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LOWER SEMICONTINUITY OF SOME FUNCTIONALS UNDER THE PDE CONSTRAINTS: $\mathcal{A}$–QUASICONVEX PAIR

ABSTRACT. The problem of establishing necessary and sufficient conditions for l.s.c. under the PDE constraints is studied for some special class of functionals:

$$(u, v, \chi) \mapsto \int\limits_{\Omega} \Big\{ \chi(x) \cdot F^{+}(x, u(x), v(x)) + (1 - \chi(x)) \cdot F^{-}(x, u(x), v(x)) \Big\} dx,$$

with respect to the convergence $u_n \to u$ in measure, $v_n \rightharpoonup v$ in $L_p(\Omega; \mathbb{R}^d)$, $\mathcal{A}v_n \to 0$ in $W^{-1,p}(\Omega)$ and $\chi_n \rightharpoonup \chi$ in $L_p(\Omega)$, where $\chi_n \in Z := \{\chi \in L_\infty(\Omega) \ : \ 0 \le \chi(x) \le 1, \ a.e.\ x\}$. Here $\mathcal{A}v = \sum_{i=1}^{N} A^{(i)} \frac{\partial v}{\partial x_i}$ is a constant rank partial differential operator.

The main result is that if the characteristic cone of $\mathcal{A}$ has the full dimension, then l.s.c. is equivalent to the fact that $F^{\pm}$ are both $\mathcal{A}$–quasiconvex and for a.e. $x \in \Omega$, for all $u \in \mathbb{R}^d$

$$F^{+}(x, u, \cdot) - F^{-}(x, u, \cdot) \equiv C(x, u).$$

As a corollary, we obtain the results for the functional

$$(u, v, \chi) \mapsto \int\limits_{\Omega} \chi(x) \cdot f(x, u(x), v(x))\, dx,$$

with respect to the same convergence. We show, that this functional is l.s.c. iff

$$f(x, u, v) \equiv g(x, u).$$

1. Introduction and statement of the problem

The property of sequential weak lower semicontinuity of a functional $I[u]$ defined on a Banach space X:

$$I[u] \le \liminf_{n \to \infty} I[u_n] \quad \text{for every sequence} \quad u_n \rightharpoonup u$$

plays a crucial role for applying Direct Methods of the Calculus of Variations. The existence of minimizers of the functional $I[u]$ can be proved by exploiting this property, it turns out to be useful for studying some properties of minimizers. There has been extensive research on this topic, we refer, for example, to [1, 3, 4, 8, 11, 12].

We treat the following problem: to establish a necessary and sufficient condition for the lower semicontinuity of the functional

$$I[u, v, \chi] = \int_\Omega g(x, u(x), v(x), \chi(x))\, dx, \tag{1}$$

where $g : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ has the following special form

$$g(x, u, v, \chi) = \chi \cdot F^+(x, u, v) + (1 - \chi) \cdot F^-(x, u, v),$$

and, thus

$$I[u, v, \chi] = \int_\Omega \Big\{ \chi(x) \cdot F^+(x, u(x), v(x)) + (1 - \chi(x)) \cdot F^-(x, u(x), v(x)) \Big\} dx, \tag{2}$$

where $F^\pm : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$ are Carathéodory functions, $\Omega \subset \mathbb{R}^N$ is a bounded domain, with respect to the convergence

$$u_n \to u \quad \text{in measure}, \qquad Z \ni \chi_n \rightharpoonup \chi \quad \text{in} \quad L_p(\Omega),$$

$$v_n \rightharpoonup v \quad \text{in} \quad L_p(\Omega; \mathbb{R}^d) \quad \text{and} \quad \mathcal{A} v_n \to 0 \quad \text{in} \quad W^{-1,p}(\Omega), \tag{3}$$

where $Z = \{\chi \in L_\infty(\Omega) \ : \ 0 \le \chi(x) \le 1, \ a.e.\ x\}$. Here

$$\mathcal{A} v = \sum_{i=1}^N A^{(i)} \frac{\partial v}{\partial x_i} \tag{4}$$

is a constant rank partial differential operator, where $A^{(i)} \in \mathrm{Lin}(\mathbb{R}^d; \mathbb{R}^l)$.

The functionals of this type find their application, for instance, in mechanics for describing the non-regularized energy of two-phase medium in the case of zero surface tension (see, for example, [12, 13]), and in the optimal design of thin films (see [2, 9]).

2. Main results

Following [8], let us introduce the following notations:

$$\mathbb{A}(w) = \sum_{i=1}^N A^{(i)} w_i \in \mathrm{Lin}(\mathbb{R}^d; \mathbb{R}^l), \quad w \in \mathbb{R}^N.$$

Recall, that the operator $\mathcal{A}$, defined in (4), is said to be a constant rank partial differential operator, if there exists $r \in \mathbb{N}$, such that

$$\operatorname{rank} \mathbb{A}(w) = r \quad \text{for all} \quad w \in S^{N-1}.$$

In the sequel we will always assume that $\mathcal{A}$ is a constant rank partial differential operator. The characteristic cone of the operator $\mathcal{A}$ is defined as follows (see also [14]):

$$\Lambda = \bigcup_{w \in S^{N-1}} \ker \mathbb{A}(w).$$

Recall the classical result (see [8]), stating a necessary and sufficient condition for lower semicontinuity of the functional

$$I_0[u, v] = \int_{\Omega} f(x, u(x), v(x))\, dx$$

under suitable growth conditions imposed on f and the constant rank assumption, with respect to the convergence

$$u_n \to u \quad \text{in measure},$$

$$v_n \rightharpoonup v \quad \text{in} \quad L_p(\Omega; \mathbb{R}^d) \quad \text{and} \quad \mathcal{A}v_n \to 0 \quad \text{in} \quad W^{-1,p}(\Omega).$$

A function f satisfying this condition is said to be $\mathcal{A}-$quasiconvex. Denote by T_N the torus $\mathbb{R}^N/\mathbb{Z}^N$, and by $C^\infty(T_N; \mathbb{R}^d)$ the space of smooth periodic functions.

Definition 2.1. *A function $f : \mathbb{R}^d \to \mathbb{R}$ is called $\mathcal{A}-$quasiconvex, if*

$$f(v) \leq \int_{T_N} f(v + w(x))\, dx$$

for all $v \in \mathbb{R}^d$ and $w \in C^\infty(T_N; \mathbb{R}^d)$ such that $\mathcal{A}(w) = 0$ and $\int_{T_N} w(x)\, dx = 0$.

Let us introduce the following definition.

Definition 2.2. *We say, that Carathéodory functions $F^\pm$ generate an $\mathcal{A}$-quasiconvex pair, if for a.e. $x \in \Omega$, for each $u \in \mathbb{R}^m$, $v \in \mathbb{R}^d$, and all*

$w \in C^\infty(T_N, \mathbb{R}^d)$, such that $\mathcal{A}w = 0$, $\int_{T_N} w(x)dx = 0$, for all $\chi \in \widehat{Z}$, the following inequality holds

$$F^+(x,u,v) \cdot \int\limits_{T_N} \chi(y)\,dy + F^-(x,u,v) \cdot \int\limits_{T_N} (1-\chi(y))\,dy \leq$$

$$\leq \int\limits_{T_N} \Big\{ \chi(y) \cdot F^+(x,u,v+w(y)) + (1-\chi(y)) \cdot F^-(x,u,v+w(y)) \Big\}\,dy,$$

where

$$\widehat{Z} = \{\chi \in L_\infty(T_N) \mid 0 \leq \chi(x) \leq 1, \text{ a.e. } x \in \Omega\}.$$

In fact, this definition is a particular case of $\mathcal{A}-$quasiconvexity, the so-called convex$-\mathcal{A}$-quasiconvexity (see, [5, 10] and also section 5 of this paper). However, this definition proves to be very convenient for establishing the fact that the difference $F^+(x,u,\cdot) - F^-(x,u,\cdot)$ is independent of v for a.e. $x \in \Omega$, for all $u \in \mathbb{R}^m$ (the implication ($2 \Rightarrow 3$) of Theorem 2.2 below).

Remark 2.1. Since $F^\pm$ are Carathéodory functions, in the above definition it is possible to replace $C^\infty(T_N; \mathbb{R}^d)$ by $L_\infty(T_N; \mathbb{R}^d)$. To prove it, for a given function $w \in L_\infty(T_N; \mathbb{R}^d) \cap \ker \mathcal{A}$ with $\int_{T_N} w\,dx = 0$, consider the mollified sequence

$$w_\varepsilon = \rho_\varepsilon * w - \int\limits_{T_N} \rho_\varepsilon * w\,dx \in C^\infty(T_N) \cap \ker \mathcal{A}, \qquad \int\limits_{T_N} w_\varepsilon\,dx = 0,$$

where ρ_ε is a mollifying kernel. Now the Lebesgue dominated convergence Theorem yields the required result.

Suppose, that $F^\pm : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$ are Carathéodory functions satisfying the growth conditions:

$$0 \leq F^\pm(x,u,v) \leq C(|v|^p + \beta(|u|)) + \alpha(x), \tag{5}$$

where C is a positive constant, $\beta(t)$ is a locally bounded function and $\alpha \in L_1(\Omega)$, $\alpha(x) \geq 0$.

The following theorems are the main results of this paper.

Theorem 2.1. *Let $F^\pm$ be Carathéodory functions satisfying the growth conditions (5). Then the following statements are equivalent:*

1. *The functional I, defined in (1), is l.s.c. with respect to the convergence (3).*

2. *The functions* $F^{\pm}$ *generate an* $\mathcal{A}$*-quasiconvex pair in the sense of Definition* 2.2.

A more constructive description of an $\mathcal{A}$-quasiconvex pair is possible in some cases. Let us suppose that the dimension of the characteristic cone of the operator $\mathcal{A}$ is full, that is $\dim \Lambda = d$.

Theorem 2.2. *Let* $F^{\pm}$ *be Carathéodory functions satisfying the growth conditions* (5). *Let us suppose that* $\dim \Lambda = d$. *Then the following statements are equivalent:*

1. *The functional* I, *defined in* (1), *is l.s.c. with respect to the convergence* (3).
2. *The functions* $F^{\pm}$ *generate an* $\mathcal{A}$*-quasiconvex pair in the sense of Definition* 2.2.
3. $F^{\pm}$ *are* $\mathcal{A}$*-quasiconvex functions (Definition 2.1) and there exists a set of full measure* Ω_0, *such that for all* $x \in \Omega_0$, $u \in \mathbb{R}^d$,

$$F^{+}(x, u, \cdot\,) - F^{-}(x, u, \cdot\,) \equiv C(x, u). \tag{6}$$

Having this Theorem, now let us turn to the problem where the integrand is linear with respect to the variable, which is free of the PDE constraints. Suppose, that we are given a function $F : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, which is of the form

$$F(x, u, v, \chi) = \chi \cdot f(x, u, v),$$

where $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$ is a Carathéodory function, satisfying the following growth conditions:

$$0 \le f(x, u, v) \le C(|v|^p + \beta(|u|)) + \alpha(x). \tag{7}$$

We are interested in the conditions for the lower semicontinuity of the functional

$$(u, v, \chi) \mapsto \int\limits_{\Omega} F(x, u(x), v(x), \chi(x))\, dx = \int\limits_{\Omega} \chi(x) \cdot f(x, u(x), v(x))\, dx, \tag{8}$$

with respect to the convergence (3).

Then the following theorem provides a necessary and sufficient condition for the lower semicontinuity of I_1 under the full rank condition imposed on the operator $\mathcal{A}$:

Theorem 2.3. *Let $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$ a Carathéodory function satisfying the growth conditions* (7). *Suppose that* $\dim \Lambda = d$. *Then the functional I_1 defined in* (8) *is lower semicontinuos with respect to the convergence* (3) *if and only if f is independent of v:*

$$f(x, u, v) = g(x, u)$$

for some Carathéodory function $g : \Omega \times \mathbb{R}^m \to \mathbb{R}$.

Proof. It suffices to apply Theorem 2.2 with $F^+(x, u, v) \equiv f(x, u, v)$ and $F^-(x, u, v) \equiv 0$. $\square$

In [12] the equivalence ($1 \Leftrightarrow 2$) was proved in the particular case $\mathcal{A} = \mathrm{curl}$ for uniformly continuous functions $F^{\pm}$, and also the equivalence ($2 \Leftrightarrow 3$) was obtained for C^1 functions $F^{\pm}$. In [4] Theorems 2.1 and 2.2 were established for $\mathcal{A} = \mathrm{curl}$ and arbitrary Carathéodory functions $F^{\pm}$.

The equivalence ($1 \Leftrightarrow 2$) can be obtained as a particular case of the $\mathcal{A}$-quasiconvexity results proved in [8]. However, in our case we are having more general growth conditions, imposed on $F^{\pm}$ (to be more precise, in our case $\alpha \in L_1(\Omega)$, and not in $L_\infty(\Omega)$). This is why we will prove the implication ($1 \Rightarrow 2$), making necessary changes, caused by this fact (which requires a more careful selection of the sets), closely following the proof in [8]. While the implication ($2 \Rightarrow 1$) can be obtained by direct application of the $\mathcal{A}$-quasiconvexity (for more details, see section 5), since in the proof in [8], the property of α to be in L_∞ was only used to establish the implication ($1 \Rightarrow 2$).

The outline of the paper is the following: in section 3 we present some results to be used in the sequel. In section 4 the implication ($1 \Rightarrow 2$) of the Theorems 2.1 and 2.2 is proved. In section 5 we will derive the implication ($2 \Rightarrow 1$) of these Theorems from $\mathcal{A}$-quasiconvexity theory. The implication ($2 \Rightarrow 3$) of the Theorem 2.2 is established in section 6.

Note, that the implication ($3 \Rightarrow 2$) is a straightforward consequence of $\mathcal{A}$-quasiconvexity of the function F^-.

3. Preliminary results

Recall some well-known properties of $\mathcal{A}$-quasiconvex functions (the following two Theorems were proved in [8]). Note, that these results heavily rely upon the constant rank condition, which we will always assume to hold.

Though the condition of $\mathcal{A}$-quasiconvexity is of integral type, there is a necessary algebraic condition for $\mathcal{A}$-quasiconvexity: convexity along lines from the characteristic cone Λ of the operator $\mathcal{A}$ (which is an analogue of rank-one convexity property of quasiconvex functions).

Theorem 3.1. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a continuous, $\mathcal{A}$-quasiconvex function. Then the restriction of f to every cone $a+\Lambda$, $a \in \mathbb{R}^d$ is a convex function, that is,*

$$f(\theta\, y + (1-\theta)\, z) \leq \theta\, f(y) + (1-\theta) f(z)$$

for all $\theta \in (0,1)$, $y, z \in \mathbb{R}^d$ such that $y - z \in \Lambda$, where

$$\Lambda = \bigcup_{w \in S^{N-1}} \ker \mathbb{A}(w).$$

We will also make use of the following theorem.

Theorem 3.2. *Let $\{u_n\}$ be a sequence such that $u_n \overset{*}{\rightharpoonup} u$ in $L_\infty(T_N;\mathbb{R}^d)$, $\mathcal{A}u_n \rightharpoonup 0$ in $L_p(T_N)$ for some $p > N$. Then there exists a sequence of functions $\{v_n\} \subset L_\infty(T_N;\mathbb{R}^d) \cap \ker \mathcal{A}$, such that*

$$\int_{T_N} v_n\, dx = \int_{T_N} u\, dx, \quad \|v_n - u_n\|_{L_\infty(T_N)} \to 0.$$

4. The implication ($1 \Rightarrow 2$) of Theorems 2.1 and 2.2

During the proof we will use methods, close to the ones exploited in [8]. The similar statement for $\alpha \in L_1(\Omega)$ for the particular case $\mathcal{A} = \mathrm{curl}$ was established in [4].

We use the following notations: $Q = (0,1)^N$ denotes the unit cube in $\mathbb{R}^N$, $Q^* = Q - (1/2,\dots,1/2)$ is Q, centered at the origin. For a point $x_0 \in \mathbb{R}^N$, $Q(x_0, r) = x_0 + rQ^*$. Given an $\mathcal{L}^N$-measurable set A, $|A|$ stands for its N-dimensional Lebesgue measure.

Remark 4.1. Without loss of generality, we will assume that $F^{\pm}$ are independent of u, that is, $F^{\pm} : \Omega \times \mathbb{R}^d \to \mathbb{R}$, and, using the rescaling arguments, we will suppose that $Q \subset \Omega$.

Theorem 4.1. *If the functional I is lower semicontinuous with respect to the convergence*

$$v_n \overset{*}{\rightharpoonup} v \quad in \quad L_\infty(\Omega;\mathbb{R}^d), \quad \mathcal{A}v_n = 0, \quad \chi_n \overset{*}{\rightharpoonup} \chi \quad in \quad L_\infty(\Omega)$$

and Carathéodory functions $F^{\pm}$ satisfy the growth condition (5)*, then the functions $F^{\pm}$ generate an $\mathcal{A}$-quasiconvex pair in the sense of Definition 2.2.*

Proof. We will carry out the proof in 2 steps. At the first step, we will carefully choose the necessary sets and construct the sequences of weakly convergent functions. At the second step, we will prove the inequality of the $\mathcal{A}$-quasiconvex pair.

1. Selection of constants and sets and construction of weakly convergent functions

By the Scorza–Dragoni Theorem (see [6]), for all $i \in \mathbb{N}$ there exists a compact set $\widetilde{K}_i \subset \Omega$ such that the restrictions of $F^{\pm}$ to $\widetilde{K}_i \times \mathbb{R}^d$ are continuous and $|\Omega \setminus \widetilde{K}_i| < 1/i$. Let $\mathcal{S}$ be a countable dense (with respect to the L_p norm) subset of $\mathbb{W} = \{w \in C^\infty(T_N) : \ \mathcal{A}w = 0, \ \int\limits_{T_N} w\,dx = 0\}$, and $\mathbb{Y}$ be a countable dense subset of $\widehat{Z}$.

Let x_0 be a Lebesgue point for the following functions:

$$\begin{gathered} x \mapsto F^{\pm}(x, v), \\ x \mapsto \int\limits_Q \Big\{ \chi(y) \cdot F^+(x, v + w(y)) + (1 - \chi(y)) \cdot F^-(x, v + w(y)) \Big\} dy, \end{gathered} \tag{9}$$

for all $v \in \mathbb{Q}^d$, $w \in \mathcal{S}$, $\chi \in \mathbb{Y}$, and let us choose x_0 to be such that the mappings $\mu \mapsto F^{\pm}(x_0, \mu)$ are continuous.

Take arbitrary $v \in \mathbb{Q}^d$, $\chi \in \mathbb{Y}$ and $w \in \mathcal{S}$. Extend w and χ periodically with period Q to $\mathbb{R}^N$.

Take $h > 0$ and fix $\varepsilon = \varepsilon(h) > 0$, $\varepsilon < \frac{1}{h}$ such that the following inequality holds

$$\int\limits_G \alpha(x) dx < \frac{1}{h^{N+1}} \quad \text{if} \quad |G| < \frac{\varepsilon}{h^N}(2\|\alpha\|_{L_1(\Omega)} + 1). \tag{10}$$

Choose $i = i(h, \varepsilon(h))$ such that $|Q(x_0, 1/h) \setminus \widetilde{K}_i| < \varepsilon/(h^{2N+1})$. Let us consider the set $E = \{x \in Q(x_0, \frac{1}{h}) \mid \alpha(x) > \frac{h^N}{\varepsilon}\}$, then the Chebyshev inequality yields

$$|E| < \frac{\varepsilon}{h^N} \cdot \int\limits_E \alpha(x) dx \le \frac{\varepsilon}{h^N} \cdot \|\alpha\|_{L_1(\Omega)}. \tag{11}$$

From these relations and (10) it follows that

$$\int_E \alpha(x)\,dx < \frac{1}{h^{N+1}}. \tag{12}$$

There exists an open set U, containing E, such that $|U \setminus E| < \frac{\varepsilon}{h^N} \cdot \|\alpha\|_{L_1(\Omega)}$. Then $|U| < 2 \cdot \frac{\varepsilon}{h^N} \cdot \|\alpha\|_{L_1(\Omega)}$.

Finally, define a new compact set $K_i = \widetilde{K}_i \cap (Q(x_0, 1/h) \setminus U)$. Since $Q \setminus U \subset Q \setminus E$, the following estimate is valid

$$\alpha(x) \leq \frac{h^N}{\varepsilon}, \quad \text{for} \quad x \in K_i, \tag{13}$$

and we have the estimate of the measure of its complement:

$$\begin{aligned} \left|Q\left(x_0, \frac{1}{h}\right) \setminus K_i\right| &\leq \left|Q\left(x_0, \frac{1}{h}\right) \setminus \widetilde{K}_i\right| + |U| \\ \leq \frac{\varepsilon}{h^N} \cdot \left(2 \cdot \int_E \alpha(x)dx + \frac{1}{h^{N+1}}\right) &\leq \frac{\varepsilon}{h^N} \cdot (2\|\alpha\|_{L_1(\Omega)} + 1). \end{aligned} \tag{14}$$

By (10) we obtain

$$\int_{Q(x_0, \frac{1}{h}) \setminus K_i} \alpha(x)dx < \frac{1}{h^{N+1}}. \tag{15}$$

Moreover, the second inequality in (14) and (12) imply, that the estimate of the complement can be improved:

$$\left|Q\left(x_0, \frac{1}{h}\right) \setminus K_i\right| \leq 3 \cdot \frac{\varepsilon}{h^N} \cdot \frac{1}{h^{N+1}}. \tag{16}$$

Remark 4.2. As a result, we get a compact set K_i, where estimate (13) of the values of α holds, whose measure of the complement is sufficiently small, and such that the integral of α over the complement is sufficiently small (by choosing $\varepsilon = \varepsilon(h)$ small enough).

Put $M_1 = \|w\|_{L_\infty(Q)} + |v|$. Let $n = n(h, \varepsilon)$ be such that

$$|x - x'| < \frac{\sqrt{N}}{n}, \quad x, x' \in K_i, \quad |\mu| \leq M_1 \Rightarrow |F^{\pm}(x, \mu) - F^{\pm}(x', \mu)| < \varepsilon. \tag{17}$$

Decompose the cube $Q(x_0, 1/h)$ as the disjoint union of n^N cubes:

$$Q\left(x_0, \frac{1}{h}\right) = \bigcup_{j=1}^{n^N} Q\left(x_j, \frac{1}{hn}\right)$$

where x_j are their centers. For each $j = 1, \ldots, n^N$, such that $|K_i \cap Q(x_j, \frac{1}{hn})| > 0$, let us choose a_j from this intersection, for which the growth estimate (5) is true.

Choose a cut-off function $\varphi \in C_0^\infty(Q(x_0, 1/h))$, such that $|Q(x_0, 1/h) \cap \{\varphi \neq 1\}| < \frac{\varepsilon}{h^{N+1}}$. Define

$$w_k(x) = \begin{cases} \varphi(x) w^*(hkn(x - x_j)), & \text{if } x \in Q\left(x_j, \frac{1}{hn}\right), \ j = 1, \ldots, n^N; \\ 0, & \text{if } x \in \mathbb{R}^N \setminus Q\left(x_0, \frac{1}{h}\right). \end{cases}$$

where $w^*(y) = w(y + (1/2, \ldots, 1/2))$ for $y \in Q^*$. The periodicity of w implies that $w_k \in C^\infty(T_N)$. It is easy to check that

$$w_k \stackrel{*}{\rightharpoonup} 0 \quad \text{in} \quad L_\infty(T_N), \qquad \mathcal{A} w_k \stackrel{*}{\rightharpoonup} 0 \quad \text{in} \quad L_\infty(T_N).$$

Analogously, define

$$\chi_k(x) = \begin{cases} \chi^*(hkn(x - x_j)), & \text{if } x \in Q\left(x_j, \frac{1}{hn}\right), \ j = 1, \ldots, n^N; \\ 0, & \text{if } x \in \mathbb{R}^N \setminus Q\left(x_0, \frac{1}{h}\right), \end{cases}$$

where $\chi^*(y) = \chi(y + (1/2, \ldots, 1/2))$ for $y \in Q^*$. It is clear, that $\chi_k \stackrel{*}{\rightharpoonup} \overline{\chi} = \int_Q \chi(y) dy$, in $L_\infty(Q(x_0, 1/h))$.

2. Proof of the inequality

We have a sequence of functions $\chi_k \stackrel{*}{\rightharpoonup} \overline{\chi}$ in $L_\infty(Q(x_0, 1/h))$ and $w_k \stackrel{*}{\rightharpoonup} 0$ in $L_\infty(T_N)$, such that $\mathcal{A} w_k \stackrel{*}{\rightharpoonup} 0$ in $L_\infty(T_N)$. By Theorem 3.2 there exists a sequence $\eta_k \in L_\infty(T_N; \mathbb{R}^d) \cap \ker \mathcal{A}$, such that $\|\eta_k - w_k\|_{L_\infty(\Omega)} \to 0$, and, hence,

$$\int_\Omega \Big\{\overline{\chi}\cdot F^+(x,v)+(1-\overline{\chi})\cdot F^-(x,v)\Big\}dx$$
$$\le \liminf_k \int_\Omega \Big\{\chi_k(x)\cdot F^+(x,v+\eta_k(x))$$
$$+(1-\chi_k(x))\cdot F^-(x,v+\eta_k(x))\Big\}dx$$
$$=\liminf_k \int_\Omega \Big\{\chi_k(x)\cdot F^+(x,v+w_k(x))$$
$$+(1-\chi_k(x))\cdot F^-(x,v+w_k(x))\Big\}dx.$$

We have

$$\int_{Q(x_0,1/h)} \Big\{\overline{\chi}\cdot F^+(x,v)+(1-\overline{\chi})\cdot F^-(x,v)\Big\}dx$$
$$\le \liminf_k \int_{Q(x_0,1/h)} \Big\{\chi_k(x)\cdot F^+(x,v+w_k(x))$$
$$+(1-\chi_k(x))\cdot F^-(x,v+w_k(x))\Big\}dx$$
$$=\liminf_k \Bigg[\sum_{j=1}^{n^N} \int_{Q(x_j,\frac{1}{hn})\cap K_i} \Big\{\chi_k(x)\cdot F^+(a_j,v+w_k(x))$$
$$+(1-\chi_k(x))\cdot F^-(a_j,v+w_k(x))\Big\}dx$$
$$+\sum_j \int_{Q(x_j,\frac{1}{hn})\cap K_i} \Big\{\chi_k(x)\Big(F^+(x,v+w_k)-F^+(a_j,v+w_k)\Big)$$
$$+(1-\chi_k(x))\Big(F^-(x,v+w_k)-F^-(a_j,v+w_k)\Big)\Big\}dx$$

$$+\sum_j \int\limits_{Q(x_j,\frac{1}{hn})\setminus K_i} \Big\{\chi_k(x)\cdot F^+(x,v+w_k(x))$$

$$+(1-\chi_k(x))\cdot F^-(x,v+w_k(x))\Big\}dx\Bigg]$$

$$=\liminf_k(I_1^k+I_2^k+I_3^k).$$

In the last inequality there are 3 sums. The growth condition imposed on $F^\pm$ and the inequalities (15) and (16) imply, that for the third sum the following estimate holds:

$$|I_3^k|\le C\cdot \int\limits_{Q(x_0,\frac{1}{h})\setminus K_i}\Big\{\alpha(x)+|M_1|^p\Big\}dx=\frac{O(1)}{h^{N+1}}. \tag{18}$$

Consider the second sum. Since $x,a_j\in K_i\cap Q(x_j,1/(hn))$, then as $\varepsilon<\frac{1}{h}$, it follows from (17)

$$|I_2^k|\le\varepsilon\cdot\left|Q\left(x_0,\frac{1}{h}\right)\cap K_i\right|<\frac{1}{h^{N+1}}.$$

Consider the first sum. Taking into account the estimate for $\{\varphi\ne 1\}$ and the fact that $a_j\in K_i$, we obtain

$$I_1^k=\sum_{j=1}^{n^N}\int\limits_{Q(x_j,\frac{1}{hn})}\Big\{\chi_k(y)\cdot F^+(a_j,v+w^*(hkn(y-x_j)))$$

$$+(1-\chi_k(y))\cdot F^-(a_j,v+w^*(hkn(y-x_j)))\Big\}dy+\frac{O(1)}{h^{N+1}}$$

$$=\sum_{j=1}^{n^N}\frac{1}{(hn)^N}\int\limits_Q\Big\{\chi(y)\cdot F^+(a_j,v+w(y))$$

$$+(1-\chi(y))\cdot F^-(a_j,v+w(y))\Big\}dy+\frac{O(1)}{h^{N+1}}.$$

At the moment we have proved that

$$\int\limits_{Q(x_0,1/h)}\Big\{\overline{\chi}\cdot F^+(x,v)+(1-\overline{\chi})\cdot F^-(x,v)\Big\}dx\le \tag{19}$$

$$\le \sum_{j=1}^{n^N} \int\limits_{Q(x_j,\frac{1}{hn})} \int\limits_{Q} \Big\{\chi(y)\cdot F^+(a_j, v+w(y))$$

$$+(1-\chi(y))\cdot F^-(a_j, v+w(y))\Big\}\,dy\,dx + \frac{O(1)}{h^{N+1}}$$

We claim that

$$\sum_{j=1}^{n^N} \int\limits_{Q(x_j,\frac{1}{hn})} \int\limits_{Q} \Big\{\chi(y)\cdot |F^+(x, v+w(y)) - F^+(a_j, v+w(y))|$$

$$+(1-\chi(y))\cdot |F^-(x, v+w(y)) - F^-(a_j, v+w(y))|\Big\}\,dy\,dx$$

$$= \left(\sum_{j=1}^{n^N} \int\limits_{Q(x_j,\frac{1}{hn})\cap K_i} \int\limits_{Q} + \sum_{j=1}^{n^N} \int\limits_{Q(x_j,\frac{1}{hn})\setminus K_i} \int\limits_{Q}\right)$$

$$\Big\{\chi(y)\cdot |F^+(x, v+w(y)) - F^+(a_j, v+w(y))|$$

$$+(1-\chi(y))\cdot |F^-(x, v+w(y)) - F^-(a_j, v+w(y))|\Big\}\,dy\,dx = J_1 + J_2 = \frac{O(1)}{h^{N+1}}.$$

Let us evaluate the second sum. The inequalities (13), (16) and the fact that $a_j \in K_i$ yield

$$\sum_{j=1}^{n^N} \int\limits_{Q(x_j,\frac{1}{hn})\setminus K_i} \alpha(a_j)\,dx \le \frac{h^N}{\varepsilon}\cdot \left|Q\left(x_0, \frac{1}{h}\right)\setminus K_i\right| \le 3\cdot \frac{1}{h^{N+1}} = \frac{O(1)}{h^{N+1}}.$$

Thus, from the growth estimate for $F^{\pm}$ and (15), (16) it follows that $J_2 = \frac{O(1)}{h^{N+1}}$.

Finally, let us consider the first sum. As $x, a_j \in K_i \cap Q(x_j, 1/(hn))$, then from (17) we deduce that $J_1 = \frac{O(\varepsilon)}{h^N} = \frac{O(1)}{h^{N+1}}$.

Continuing the estimate (19):

$$\int\limits_{Q(x_0,1/h)} \Big\{\overline{\chi}\cdot F^+(x,v) + (1-\overline{\chi})\cdot F^-(x,v)\Big\}\,dx \le$$

$$\sum_{j=1}^{n^N} \int\limits_{Q(x_j,\frac{1}{hn})} \int\limits_{Q} \Big\{\chi(y)\cdot F^+(x,v+w(y)) + (1-\chi(y))$$

$$\cdot F^-(x,v+w(y))\Big\}dy\,dx + \frac{O(1)}{h^{N+1}} \le$$

$$\le \int\limits_{Q(x_0,\frac{1}{h})} \int\limits_{Q} \Big\{\chi(y)\cdot F^+(x,v+w(y)) + (1-\chi(y))$$

$$\cdot F^-(x,v+w(y))\Big\}dy\,dx + \frac{O(1)}{h^{N+1}}.$$

Multiply this inequality by h^N and let $h \to \infty$. Since x_0 is a Lebesgue point for the functions in (9), we have

$$F^+(x_0,v)\cdot \int\limits_Q \chi(y)dy + F^-(x_0,v)\cdot \int\limits_Q (1-\chi(y))dy$$

$$\le \int\limits_Q \Big\{\chi(y)\cdot F^+(x_0,v+w(y)) + (1-\chi(y))\cdot F^-(x_0,v+w(y))\Big\}dy,$$

for all $v \in \mathbb{Q}^d$, $\chi \in \mathbb{Y}$ and $w \in \mathcal{S}$.

As $(\lambda,\mu) \to F^{\pm}(x_0,\lambda,\mu)$ are continuous, this inequality is valid for all $v \in \mathbb{R}^d$, $\chi \in \widehat{Z}$ and $w \in \mathbb{W}$. $\square$

5. The implication ($2 \Rightarrow 1$) of Theorems 2.1, 2.2

Now let us take the following viewpoint: suppose, we are given a function

$$g(x,u,v,\chi) = \chi\cdot F^+(x,u,v) + (1-\chi)\cdot F^-(x,u,v), \tag{20}$$

and we are interested in the conditions for lower semicontinuity of the functional

$$(u,v,\chi) \mapsto \int\limits_\Omega g(x,u(x),v(x),\chi(x))\,dx$$

with respect to the convergence (3). It is easy to see that it is a particular case of $\mathcal{A}$-quasiconvexity, where χ is free of PDE constraints, and the proof of the sufficiency of $\mathcal{A}$-quasiconvexity for lower semicontinuity presented in [8], is valid for the case $\alpha \in L_1(\Omega)$. Let us check, how this condition looks like in this particular case:

$$\overline{\chi} \cdot F^+(x,u,v) + (1-\overline{\chi}) \cdot F^-(x,u,v) = g(x,u,v,\overline{\chi})$$
$$\leq \int_\Omega g(x,u,v+w(y),\overline{\chi}+\chi(y))\,dy$$
$$= \int_\Omega \Big\{ (\overline{\chi}+\chi(y)) \cdot F^+(x,u,v+w(y)) + (1-\overline{\chi}-\chi(y)) \cdot F^-(x,u,v+w(y)) \Big\}\,dy$$

for a.e. $x \in \Omega$, for all $u \in \mathbb{R}^m$, $v \in \mathbb{R}^d$ and $w \in C_0^\infty(T_N)$, $\mathcal{A}w = 0$, $\int_{T_N} w\,dy = 0$ and $\chi \in C_0^\infty(T_N)$, $\int_{T_N} \chi\,dy = 0$, such that $0 \leq \overline{\chi}+\chi(y) \leq 1$. As it is easy to see, the definition of $\mathcal{A}$-quasiconvex pair implies that the function g, defined in (20) is $\mathcal{A}$-quasiconvex.

6. Proof of the implication $(2 \Rightarrow 3)$ of Theorem 2.2

First, we will establish one inequality, resembling rank-one convexity inequality for quasiconvex functions. Then we will obtain the desired result. The proof is carried out closely following the ideas, presented in [4] for the case $\mathcal{A} = \operatorname{curl}$.

Theorem 6.1. *Let $F^\pm$ be Caratheodory functions, generating an $\mathcal{A}$-quasiconvex pair. Then there exists a set of full measure Ω_0, such that for all $x \in \Omega_0$, for each $u \in \mathbb{R}^m$, $0 < \alpha < 1$, for all $y, z \in \mathbb{R}^d$, such that $y - z \in \Lambda$, the following inequality holds:*

$$\alpha \cdot F^+(x,u,y) + (1-\alpha) \cdot F^-(x,u,y)$$
$$\leq \alpha \cdot F^+(x,u,z) + (1-\alpha) \cdot F^-\left(x,u,y+\frac{\alpha}{1-\alpha}(y-z)\right). \tag{21}$$

Proof. Let us define the functions

$$\varphi_1(t) = \begin{cases} 1, & t \in (0,\alpha]; \\ 0, & t \in (\alpha,1); \end{cases} \qquad \varphi_2(t) = \begin{cases} 1, & t \in (0,\alpha]; \\ -\frac{\alpha}{1-\alpha}, & t \in (\alpha,1); \end{cases}$$

and extend them periodically on $\mathbb{R}$. Let $w \in S^{N-1}$ be such that $y - z \in \ker \mathbb{A}(w)$. Put

$$\chi_k(\eta) = \varphi_1(k\,\eta \cdot w)$$
$$v_k(\eta) = (z-y)\,\varphi_2(k\,\eta \cdot w).$$

Since $y - z \in \ker \mathbb{A}(w)$, it is easy to check that

$$\mathcal{A} v_k = 0 \quad \text{in} \quad W^{-1,p}(Q).$$

Also, it is easy to see, that $\chi_k \rightharpoonup \alpha$ in $L_p(Q)$ and $v_k \overset{*}{\rightharpoonup} 0$ in $L_\infty(Q)$. Besides, for every function $g \in C(\mathbb{R}^d \times \mathbb{R})$, the following holds:

$$g(v_k, \chi_k) \overset{*}{\rightharpoonup} \alpha \cdot g((z-y), 1) + (1-\alpha) \cdot g\left(-\frac{\alpha}{1-\alpha}\,(z-y), 0 \right) \quad \text{in} \quad L_\infty(Q).$$

Now, let us take functions $0 \leq \varphi, \psi_k \leq 1$, such that $\psi_k, \varphi \in C_0^\infty(Q; [0,1])$, and $|\{\psi_k = 1\}| = 1 - 2^{-k}$ and $|\{\varphi = 1\}| = 1 - \varepsilon$, $\varepsilon > 0$. Then it follows that

$$\overline{\chi}_k := \psi_k \, \chi_k \rightharpoonup \alpha \quad \text{in} \quad L_p(T_N),$$

and since $v_k \overset{*}{\rightharpoonup} 0$ in L_∞, we obtain

$$\mathcal{A}(\varphi \, v_k) = \sum_{i=1}^{N} A^{(i)} v_k \frac{\partial \varphi}{\partial x_i} \overset{*}{\rightharpoonup} 0 \quad \text{in} \quad L_\infty(T_N).$$

By Theorem 3.2, there exists a sequence $\overline{v}_k \in L_\infty(T_N; \mathbb{R}^d) \cap \ker \mathcal{A}$, such that

$$\int_{T_N} \overline{v}_k = 0, \quad \|\overline{v}_k - \varphi \, v_k\|_{L_\infty(Q)} \to 0.$$

Let Ω_0 consist of the points $x \in \Omega$ for which the functions $F^{\pm}(x, \cdot\,, \cdot\,)$ are continuous. For every $x \in \Omega_0$ and each $u \in \mathbb{R}^d$ let us introduce the continuous function

$$g(\mu, \xi) = \xi \cdot F^{+}(x, u, \mu) + (1 - \xi) \cdot F^{-}(x, u, \mu).$$

We have

$$\begin{aligned}
\lim_k \int_{T_N} g(y + \overline{v}_k(\eta), \, \overline{\chi}_k(\eta)) \, d\eta &= \lim_k \int_{T_N} g(y + \varphi(\eta) \, v_k(\eta), \; \psi_k(\eta) \, \chi_k(\eta)) \, d\eta \\
&\leq \lim_k \int_{T_N} g(y + v_k(\eta), \; \chi_k(\eta)) \, d\eta + C \, \varepsilon \\
&= \alpha \cdot F^{+}(x, u, z) + (1 - \alpha) \cdot F^{-}\left(x, u, y + \frac{\alpha}{1-\alpha}\,(y - z) \right) + C \, \varepsilon, \quad (22)
\end{aligned}$$

where $C = 2$, $\max\{|g(\mu,\xi)| \,:\, |\mu| \le |y| + 2\|v_k\|_{L_\infty},\ |\xi| \le 1\}$.

However, the definition of $\mathcal{A}$-quasiconvex pair and Remark 2.1, imply, that

$$\int\limits_{T_N} g(y + \overline{v}_k(\eta), \overline{\chi}_k(\eta))\, dx$$

$$= \int\limits_{T_N} \Big\{ \overline{\chi}_k(\eta) \cdot F^+(x,u,y+\overline{v}_k(\eta)) + (1-\overline{\chi}_k(\eta)) \cdot F^-(x,u,y+\overline{v}_k(\eta)) \Big\}$$

$$\ge F^+(x,u,y) \int\limits_{T_N} \overline{\chi}_k(\eta)\, d\eta + F^-(x,u,y) \int\limits_{T_N} (1-\overline{\chi}_k(\eta))\, d\eta$$

$$\to \alpha \cdot F^+(x,u,y) + (1-\alpha)\cdot F^-(x,u,y). \tag{23}$$

The inequalities (22), (23) and the fact that ε is arbitrary small yield the required result. $\square$

Theorem 6.2. *If $F^\pm$ are Caratheodory functions, generating an $\mathcal{A}$-quasiconvex pair, and $F^\pm$ satisfy the growth conditions (5), then for a.e. $x \in \Omega$, for all $u \in \mathbb{R}^m$ the difference does not depend upon the third variable:*

$$F^+(x,u,\cdot\,) - F^-(x,u,\cdot\,) = C(x,u).$$

Proof. The definition of $\mathcal{A}$-quasiconvex pair implies, that the functions $F^\pm$ are $\mathcal{A}$-quasiconvex. Thus, they are convex along every direction from the characteristic cone Λ of the operator $\mathcal{A}$.

We aim at proving, that $F^+ - F^-$ is constant along every line, such that $y - z \in \Lambda$. Since $\dim \Lambda = d$, every two points of $\mathbb{R}^d$ can be connected by a finite number of line segments of this type, and the theorem will be proved.

Let us take an arbitrary point $y_0 \in \mathbb{R}^d$, and a "direction" $y - z \in \Lambda$. Let us prove that the function $F^+(x,u,\cdot\,) - F^-(x,u,\cdot\,)$ is constant along this line.

Let us introduce the following notations:

$$f^+(\lambda) = F^+(x,u,y_0 + \lambda(y-z)), \quad \lambda \in \mathbb{R},$$

$$f^-(\lambda) = F^-(x,u,y_0 + \lambda(y-z)), \quad \lambda \in \mathbb{R}$$

and

$$g(\lambda) = F^+(x,u,y_0 + \lambda(y-z)) - F^-(x,u,y_0 + \lambda(y-z)), \quad \lambda \in \mathbb{R}.$$

It is clear, that $f^{\pm}$ are convex and continuous on the whole real line $\mathbb{R}$, and g is locally Lipschitz (as the difference of two convex, and, hence, locally Lipschitz functions) and of the class C^2 at the points of C^2-smoothness of the functions $f^{\pm}$. Then the following identity is valid:

$$g(t_1) - g(t_2) = \int\limits_{t_2}^{t_1} g'(\tau) d\tau.$$

Let us consider the set $\Sigma \subset \mathbb{R}$ of points, where the functions $f^{\pm}$ are of the class C^2 in the sense that the Taylor expansion holds at these points. Due to Alexandrov's Theorem (see, for instance, [7]), Σ is a set of full measure. We aim at proving that g is a constant, and it is sufficient to show, that $g'(t) = 0$, if $t \in \Sigma$.

So, let $t \in \Sigma$ be an arbitrary point.

Take an arbitrary $\lambda \in \mathbb{R}$. Denote by

$$y_1 = y_0 + t\,(y - z), \quad y_2 = y_0 + (t + \lambda)\,(y - z).$$

Now we will write an identity, and then use the inequality (21) with some fixed $\alpha \in (0, 1)$ (for instance, $\alpha = \frac{1}{2}$):

$$\begin{aligned}
0 = {} & \alpha\, F^+(x, u, y_1) + (1-\alpha) F^-(x, u, y_1) - \alpha F^+(x, u, y_1) - (1-\alpha)\, F^-(x, u, y_1) \\
& \leq \alpha \cdot F^+(x, u, y_2) + (1 - \alpha) \cdot F^-\left(x, u, y_1 + \frac{\alpha}{1-\alpha}(y_1 - y_2)\right) \\
& \qquad - \alpha\, F^+(x, u, y_1) - (1 - \alpha)\, F^-(x, u, y_1) \\
& = \alpha \cdot \Big(F^+(x, u, y_2) - F^-(x, u, y_2) - F^+(x, u, y_1) + F^-(x, u, y_1) \Big) \\
& \quad + \alpha\, F^-(x, u, y_2) + (1 - \alpha) \cdot F^-\left(x, u, y_1 + \frac{\alpha}{1-\alpha}(y_1 - y_2)\right) \\
& \qquad - F^-(x, u, y_1) = \alpha \cdot \Big(g(t + \lambda) - g(t) \Big) \\
& \quad + \alpha\, f^-(t + \lambda) + (1 - \alpha)\, f^-\left(t - \frac{\alpha}{1-\alpha}\,\lambda\right) - f^-(t). \qquad (24)
\end{aligned}$$

Since t is a point of C^2-smoothness for f^-, then

$$f^-(t + \lambda) = f^-(t) + Df^-(t)(\lambda) + \frac{1}{2} D^2 f^-(t) \lambda^2 + o(\lambda^2),$$

and

$$f^-\Big(t-\frac{\alpha}{1-\alpha}\,\lambda\Big)=f^-(t)+Df^-(t)\Big(-\frac{\alpha}{1-\alpha}\,\lambda\Big)$$
$$+\frac{1}{2}D^2f^-(t)\Big(\frac{\alpha}{1-\alpha}\,\lambda\Big)^2+o(\lambda^2).$$

Using these inequalities and the linearity of the differential, we obtain that the inequality (24) takes the form

$$0\le\alpha\cdot\Big(g(t+\lambda)-g(t)\Big)+\frac{\alpha}{2}D^2f^-(t)\cdot\lambda^2+$$
$$(1-\alpha)\frac{\alpha^2}{2(1-\alpha)^2}D^2f^-(t)\lambda^2+o(\lambda^2),$$

that is,

$$g(t)-g(t+\lambda)\le O(\lambda^2)\quad\text{for sufficiently small}\quad\lambda.$$

But now we recall, that t is a point, where there exists the derivative of the function g.

Finally, dividing the last inequality by λ for $\lambda>0,\ \lambda<0$, we have $g'(t)=0$. $\square$

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