V}(t,y)|_{t=0} &= \mathbf{V}(0) = \widetilde{\mathbf{V}}_0(y) + \frac{\Omega}{2} e_3 \times y, \\ \mathbf{B}(t,y)|_{t=0} &= \mathbf{B}(0) = B_T e_3 + \widetilde{\mathbf{B}}_0(y), \end{aligned} \tag{1.4}$$

where $y = (y_1, y_2, y_3)$, $\mathbf{V}(t, y) = (V_1, V_2, V_3)$ is the velocity field and p is the pressure; $\mathbf{B}(t, y) = (B_1, B_2, B_3)$ is the magnetic field. In Eqs. (1.1) e_3 denotes the vertical unit vector; Ω and B_T are constant parameters. The fields $\widetilde{\mathbf{V}}_0(y)$ and $\widetilde{\mathbf{B}}_0(y)$ depend on three variables y_1, y_2 and y_3. Since $curl(\frac{\Omega}{2} e_3 \times y) = \Omega e_3$, the vorticity vector at initial time $t = 0$ is

$$\operatorname{curl} \mathbf{V}(0, y) = \operatorname{curl} \widetilde{\mathbf{V}}_0(y) + \Omega e_3, \tag{1.5}$$

and the initial vorticity has a large component weakly aligned along e_3, when $\Omega >> 1$. Similarly, when $B_T >> 1$, the initial magnetic field has a large component weakly aligned along e_3. These are fully three-dimensional large initial data with large initial 3D vortex stretching. Recall that the current $\mathbf{J}$ is given by $\mathbf{J} = \operatorname{curl} \mathbf{B}$.

Eqs. (1.1) are studied in cylindrical domains

$$\mathbf{C} = \{(y_1, y_2, y_3) \in \mathbf{R}^3 : 0 < y_3 < 2\pi/\alpha, y_1^2 + y_2^2 < R^2\} \tag{1.6}$$

where α and R are positive real numbers. If L is the height of the cylinder, $\alpha = 2\pi/L$. Let

$$\Gamma = \{(y_1, y_2, y_3) \in \mathbf{R}^3 : 0 < y_3 < 2\pi/\alpha, y_1^2 + y_2^2 = R^2\}. \tag{1.7}$$

Without loss of generality, we can assume that $R = 1$. Eqs. (1.1) are considered with periodic boundary conditions in y_3

$$\mathbf{V}(y_1, y_2, y_3) = \mathbf{V}(y_1, y_2, y_3 + 2\pi/\alpha), \mathbf{B}(y_1, y_2, y_3) = \mathbf{B}(y_1, y_2, y_3 + 2\pi/\alpha) \tag{1.8}$$

and vanishing normal component of velocity on Γ

$$\mathbf{V} \cdot \mathbf{N} = \widetilde{\mathbf{V}} \cdot \mathbf{N} = 0, \mathbf{B} \cdot \mathbf{N} = \widetilde{\mathbf{B}} \cdot \mathbf{N} = 0 \text{ on } \quad \Gamma; \tag{1.9}$$

where $\mathbf{N}$ is the normal vector to Γ. From the invariance of 3D MHD equations under the symmetry $y_3 \to -y_3$, $V_1 \to V_1$, $V_2 \to V_2$, $V_3 \to -V_3$ (similarly, for $\mathbf{B}$), all results in this paper extend to cylindrical domains bounded by two horizontal plates. Then the boundary conditions in the vertical direction are zero flux on the vertical boundaries (zero vertical velocity on the plates). One only needs to restrict vector fields to be even in y_3 for V_1, V_2 and odd in y_3 for V_3, and double the cylindrical domain to $-L \leq y_3 \leq +L$.

With periodic boundary conditions in y_3, small α and large L the periodic cylinder can be considered as a local approximation to Tokamak toroidal configurations. The scalar parameter $B_T >> 1$ is equivalent to an imposed large toroidal magnetic field component, typical of operating configurations for Tokamaks; it is well known that under such a constraint, the incompressible MHD Equations (1.1) are a good approximation to the exact compressible ones and the component $B_T e_3$ excites high frequency Alfven waves.

We choose $\widetilde{\mathbf{V}}_0(y)$ and $\widetilde{\mathbf{B}}_0(y)$ in $\mathbf{L}_2(\mathbf{C})$. We introduce $\widetilde{\mathbf{V}}(t, y)$ and $\widetilde{\mathbf{B}}(t, y)$ such that

$$\begin{aligned} &\mathbf{V}(t, y) = \widetilde{\mathbf{V}}(t, y) + \frac{\Omega}{2} e_3 \times y, \operatorname{curl} \mathbf{V}(t, y) = \\ &\operatorname{curl} \widetilde{\mathbf{V}}(t, y) + \Omega e_3, \mathbf{B}(t, y) = B_T e_3 + \widetilde{\mathbf{B}}(t, y) \end{aligned} \tag{1.10}$$

For the vorticity field $\omega = \operatorname{curl} \mathbf{V}$ Eqs. (1.1) become

$$\frac{\partial}{\partial t}\omega + \mathbf{V} \cdot \nabla\omega - \mathbf{B} \cdot \nabla\mathbf{J} = \omega \cdot \nabla\mathbf{V} - \mathbf{J} \cdot \nabla\mathbf{V}, \tag{1.11}$$

$$\omega(0, y) = \operatorname{curl} \widetilde{\mathbf{V}}_0(y) + \Omega e_3, \tag{1.12}$$

and the initial condition induces large initial vortex and current stretching.

We establish regularity for arbitrarily large finite times for the 3D ideal MHD equations for Ω and B_T both large, but *finite*, with the ratio

Ω/B_T fixed and finite. Our solutions are not close in any sense to those of the 2D or "quasi 2D" MHD equations and they are characterized by fast oscillations in the e_3 direction, together with a large vortex stretching term $\omega(t,y)\cdot\nabla\mathbf{V}(t,y) = \omega_1\frac{\partial V_1}{\partial y_1} + \omega_2\frac{\partial V_2}{\partial y_2} + \omega_3\frac{\partial V_3}{\partial y_3}$, $t \geq 0$ with leading component $\Omega\frac{\partial}{\partial y_3}V_3(t,y) >> 1$. There are no assumptions on oscillations in y_1, y_2 for our solutions (nor for the initial condition $\widetilde{\mathbf{V}}_0(y)$).

Our approach is entirely based on fast singular oscillating limits of Eqs. (1.11)–(1.12), nonlinear averaging and cancelation of oscillations in the nonlinear interactions for the vorticity field for large Ω. This has been developed in [3], [4], [5] and [25] for the cases of periodic lattice domains and the infinite space $\mathbf{R}^3$. Through the canonical transformation (1.24) in both the field $\mathbf{V}(t,y)$, $\mathbf{B}(t,y)$ and the space coordinate $y=(y_1,y_2,y_3)$ for every Ω and B_T (not necessary large) we map every solution $\mathbf{V}(t,y)$, respectively $\mathbf{B}(t,y)$ of Eqs. (1.1) one-to-one to a solution $\mathbf{U}(t,x)$, respectively $\mathcal{H}(t,x)$, $x=(x_1,x_2,x_3)$ of

$$\begin{aligned}
&\partial_t\mathbf{U} + (\mathbf{U}\cdot\nabla_x)\mathbf{U} - (\mathcal{H}\cdot\nabla_x)\mathcal{H} + \Omega\mathbf{e}_3\times\mathbf{U} = \\
&-\nabla_x\Big(p - \frac{\Omega^2}{4}|x_h|^2 - \frac{1}{2}|\mathbf{U}|^2 + \frac{1}{2}|\mathcal{H}|^2\Big),\\
&\partial_t\mathcal{H} = \mathrm{curl}_x(\mathbf{U}\times\mathcal{H}),\\
&\nabla_x\cdot\mathbf{U} = 0, \nabla_x\cdot\mathcal{H} = 0,\\
&\mathbf{U}(t,x)|_{t=0} = \mathbf{U}(0,x) = \widetilde{\mathbf{V}}_0(x),\\
&\mathcal{H}(t,x)|_{t=0} = \mathcal{H}(0,x) = \mathcal{H}_0(x) = B_T e_3 + \widetilde{\mathbf{B}}_0(x)
\end{aligned}$$

where $x=y$ at $t=0$ and $x_h=(x_1,x_2)$.

Let $\epsilon = 1/\Omega$, $B_T = \mathcal{H}_T/\epsilon$; from now on we renormalize the field $\mathcal{H}$ as

$$\mathcal{H}(t,x) = \mathcal{H}_T/\epsilon + \mathbf{H}(t,x). \tag{1.13}$$

From now on we set $\mathcal{H}_T/\epsilon = B_T$, $B_T/\Omega = \mathcal{H}_T$, to account for the context $\Omega >> 1$, $B_T >> 1$, B_T/Ω finite.

The above equations in a rotating frame of reference are obtained by the canonical field transformation

$$R_{\Omega t} = \exp(\mathbf{S}\Omega t/2), \tag{1.14}$$

where $\mathbf{S} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $R_{\Omega t}$ is the rotation matrix of frequency $\Omega/2$ around the cylinder axis.

With $x(t)$ and $y(t)$ related via

$$y(t) = R_{\Omega t} x(t), \dot{y}(t) = \mathbf{V}(t, y(t)), \dot{x}(t) = \mathbf{U}(t, x(t)), \tag{1.15}$$

we have

$$\begin{aligned} \mathbf{V}(t, y(t)) &= R_{\Omega t}(\mathbf{U}(t, x(t)) + \Omega e_3 \times x(t)), \\ \mathbf{B}(t, y(t)) &= R_{\Omega t}(\mathcal{H}(t, x(t)) = B_T e_3 + R_{\Omega t}(\mathbf{H}(t, x(t)). \end{aligned} \tag{1.16}$$

The transformed MHD equation can also be written as

$$\begin{aligned} &\partial_t \mathbf{U} + \operatorname{curl}_x \mathbf{U} \times \mathbf{U} + \Omega e_3 \times \mathbf{U} + \nabla_x q = \operatorname{curl}_x \mathcal{H} \times \mathcal{H}, \\ &\partial_t \mathcal{H} = \operatorname{curl}_x (\mathbf{U} \times \mathcal{H}), \end{aligned} \tag{1.17}$$

$$\operatorname{div}_x \mathbf{U} = \operatorname{div}_x \mathcal{H} = 0, \tag{1.18}$$

with $q = p - \frac{1}{4}\Omega^2 |x_h|^2$, $|x_h|^2 = x_1^2 + x_2^2$. Now with $\epsilon = 1/\Omega$, $\mathcal{H}_T/\epsilon = B_T$, and $\mathcal{H} = \frac{1}{\epsilon}\mathcal{H}_T e_3 + \mathbf{H}(t, x(t))$, Eq. (1.17)–(1.18) can further be written as:

$$\begin{aligned} &\partial_t \mathbf{U} + \operatorname{curl} \mathbf{U} \times \mathbf{U} + \frac{1}{\epsilon} e_3 \\ &\times \mathbf{U} - \operatorname{curl} \mathbf{H} \times \mathbf{H} + \frac{1}{\epsilon}\mathcal{H}_T e_3 \times \operatorname{curl} \mathbf{H} + \nabla q = 0, \\ &\partial_t \mathbf{H} - \frac{1}{\epsilon}\mathcal{H}_T \frac{\partial \mathbf{U}}{\partial x_3} = \operatorname{curl}(\mathbf{U} \times \mathbf{H}), \\ &\operatorname{div} \mathbf{U} = \operatorname{div} \mathbf{H} = 0, \end{aligned} \tag{1.19}$$

and

$$\mathbf{U}(t, x)|_{t=0} = \widetilde{\mathbf{V}}(0, y) = \widetilde{\mathbf{V}}_0(y), \tag{1.20}$$

$$\mathbf{H}(t, x)|_{t=0} = \widetilde{\mathbf{B}}(0, y) = \widetilde{\mathbf{B}}_0(y). \tag{1.21}$$

The renormalized Eq. (1.19) evidence that the original singular initial value problem (weakly aligned uniformly large initial vorticity, large poloidal component of the initial magnetic field) is recast into a problem with bounded initial conditions $\widetilde{\mathbf{V}}_0(y)$, $\widetilde{\mathbf{B}}_0(y)$, albeit with singular linear operators:

(i) the singular Coriolis operator $\frac{1}{\epsilon} e_3 \times \mathbf{U}$;
(ii) the singular gyromagnetic operator $\frac{1}{\epsilon}\mathcal{H}_T e_3 \times \operatorname{curl} \mathbf{H}$;
(iii) the singular operator $\frac{1}{\epsilon}\mathcal{H}_T \frac{\partial \mathbf{U}}{\partial x_3}$;

the last one is especially of interest, as even small vertical shear of the velocity field $\mathbf{U}$ is translated into large deformations of the magnetic field lines of $\mathbf{H}$; this is a direct consequence of the large poloidal component $B_T e_3$ of the initial magnetic field. The linear component of the Eq. (1.19):

$$\begin{aligned} &\partial_t \mathbf{U} + \frac{1}{\epsilon} e_3 \times \mathbf{U} + \frac{1}{\epsilon} \mathcal{H}_T e_3 \times \operatorname{curl} \mathbf{H} + \nabla \pi = 0, \\ &\partial_t \mathbf{H} - \frac{1}{\epsilon} \mathcal{H}_T \frac{\partial \mathbf{U}}{\partial x_3} = 0, \\ &\operatorname{div} \mathbf{U} = div \mathbf{H} = 0, \end{aligned} \tag{1.22}$$

turns out to be a non-local wave operator, generalizing the Poincare-Sobolev wave operator, and reducing to the latter when $\mathcal{H}_T = \mathbf{H} = 0$. We will call the operator in (1.22) the *Poincare–Sobolev–Alfven* wave operator, as it embodies interactions between high frequency Poincare–Sobolev waves and Alfven waves. For the full nonlinear equations (1.19), these fast oscillating high frequency vorticity-Alfven waves interact non-linearly with the fields $\mathbf{U}$, $\operatorname{curl} \mathbf{U}$, $\mathbf{H}$ and $\mathbf{J} = \operatorname{curl} \mathbf{H}$. The properties of these waves and their nonlinear interactions will be detailed in Sections 2 and 3. In particular, it will be demonstrated that such vorticity-Alfven waves are not just solutions to the formal linearization Eq. (1.22), but they are exact time-dependent solutions of the full original ideal MHD equations, with $\operatorname{curl} \widetilde{\mathbf{V}}(t,y) || \widetilde{\mathbf{V}}(t,y)$ and $\mathbf{J} = \operatorname{curl} \widetilde{\mathbf{B}}(t,y) || \widetilde{\mathbf{B}}(t,y)$ (Arnold–Beltrami–Childress like flows), and

$$\mathbf{V}(t,y) = \widetilde{\mathbf{V}}(t,y) + \frac{\Omega}{2} e_3 \times y, \mathbf{B}(t,y) = \widetilde{\mathbf{B}}(t,y) + B_T e_3. \tag{1.23}$$

We demonstrate that nonlinear interactions with the fast oscillating vorticity-Alfven waves prevent finite-time blow-up of the solution to the Cauchy problem (1.1) and ensure regularity of the solutions on arbitrary large times as $\epsilon \longrightarrow 0$. The strategy of the proof is as follows:
(A) Using the linear propagator $\mathbf{E}(t/\epsilon)$ (unitary wave operator) of the Poincare-Sobolev-Alfven equations, we perform a Van der Pol transformation of the fields $\mathbf{U}$ and $\mathbf{H}$, that is

$$\mathbf{U}(t,x) = \mathbf{E}(-t/\epsilon)\mathbf{u}(t,x), \mathbf{H}(t,x) = \mathbf{E}(-t/\epsilon)\mathbf{b}(t,x) \tag{1.24}$$

this effectively factorizes the fields into a "slow"-modulated envelope and fast oscillations;
(B) The MHD equations (1.19) transform into equations with explicit fast oscillations in the quadratic nonlinear terms; we apply methods of

averaging in the sense of almost periodic functions in t, with a *countable* set of frequencies. The averaging theorem for such functions and nonlinear resonances yield the *resonant MHD equations.*;
(C) These resonant MHD equations split between two-dimensional, three-components ideal MHD equations, and fully 3D equations coupled with the former. For almost all cylinder heights L, we prove that the fully 3D part of the resonant equations admit *new conservation laws* beyond the the classical ones for energy, cross-helicity and magnetic helicity. We obtain long-time regularity for the resonant MHD equations for almost all cylinder heights L.
(D) We control the fast oscillating error term between the solutions to the resonant MHD equations and the full MHD equations, and prove that the error goes to zero in appropriate strong norms, when $\epsilon \longrightarrow 0$.
(E) This allows us to bootstrap into the long-time regularity of solutions to the full MHD equations in the rotating frame; and by isometry properties of the rotation operator $R_{\Omega t}$ and the unitary wave operator $\mathbf{E}(t/\epsilon)$ in appropriate function spaces norm-equivalent to the usual Sobolev spaces, to finally bootstrap into th elong-time regularity of the Cauchy problem for the original 3D ideal MHD equations (1.1)–(1.4) in the original frame of reference, for almost all cylinder heights L, for ϵ small enough.

We now state our non-blowup theorem. Here $\mathbf{H}^s_\sigma(C)$ denotes the usual Sobolev space of solenoidal fields in the cylinder, $s \geq 0$:

Theorem 1.1. *Consider the Cauchy problem for the ideal 3D MHD equations* (1.1)–(1.2)*, in the cylinder C of radius R, height L, with the initial values:*

$$\operatorname{curl} \mathbf{V}(0,y) = \Omega e_3 + \operatorname{curl} \widetilde{\mathbf{V}}_0(y), \mathbf{B}(0,y) = B_T e_3 + \widetilde{\mathbf{B}}_0(y), \qquad (1.25)$$

where $\widetilde{\mathbf{V}}_0(y)$, $\widetilde{\mathbf{B}}_0(y)$ are fully 3D fields such that $\widetilde{\mathbf{V}}_0(y), \widetilde{\mathbf{B}}_0(y) \in \mathbf{H}^s_\sigma(C)$, $s \geq 4$,
$||\widetilde{\mathbf{V}}_0(y)||_{\mathbf{H}^s_\sigma} \leq M^0_s$, $||\widetilde{\mathbf{B}}_0(y)||_{\mathbf{H}^s_\sigma} \leq M^0_s$, and $\operatorname{curl} \widetilde{\mathbf{B}}_0(y) \cdot N = 0$, $\operatorname{curl}^j \widetilde{\mathbf{V}}_0(y) \cdot N = 0$ on Γ, $0 \leq j \leq s$. Further consider the following class of initial conditions for $\widetilde{\mathbf{B}}_0(y)$:

$$\frac{1}{L}\int_0^L \widetilde{\mathbf{B}}_0(y_1,y_2,y_3)dy_3 \times e_3 = 0, \qquad (1.26)$$

that is the mean along the periodic toroidal axis e_3 is parallel to this axis. Let $T_M > 0$ fixed, arbitrary large. Then for $L/R \notin K^*$, where K^*

is a countable, discrete parametric resonance set for L/R, there exists $\Omega^*(L/R, T_M, M_s^0)$, $B_T^*(L/R, T_M, M_s^0)$ such that for every fixed $\Omega \geq \Omega^*$, $B_T \geq B_T^*$, with the ratio B_T/Ω finite, the solution to the Cauchy problem (1.1)–(1.4) exists and is regular on $0 \leq t \leq T_M$:

$$||\mathbf{V}(t,y) - \frac{\Omega}{2} e_3 \times y||_{\mathbf{H}_\sigma^s} \leq \widetilde{M}_s(L/R, T_M, M_s^0), \tag{1.27}$$

$$||\mathbf{B}(t,y) - B_T e_3||_{\mathbf{H}_\sigma^s} \leq \widetilde{M}_s(L/R, T_M, M_s^0). \tag{1.28}$$

Moreover, $\operatorname{curl}^j \mathbf{V}(t,y) \cdot N = 0$, $\operatorname{curl}^j \mathbf{B}(t,y) \cdot N = 0$ on Γ, $0 \leq j \leq s$, $0 \leq t \leq T_M$. Furthermore, for M_s^0 fixed, $T_M \to +\infty$ as $1/\Omega^*$, $1/B_T^* \to 0$. Alternatively, we can take arbitrary large bounded sets of initial data: $M_s^0 \to +\infty$ if $1/\Omega^*$, $1/B_T^* \to 0$, T_M fixed.

Remark 1.2. The theorem extends to the ideal 3D MHD equations similar results for 3D Euler equations in cylindrical domains by [26].

The theorem also holds true when $B_T = 0$, i.e. $\mathbf{B}(0,y) = \widetilde{\mathbf{B}}_0(y)$, $||\widetilde{\mathbf{B}}_0(y)||_{\mathbf{H}_\sigma^s} \leq M_s^0$, and only the initial vorticity is near-singular. In that case, only the usual Poincare–Sobolev waves are present, and the result is a straightforward extension of [26].

Remark 1.3. Since $\operatorname{curl} \mathbf{B} = \mathbf{J}$, the class of initial conditions (1.26) is simply equivalent to:

$$e_3 \cdot \left(\frac{1}{L} \int_0^L \mathbf{J}(y_1, y_2, y_3)|_{t=0} dy_3\right) = 0; \tag{1.29}$$

that is the mean along the periodic toroidal axis of the initial current $\mathbf{J}|_{t=0}$ has zero axial (toroidal) component. Such an initial condition is not preserved by the full 3D ideal MHD equations (1.1)–(1.2), that is this set is not an invariant manifold of (1.1)–(1.2). In particular, the fully 3D part of the resonant MHD equations admit solutions $\mathbf{H}(t,x)$ which are fully 3D. The restriction affects only the 2D-3C component part of the resonant MHD equations, and its lifting is prevented by the fact that no global existence theorem has ever been proven for the most general 2 dimensional, 2 components *ideal* MHD equations initial value problem. This is in sharp contrast with the 2D Euler equations and is a major open mathematical problem.

2. The Poincare–Sobolev–Alfven Equations in Cylindrical Domains

In cylindrical domains, the propagator $\mathbf{E}(-t/\epsilon)$ is defined as the unitary operator solution of the system

$$\begin{aligned}
&\partial_t \mathbf{U} + \frac{1}{\epsilon} e_3 \times \mathbf{U} + \frac{1}{\epsilon}\mathcal{H}_T e_3 \times \operatorname{curl} \mathbf{H} + \nabla \pi = 0, && (2.1)\\
&\partial_t \mathbf{H} - \frac{1}{\epsilon}\mathcal{H}_T \frac{\partial \mathbf{U}}{\partial x_3} = 0, &&\\
&\operatorname{div} \mathbf{U} = \operatorname{div} \mathbf{H} = 0, &&\\
&\mathbf{U}(0,x) = \mathbf{U}_0, \mathbf{H}(0,x) = \mathbf{H}_0, && (2.2)
\end{aligned}$$

and

$$\mathbf{E}(-t/\epsilon)(\mathbf{U}_0, \mathbf{H}_0) = (\mathbf{U}(t), \mathbf{H}(t)). \tag{2.3}$$

This system is equivalent to the non-local Telegrapher's wave equation

$$(\partial_{tt} - \frac{1}{\epsilon^2}\mathcal{H}_T \partial_{x_3x_3})\mathbf{U} + \frac{1}{\epsilon} e_3 \times \partial_t \mathbf{U} + \nabla \widetilde{\pi} = 0, \operatorname{div} \mathbf{U} = 0, \tag{2.4}$$

or equivalently to:

$$\partial_{tt} \operatorname{curl}^2 \mathbf{U} - -\frac{1}{\epsilon^2}\partial_{x_3x_3} \mathbf{U} - \frac{1}{\epsilon^2}\mathcal{H}_T^2 \partial_{x_3x_3} \operatorname{curl}^2 \mathbf{U} - \frac{1}{\epsilon^2}\mathcal{H}_T \partial_{x_3x_3} \operatorname{curl} \mathbf{H} = 0. \tag{2.5}$$

the latter equation reduces to the Poincare-Sobolev wave equation when $\mathcal{H}_T = 0$.

We look for the eigenfunctions and eigenvalues of the above linear propagator of the type:

$$\begin{aligned}
&\mathbf{U}_n(x) e^{\pm i\mu_n t/\epsilon}, \mathbf{H}_n(x) e^{\pm i\mu_n t/\epsilon};\\
&\operatorname{div} \mathbf{U}_n = \operatorname{div} \mathbf{H}_n = 0;
\end{aligned} \tag{2.6}$$

the dispersion relation for μ_n turns out as:

$$\mu_n = -\frac{\check{n}_3}{2\lambda_n}(1 \pm \sqrt{1 + 4\lambda_n^2 \mathcal{H}_T^2}), \tag{2.7}$$

where $\check{n}_3 = \alpha n_3 = 2\pi n_3/L$ denoting the vertical Fourier wavenumber along the toroidal axis e_3; λ_n are the eigenvalues of the curl operator. Moreover, one demonstrates that $\mathbf{U}_n(x) = \Phi_n(x)$, $\mathbf{H}_n(x) = \gamma_n \Phi_n(x)$, where $\Phi_n(x)$ are the eigenfunctions of the curl operator in the cylinder:

$$\operatorname{curl} \Phi_n = \lambda_n \Phi_n, \quad \Phi_n \cdot N = 0 \ \text{ on } \ \Gamma. \tag{2.8}$$

For a full discussion of these *curl* eigenfunction-eigenvalues, see [26]. The eigenvalues occur by pairs $\pm\lambda_n$, such that $0 < |\lambda_1| \le |\lambda_2| \le \ldots \le |\lambda_n|, \ldots$ and $|\lambda_n| \to +\infty$ as $n \to +\infty$. The eigenvalues of $\mathbf{E}(t)$ are $e^{\pm i\mu_n t}$. More precisely,

Theorem 2.1. (i) *On the unit complex circle the spectrum of the Poincare–Sobolev–Alfven wave operator consists of dense, but countable eigenvalues* $e^{\pm i\mu_n t}$, μ_n *given by* (2.7) *with finite dimensional eigenspaces for each eigenvalue.*

(ii) $\mathbf{E}(t) = \mathrm{Id}$ *for* $(\mathbf{U}, \mathbf{H})$ *such that*

$$\begin{aligned}
&\mathbf{U} = \mathbf{U}(x_1, x_2) = (U_1(x_1, x_2), U_2(x_1, x_2), U_3(x_1, x_2)),\\
&\mathbf{H} = \mathbf{H}(x_1, x_2) = (H_1(x_1, x_2), H_2(x_1, x_2), H_3(x_1, x_2));\\
&\operatorname{div} \mathbf{U} = \operatorname{div} \mathbf{H} = 0,
\end{aligned} \tag{2.9}$$

that is $\mathbf{E}(t)$ *reduces to the identity on* 2 *dimensional,* 3 *components solenoidal fields.*

(iii) *On fields orthogonal to the 2d-3C fields of* (ii), $\mathbf{E}(t)$ *diagonalizes in the* curl *eigenvector basis.*

Remark 2.2. In the limit of small vertical scales $n_3 \to +\infty$, we know that $\lambda_n \to \pm\infty$, hence: $\mu_n/\epsilon \to \pm\check{n}_3/\epsilon\mathcal{H}_T$, and the eigenfunctions are equivalent to, as $n_3 \to +\infty$: $\Psi_n(r, m_2) exp(i\alpha(\pm n_3/\epsilon\mathcal{H}_T t + n_3 x_3))$, which are exactly the Alfven waves; here $\psi_n(r, m_2)$ are the radial-poloidal components of the curl eigenfunctions, m_2 are the poloidal (azimuthal) wavenumber.

Hence, we are fully justified calling the above waves Poincare–Sobolev–Alfven waves. Similarly, for λ_n and n_3 fixed, $\mathcal{H}_T \to +\infty$, we have again $\mu_n/\epsilon \to \pm\check{n}_3\mathcal{H}_T/\epsilon$, and convergence to high frequency Alfven waves. The latter limit is the case $\Omega \to +\infty$, $B_T/\Omega \to +\infty$.

Remark 2.3. We note that each Poincare–Sobolev–Alfven wave generates an exact solution to the full nonlinear, ideal 3D MHD equations in the original frame of reference, Eqs. (1.1)-(1.3):

$$\begin{aligned}
\mathbf{V}(t, y) &= \frac{\Omega}{2} e_3 \times y + R_{\Omega t}\Phi_n(R_{-\Omega t} y) exp(\pm i\mu_n \Omega t),\\
\mathbf{B}(t, y) &= B_T e_3 + \gamma_n R_{\Omega t}\Phi_n(R_{-\Omega t} y) exp(\pm i\mu_n \Omega t).
\end{aligned} \tag{2.10}$$

Such exact solutions are respectively the sum of a poloidal velocity field (resp. a toroidal magnetic field), plus fields of Arnold–Beltrami–Childress type; that is, where fields vector and its *curl* are parallel. Of course, the sum of two such solutions is not a solution of the nonlinear MHD equations.

3. The Limit Resonant MHD Equations and Long-Time Regularity

We first specify the function spaces appropriate for our problem. Let $H^s_\sigma(C)$ denote the usual Sobolev space of order $s \geq 0$, s integer, of solenoidal vector functions in the cylinder. Let $\mathbf{P}_\sigma$ denotes the usual projection on solenoidal vector fields with null normal component to the boundary Γ. We cannot use the above Sobolev spaces since $\mathbf{P}_\sigma$ does not commute with ∇, and $\mathbf{E}(t/\epsilon)$ is not an isometry on such spaces, contrary to the usual case, in periodic lattices or infinite domain. Inspired by the pioneering work of O. A. Ladyzhenskaya and coworkers [20], [23] we introduce *curl* norm spaces:

$$\begin{aligned} H^s_\nu = \{&\mathbf{W} : \operatorname{div} \mathbf{W} = 0, \\ &\operatorname{curl}^j \mathbf{W} \in L^2(C), \operatorname{curl}^j \mathbf{W} \cdot N = 0 \quad \text{on} \quad \Gamma, 0 \leq j \leq s\}, \end{aligned} \tag{3.1}$$

together with the norm

$$||\mathbf{W}||_s = ||\operatorname{curl}^s \mathbf{W}||_{L^2}, \tag{3.2}$$

which is equivalent to the usual Sobolev norm, together with Poincare inequality:

Proposition 3.1. *There exists constants $C_1, C_2 > 0$ such that*

$$C_1||\mathbf{W}||_{H^s_\sigma} \leq ||\mathbf{W}||_{H^s_\nu} \leq C_2||\mathbf{W}||_{H^s_\sigma}. \tag{3.3}$$

In [23] O. A. Ladyzhenskaya and V. A. Solonnikov use the above to establish short time existence in 3D as well long time existence in 2D for the viscous MHD equations. Using Kato's methods for 3D Euler, a short time existence has been established in bounded domains by P. G. Schmidt [31]; however, such estimates are not uniform in Ω (adding the singular Coriolis–Gyromagnetic terms) in the usual Sobolev spaces. In our curl *spaces short time estimates are uniform in ϵ and $\mathbf{E}(t/\epsilon)$ is an isometry [26].*

We denote the orthogonal decomposition $\mathbf{w} = \overline{\mathbf{w}} + \mathbf{w}^\perp$ where $\overline{\mathbf{w}}$ is the barotropic projection (vertical averaging)

$$\overline{\mathbf{w}}(x_1, x_2) = \frac{1}{L}\int_0^L \mathbf{w}(x_1, x_2, x_3)dx_3, \tag{3.4}$$

and the orthogonal field $\mathbf{w}^{\perp}(x_1, x_2, x_3)$ verifies $\overline{\mathbf{w}}^{\perp}(x_1, x_2, x_3) = 0$. Then

$$\mathbf{w} = \overline{\mathbf{w}} + \mathbf{w}^{\perp}. \tag{3.5}$$

Recall that for such projected 2d-3C fields, $\mathbf{E}(t/\epsilon) = \mathrm{Id}$. We introduce the Van der Pol transformation by setting

$$\mathbf{U}(t) = \mathbf{E}(-t/\epsilon)\mathbf{u}(t), \mathbf{B}(t) = \mathbf{E}(-t/\epsilon)\mathbf{b}(t), \tag{3.6}$$

where $\mathbf{u}(t)$, $\mathbf{b}(t)$ are the *slow envelope* variables also denoted *Poincare variable.* Eqs. (1.19) in the new 'slow' fields become

$$\begin{aligned} &\partial_t \mathbf{u}(t) = \mathbf{E}(t/\epsilon)(-\operatorname{curl} \mathbf{E}(-t/\epsilon)\mathbf{u} \\ &\times \mathbf{E}(-t/\epsilon)\mathbf{u} + \operatorname{curl} \mathbf{E}(-t/\epsilon)\mathbf{b} \times \mathbf{E}(-t/\epsilon)\mathbf{b}), \\ &\partial_t \mathbf{b}(t) = \mathbf{E}(t/\epsilon)(\operatorname{curl} \mathbf{E}(-t/\epsilon)\mathbf{u} \times \mathbf{E}(-t/\epsilon)\mathbf{u}). \end{aligned} \tag{3.7}$$

More concisely,

$$\partial_t(\mathbf{u}(t), \mathbf{b}(t)) = \mathbf{Q}_\epsilon(t/\epsilon, \mathbf{u}(t), \mathbf{b}(t)). \tag{3.8}$$

The fast singular limit equations by averaging the above $\mathbf{Q}_\epsilon$ in the sense of almost periodic functions with values in Banach spaces $AP(\mathbf{R}; C^0([0,T]; H^s_\nu))$ ([9], [16]):

$$\widetilde{\mathbf{Q}}(\mathbf{u}(t), \mathbf{b}(t)) = lim_{\epsilon\to 0}\frac{1}{T}\int_0^T \mathbf{Q}_\epsilon(s/\epsilon, \mathbf{u}(t), \mathbf{b}(t))ds, \tag{3.9}$$

which is permissible since the spectrum of frequencies μ_n/ϵ of $\mathbf{E}(s/\epsilon)$ is countable. For a full discussion, see [26].

To explicit the resonant limit MHD operator $\widetilde{\mathbf{Q}}(\mathbf{u}(t), \mathbf{b}(t))$, we decompose both fields as $\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}^{\perp}$, $\mathbf{b} = \overline{\mathbf{b}} + \mathbf{b}^{\perp}$, and expand $\mathbf{u}^{\perp}$, $\mathbf{b}^{\perp}$ in $\mathbf{Q}_\epsilon(s/\epsilon, \mathbf{u}(t), \mathbf{b}(t))$ along the eigenfunctions of curl and $\mathbf{E}(t/\epsilon)$. With indices k, m, n of the eigenfunction expansions of the quadratic forms in $\mathbf{Q}_\epsilon$, phase factors of the type $\exp(i(\pm\mu_k \pm \mu_m \pm \mu_n)t/\epsilon)$ appear and the resonant operator $\widetilde{\mathbf{Q}}$ is obtained by setting the resonances $\pm\mu_k \pm \mu_m \pm \mu_n = 0$. From the expressions for μ_n in Section 2, note that $n_3 = 0$ components correspond to the projection of $\mathbf{u}$, $\mathbf{b}$ on $\overline{\mathbf{u}}$, $\overline{\mathbf{b}}$, the 2d-3C components, that is vertical (toroidal) averaging. This leads to a classification of resonances:
(i) strict 3 waves resonances such that $\pm\mu_k \pm \mu_m \pm \mu_n = 0$ and $\mu_k\mu_m\mu_n \neq 0$;
(ii) 2 wave resonances such that $\mu_n = 0$ and $\mu_k\mu_m \neq 0$; $\mu_k = 0$ and

$\mu_n\mu_m \neq 0$; $\mu_m = 0$ and $\mu_k\mu_n \neq 0$;
(iii) trivial resonances, that is $\mu_k = \mu_m = \mu_n = 0$ together with $k_3 = m_3 = n_3 = 0$.

In fact, 2 wave resonances of the type $\mu_n = 0$, $\mu_k\mu_m \neq 0$ are eliminated by

Theorem 3.2. (*Operator Splitting*)

$$\overline{\widetilde{\mathbf{Q}}(\mathbf{u},\mathbf{b})} = \widetilde{\mathbf{Q}}(\overline{\mathbf{u}},\overline{\mathbf{b}}) = \mathbf{Q}_{2D}(\overline{\mathbf{u}},\overline{\mathbf{b}}), \tag{3.10}$$

where $\mathbf{Q}_{2D}$ stands for the 2D-3C ideal MHD operator:

$$\partial_t\overline{\mathbf{u}} + \overline{\mathbf{u}}\cdot\nabla_h\overline{\mathbf{u}} - \overline{\mathbf{b}}\cdot\nabla_h\overline{\mathbf{b}} + \nabla_h\overline{p} = 0, \tag{3.11}$$

$$\partial_t\overline{\mathbf{b}} = \overline{\mathbf{b}}\cdot\nabla_h\overline{\mathbf{u}} - \overline{\mathbf{u}}\cdot\nabla_h\overline{\mathbf{b}},$$

$$\operatorname{div}\overline{\mathbf{u}} = \operatorname{div}\overline{\mathbf{b}} = 0, \tag{3.12}$$

and h designates derivatives with respect to the poloidal coordinates $(x_1, x_2) = x_h$.

Theorem 3.3. *For $L/R \notin K^*$, where K^* is some countable,discrete set of cylinder aspect ratios, there are no 3 wave resonances, and no contribution to $\widetilde{\mathbf{Q}}(\mathbf{u},\mathbf{b})$ by 3-wave resonances.*

Theorem 3.4. *For aspect ratios $L/R \notin K^*$, the resonant limit equations for $\mathbf{u}^\perp$, $\mathbf{b}^\perp$ are:*

$$\partial_t(\mathbf{u}^\perp,\mathbf{b}^\perp) = \widetilde{\mathbf{Q}}_2((\overline{\mathbf{u}},\overline{\mathbf{b}}),(\mathbf{u}^\perp,\mathbf{b}^\perp)), \tag{3.13}$$

where the vector valued operator $\widetilde{\mathbf{Q}}_2$ is bilinear in $(\overline{\mathbf{u}},\overline{\mathbf{b}})$ and $(\mathbf{u}^\perp,\mathbf{b}^\perp)$. Eqs. (3.13) admit the new conservation laws for $(\mathbf{u}^\perp,\mathbf{b}^\perp)$:

$$\frac{d}{dt}(||\partial_{x_3}\mathbf{u}^\perp||^2_{L^2} + ||\partial_{x_3}\mathbf{b}^\perp||^2_{L^2}) = 0, \tag{3.14}$$

$$\frac{d}{dt}(||curl^j\mathbf{u}^\perp||^2_{L^2} + ||curl^j\mathbf{b}^\perp||^2_{L^2}) = 0 \tag{3.15}$$

for j even; $j = 0, 2, 4, \ldots$.

Remark 3.5. The resonant limit equations (3.13) involve only two-wave resonances and couple the fields $(\mathbf{u}^\perp,\mathbf{b}^\perp)$ with the 2D-3C fields $(\overline{\mathbf{u}},\overline{\mathbf{b}})$. These solutions are unique, and as smooth as the solutions of the 2D-3C ideal MHD equations, on the time of existence of the latter.

Given the time T_M of regularity of the 2D-3C ideal MHD Equations, with initial conditions $\overline{\mathbf{u}}(0) = \widetilde{\overline{\mathbf{V}}}_0$, $\overline{\mathbf{b}}(0) = \widetilde{\overline{\mathbf{B}}}_0$, we deduce the regularity of the original 3D ideal MHD equations (1.1)–(1.3), for $\Omega \geq \Omega^*$ and $B_T \geq B_T^*$, on the same time interval $0 \leq t \leq T_M$. This is achieved by uniform control of fast oscillating error terms and bootstrapping methods, similarly to Section 4 in [26]. However, *long-time* regularity of even the 2D-2C components ideal MHD equations is a major open problem [10], [32], in contrast to the viscous case [20], [23]. In general, global existence has only been proven for small enough data [32]. However, physicists are unable to observe singularities for 2D ideal MHD turbulence, even with high resolution numerical simulations [28] and are now of the opinion that 'current sheets' do not generate singularities in 2D MHD turbulence [17].

The mathematical difficulty is the lack of control of enstrophy. There is however a reasonable set of initial conditions for which this is still true. Denote by $\overline{\mathbf{u}}_h$ the horizontal poloidal component of $\overline{\mathbf{u}} = (\overline{\mathbf{u}}_h, \overline{u}_3)$, respectively, $\overline{\mathbf{b}} = (\overline{\mathbf{b}}_h, \overline{b}_3)$.

Proposition 3.6. *Let $\overline{\mathbf{b}}(0, x_h) = 0$, where*

$$\overline{\mathbf{b}}(t, x_h) = \frac{1}{L}\int_0^L \mathbf{H}(t, x)dx_3, \tag{3.16}$$

then $\overline{\mathbf{b}}(t, x_h) = 0$ for all t, for the solutions of the 2d-3C limit resonant MHD equations.

The latter reduce to

$$\begin{aligned} &\partial_t \overline{\mathbf{u}}_h + \overline{\mathbf{u}}_h \cdot \nabla_h \overline{\mathbf{u}}_h + \nabla_h \overline{p} = 0, \\ \partial_t u_3 + \overline{\mathbf{u}}_h \cdot \nabla_h u_3 = 0 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.17) \\ &\partial_t b_3 = -\overline{\mathbf{u}}_h \cdot \nabla_h b_3 \\ \operatorname{div} \overline{\mathbf{u}}_h = div \overline{\mathbf{b}}_h = 0, & \qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.18) \end{aligned}$$

which admit smooth solutions on arbitrary large time intervals.

Proposition 3.6 is at the core of Theorem 1.1.

Remark 3.7. The solutions to the full resonant 3D MHD equations do admit a non-trivial poloidal component for $\mathbf{b}^{\perp}(t, x)$ with $\overline{\mathbf{b}}^{\perp} = 0$. They are genuinely three-dimensional.

Note that for the solutions $\mathbf{U}(t,x)$, $\mathbf{H}(t,x)$ of the full 3D Equations (1.19), even for ϵ small, the initial condition is not invariant under the 3D MHD dynamics. A poloidal component will build up.

Remark 3.8. If the 2D-2C components MHD equations possess an arbitrary long time of existence for general initial conditions, then Theorem 1.1 can be replaced by a conditional theorem of non blowup. That is, 3D blow-up occurs if and only if there is blow-up for the 2D-2C MHD equations.

Remark 3.9. For the null measure set of parametric resonances for L/R, we are investigating the possibility of finite time blow-up, even when the 2D-2C MHD equations are regular.

Acknowledgment. We would like to thank Professors V. I. Arnold, C. Foias, V. A. Solonnikov, H. K. Moffatt, G. Papanicolaou and Ya. Sinai for the very useful discussions. O. A. Ladyzhenskaya encouraged this work. We acknowledge the support of the AFOSR grant F49620-02-10026.

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Поступило 20 декабря 2004 г.

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