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BOUNDARY PARTIAL REGULARITY FOR THE NAVIER–STOKES EQUATIONS

Abstract. We prove two conditions of local Hölder continuity for suitable weak solutions to the Navier–Stokes equations near the smooth curved part of the boundary of a domain. One of these condition has the form of the Caffarelli–Kohn–Nirenberg condition for the local boundedness of suitable weak solutions at the interior points of the space-time cylinder. The corresponding results near the plane part of the boundary were established earlier by G. Seregin.

1. Introduction and main results

Let $\Omega \subset \mathbb{R}^3$ be a domain of class C^2 and $Q_T = \Omega \times (0,T)$. Assume $\Gamma \subset \partial\Omega$ is a (relatively) open part of the boundary of Ω and consider the Navier–Stokes system near Γ:

$$\begin{gathered}\frac{\partial v}{\partial t} + (v\cdot\nabla)v - \Delta v + \nabla p = 0, \quad \nabla\cdot v = 0, \ \text{ in } \ Q_T, \\ v|_{\Gamma\times(0,T)} = 0,\end{gathered} \tag{1.1}$$

where $v = v(x,t) = (v_1(x,t), v_2(x,t), v_3(x,t))$ and $p = p(x,t)$ denote the unknown velocity vector and the unknown pressure of the fluid at the point $(x,t) \in Q_T$, respectively.

In this paper, we assume that Γ is C^2-uniform. This means that any point of Γ has some neighborhood of a fixed radius (which is the same for all points of Γ) in which $\partial\Omega$ can be represented as a graph of some C^2-function, and C^2-norms of these functions are uniformly bounded with respect to the points of Γ. To present this property in a more formal way we introduce some notations:

List of Basic Notation for Sets. For $x_0 \in \mathbb{R}^3$, $z_0 = (x_0, t_0)$, $y_0' \in \mathbb{R}^2$, $\rho > 0$ we introduce sets

$\mathbb{R}^3_+ = \{x \in \mathbb{R}^3 : x_3 > 0\}$,

$B(x_0,\rho) = \{x \in \mathbb{R}^3 : |x - x_0| < \rho\}$, $B_\rho = B(\rho) = B(0,\rho)$,

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$B^+(x_0,\rho)=\{x\in B(x_0,\rho): x_3>0\}$, $B^+_\rho=B^+(\rho)=B^+(0,\rho)$,
$Q(z_0,\rho)=B(x_0,\rho)\times(t_0-\rho^2,t_0)$, $Q(\rho)=Q(0,\rho)$,
$Q^+(z_0,\rho)=B^+(x_0,\rho)\times(t_0-\rho^2,t_0)$, $Q^+(\rho)=Q^+(0,\rho)$,
$\Omega(x_0,\rho)=\Omega\cap B(x_0,\rho)$, $\omega(z_0,\rho)=\Omega(x_0,\rho)\times(t_0-\rho^2,t_0)$,
$\Pi_\rho\equiv\Pi(\rho)=\mathbb{R}^3\times(-\rho^2,0)$, $\Pi^+_\rho\equiv\Pi^+(\rho)=\mathbb{R}^3_+\times(-\rho^2,0)$,
$K(y'_0,\rho)=\{y'\in\mathbb{R}^2: |y'-y'_0|<\rho\}$, $K_\rho=K(\rho)=K(0,\rho)$,

We denote also by $\partial' Q(z_0,\rho)$ the parabolic boundary of $Q(z_0,\rho)$: $\partial' Q(z_0,\rho)=(\partial B(x_0,\rho)\times(t_0-\rho^2,t_0))\cup(B(x_0,\rho)\times\{t=t_0-\rho^2\})$.

So, we assume that Γ possesses the following property:

Main Condition on Γ. *There exist positive numbers M and R_0 depending only on Γ such that for each point $x_0\in\Gamma$ we can choose a Cartesian coordinate system $\{y_i\}_{i=1}^3$ associated to the origin x_0, and some function $\varphi_{x_0}\in C^2(\overline{K}_{R_0})$ such that*

$$\Omega(x_0,R_0)\equiv\Omega\cap B(x_0,R_0)=\{y\in B_{R_0}: y_3>\varphi_{x_0}(y_1,y_2)\ \},\tag{1.2}$$

and

$$\varphi_{x_0}(0)=0,\quad \nabla\varphi_{x_0}(0)=0,\quad \sup_{|y'|\leqslant R_0}|\nabla^2\varphi_{x_0}|\leqslant M.\tag{1.3}$$

We emphasize that Main Condition on Γ provides the uniform estimate of C^2-norms of functions φ_{x_0}:

$$\sup_{x_0\in\Gamma}\|\varphi_{x_0}\|_{C^2(\overline{K}_{R_0})}\leqslant M'\equiv M(1+R_0+R_0^2).\tag{1.4}$$

We introduce also some more notations:

Algebraic and Other Notation. We use a convention on summation over repeated indexes. For u, $v\in\mathbb{R}^3$, A, $B\in\mathbb{M}^{3\times3}$ we denote

$$u\cdot v=u_iv_i\equiv\sum_{i=1}^3u_iv_i,\quad A:B=A_{ij}B_{ij},\quad u\otimes v=(u_iv_j)\in\mathbb{M}^{3\times3},$$

$$v_{,k}=\frac{\partial v}{\partial x_k},\quad \nabla v=(v_{i,j}),\quad |\Omega|=\text{meas }\Omega,$$

$\rightharpoonup$ and $\to$ are the weak and strong convergence, respectively,

For $\Omega\subset\mathbb{R}^3$ and $\omega\subset\mathbb{R}^3\times\mathbb{R}^1$ we denote by $[p]_\Omega$ and $(v)_\omega$ the spatial and total averages, respectively. For instance,

$$[p]_\Omega=\frac{1}{|\Omega|}\int\limits_\Omega p(x,t)\,dx,\quad (v)_\omega=\frac{1}{|\omega|}\int\limits_\omega v(x,t)\,dxdt.$$

Notation for Functional Spaces.

- $L_q(\Omega)$, $L_q(Q_T)$, $W^k_q(\Omega)$, $\overset{\circ}{W}{}^k_q(\Omega)$, $W^{-k}_q(\Omega)$ are the usual Lebesgue and Sobolev spaces, $L_q(\Omega,\mathbb{R}^k)$ is the Lebesgue space of functions on Ω with values in $\mathbb{R}^k$ etc, but (when it is clear from the context) we shall often omit the tangent space in notation for the spaces of vector-valued functions,
- $L_{s,r}(Q_T)\equiv L_r(0,T;L_s(\Omega))$, $\quad L_{s,\infty}(Q_T)\equiv L_\infty(0,T;L_s(\Omega))$,
  $\|f\|_{L_{s,r}(Q_T)}\equiv\Big(\int\limits_0^T\|f(\cdot,t)\|^r_{s,\Omega}\,dt\Big)^{1/r}$,
  $\|f\|_{L_{s,\infty}(Q_T)}\equiv\operatorname{esssup}_{t\in(0,T)}\|f(\cdot,t)\|_{L_s(\Omega)}$,
- $W^{1,0}_{s,r}(Q_T)\equiv L_r(0,T;W^1_s(\Omega))=\{u\in L_{s,r}(Q_T):\nabla u\in L_{s,r}(Q_T)\}$,
  $\|u\|_{W^{1,0}_{s,r}(Q_T)}\equiv\|u\|_{L_{s,r}(Q_T)}+\|\nabla u\|_{L_{s,r}(Q_T)}$,
- $W^{2,1}_{s,r}(Q_T)=\{u\in W^{1,0}_{s,r}(Q_T):\ \nabla^2u,\ \partial_tu\in L_{s,r}(Q_T)\}$, $[u]_{W^{2,1}_{s,r}(Q_T)}\equiv$ $\|\nabla^2u\|_{L_{s,r}(Q_T)}+\|\partial_tu\|_{L_{s,r}(Q_T)}$,
  $\|u\|_{W^{2,1}_{s,r}(Q_T)}\equiv\|u\|_{W^{1,0}_{s,r}(Q_T)}+[u]_{W^{2,1}_{s,r}(Q_T)}$.

Under appropriate conditions on Ω (see [7, 8]) existence of weak solutions of the initial-boundary value problem to system (1.1) is known. In this paper we study regularity of the so-called *boundary suitable weak solutions.*

Definition 1.1. *Assume $\Gamma\subset\partial\Omega$. We call a pair of functions (v,p) a boundary suitable weak solution to system* (1.1) *near $\Gamma_T\equiv\Gamma\times(0,T)$ if*

$$v\in L_{2,\infty}\cap W^{1,0}_2\cap W^{2,1}_{\frac98,\frac32}(Q_T;\mathbb{R}^3),\qquad p\in L_{\frac32}\cap W^{1,0}_{\frac98,\frac32}(Q_T),$$

(v,p) satisfy (1.1)*, and also satisfy the following local energy inequality near Γ_T:*

$$\begin{gathered}\int\limits_\Omega\zeta(x,t)|v(x,t)|^2\,dx+2\int\limits_0^t\int\limits_\Omega\zeta|\nabla v|^2\,dxdt'\leqslant\\ \leqslant\int\limits_0^t\int\limits_\Omega\left\{|v|^2\left(\frac{\partial\zeta}{\partial t}+\Delta\zeta\right)+v\cdot\nabla\zeta\left(|v|^2+2p\right)\right\}\,dxdt'.\end{gathered}\tag{1.5}$$

for a.e. $t\in(0,T)$ and all nonnegative functions $\zeta\in C^\infty_0(\mathbb{R}^3\times(0,T))$ vanishing in a neighborhood of the set $(\partial\Omega\setminus\Gamma)\times(0,T)$.

Our definition of suitable weak solutions near the boundary was originally introduced in [14], see also a discussion there. We only note that this class of suitable weak solutions is somewhat different from the one introduced in [2] in the internal case, as here we also require existence of integrable second spatial and first time derivative of v as well as first spatial derivatives of p.

Existence of boundary suitable weak solutions can be proved for various particular classes of domains, for example, for bounded domains with C^2-boundaries. The main results of the present paper are the following theorems on boundary regularity of such solutions:

Theorem 1.1. *Assume Γ satisfies Main Condition and let the numbers M and R_0 be fixed by (1.3), (1.4). There exist an absolute constant $\varepsilon_* > 0$ and a constant $R_* \in (0, R_0)$ depending only on M and R_0 with the following property. Assume that a boundary suitable weak solution (v, p) of the Navier–Stokes equations (1.1) near Γ_T and a point $z_0 = (x_0, t_0)$ with $x_0 \in \Gamma$ and $0 < t_0 \leqslant T$ satisfy the conditon*

$$\frac{1}{\rho_*^2} \int\limits_{t_0-\rho_*^2}^{t_0} \int\limits_{\Omega(x_0,\rho_*)} \left(|v|^3 + |p|^{\frac{3}{2}}\right) \, dydt < \varepsilon_* \tag{1.6}$$

for some $\rho_* \leqslant R_*$. Then v is Hölder continuous on $\overline{\Omega}(x_0, \frac{\rho_*}{8}) \times [t_0 - \frac{\rho_*^2}{16}, t_0]$.

Theorem 1.2. *Let all assumptions of Theorem 1.1 be satisfied. There exists an absolute constant $\varepsilon_0 > 0$ with the following property. Assume that a boundary suitable weak solution (v, p) of the Navier–Stokes equations (1.1) near Γ_T and a point $z_0 = (x_0, t_0)$ with $x_0 \in \Gamma$ and $0 < t_0 \leqslant T$ satisfy the condition*

$$\limsup_{\rho \to 0} \frac{1}{\rho} \int\limits_{t_0-\rho^2}^{t_0} \int\limits_{\Omega(x_0,\rho)} |\nabla v|^2 \, dydt < \varepsilon_0. \tag{1.7}$$

Then v is Hölder continuous in some neighborhood of z_0.

From Theorem 1.2 the following result easily follows:

Theorem 1.3. *Assume Ω is of class C^2 and its boundary $\partial\Omega$ satisfies Main Condition with $\Gamma \equiv \partial\Omega$. Let (v, p) be a boundary suitable weak solution to the Navier–Stokes system near $\partial\Omega \times (0, T)$. Then there exists a closed set $\Sigma \subset \partial\Omega \times (0, T]$ such that for any point $z_0 \in (\partial\Omega \times (0, T]) \setminus \Sigma$ the function v is Hölder continuous in some neighborhood of z_0 and, moreover,*

$$\mathcal{P}^1(\Sigma) = 0,$$

where $\mathcal{P}^1(\Sigma)$ is the one-dimensional parabolic Hausdorff measure of Σ.

The study of partial regularity for weak solution of the Navier–Stokes system satisfying the local energy inequality was initiated by V. Scheffer in [12]. L. Caffarelli, R. Kohn, and L. Nirenberg in [2] proved an ε-condition of the local boundedness for interior points of parabolic cylinder which is analogues to (1.7) in the present paper. Later, authors of [11, 9, 3] gave some improvements and simplifications in the proof of the CKN-condition as well as some new local conditions of ε-regularity for solutions to the Navier–Stokes system.

The study of partial regularity near the boundary was started by G. A. Seregin. In [14, 15, 17] he considered the Navier–Stokes system near the plane part of the boundary of a domain and for that case he proved analogs of ours Theorems 1.1–1.2 above. The present paper naturally extends results of [14, 15, 17] for the case of the points on a curved part of the boundary.

Our paper is organized as following. In Sec. 2, we introduce some notations and specify certain properties of Γ which are provided by the Main Condition of the present section. We introduce also the procedure of straightening of the boundary which "spoils" the coefficient of the original Navier–Stokes system. The new system we obtain we will call *the Perturbed Navier–Stokes system.* In Sec. 3, we develop the theory of global and local $L_{s,r}$-estimates for the linearization of the Perturbed Navier–Stokes system. In Sec. 4, we prove Theorem 1.1. The proof of the main Lemma 4.1 we present here follows the method developed in [17] for the case of the flat boundary. Finally, in Sec. 5, we present a proof of Theorem 1.2.

2. Preliminaries

Assume first $R > 0$ is arbitrary, Let $\varphi \in C^2(\overline{K}_R)$ be such that

$$\varphi(0) = 0, \quad \nabla\varphi(0) = 0, \quad \|\varphi\|_{W^2_\infty} \leqslant \mu. \tag{2.1}$$

Assume the set $\Omega_R \equiv \Omega \cap B_R$ is represented in a Cartesian coordinates $\{y_i\}_{i=1}^3$ in the following way:

$$\Omega_R = \{y \in B_R : y_3 > \varphi(y_1, y_2)\}.$$

We define new coordinates $x = \psi(y)$ by formulas

$$x = \psi(y) \equiv \begin{pmatrix} y_1 \\ y_2 \\ y_3 - \varphi(y_1, y_2) \end{pmatrix}. \tag{2.2}$$

The diffeomorphism (2.2) transforms the set Ω_R onto some subdomain $\psi(\Omega_R)$ of $\mathbb{R}^3_+ \equiv \{x \in \mathbb{R}^3 : x_3 > 0\}$. System (1.1) in Ω_R after the change of variables (2.2) takes the form

$$\begin{gathered} \frac{\partial \widehat{v}}{\partial t} + (\widehat{v} \cdot \widehat{\nabla})\widehat{v} - \widehat{\Delta}\widehat{v} + \widehat{\nabla}\widehat{p} = 0, \quad \widehat{\nabla} \cdot \widehat{v} = 0 \\ \widehat{v}|_{x_3=0} = 0. \end{gathered} \tag{2.3}$$

Here $\widehat{v} = v \circ \psi^{-1}$, $\widehat{p} = p \circ \psi^{-1}$, and $\widehat{\nabla}$ and $\widehat{\Delta}$ are the differential operators with variable coefficients defined by formulas

$$\begin{aligned} &\widehat{\nabla} \equiv \widehat{\nabla}_\varphi = \left(\frac{\partial}{\partial x_1} - \frac{\partial \varphi}{\partial x_1}\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_2} - \frac{\partial \varphi}{\partial x_2}\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_3} \right), \\ &\widehat{\Delta} \equiv \widehat{\Delta}_\varphi = a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + b_i(x)\frac{\partial}{\partial x_i}, \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} &a_{11}(x) = a_{22}(x) = 1, \quad a_{33}(x) = 1 + (\varphi_{,1})^2 + (\varphi_{,2})^2, \\ &a_{12}(x) = a_{21}(x) = 0, \quad a_{13}(x) = a_{31}(x) = -\varphi_{,1}, \\ &a_{23}(x) = a_{32}(x) = -\varphi_{,2}, \\ &b_1(x) = b_2(x) = 0, \quad b_3(x) = -\varphi_{,11} - \varphi_{,22}. \end{aligned} \tag{2.5}$$

It is easy to see that if C^1-norm of φ is sufficiently small, say, less than $\frac{1}{100}$ then the operator $\widehat{\nabla}$ is equivalent to ∇ in the following sense:

$$\frac{1}{2}|\nabla v|^2 \leqslant |\widehat{\nabla} v|^2 \leqslant 2|\nabla v|^2. \tag{2.6}$$

From (2.1), using Taylor formula, we obtain

$$\|\varphi\|_{C^1(\overline{K}_R)} \leqslant 2\mu R(1+R), \tag{2.7}$$

and hence if we put

$$\overline{R}_1(\mu) \equiv \frac{1}{400}(\mu+1)^{-1},$$

then for any $R \leqslant \overline{R}_1(\mu)$ inequalities (2.7) are true. Moreover, for $R \leqslant \overline{R}_1(\mu)$ we have

$$B^+_{\frac{R}{2}} \subset \psi(\Omega_R) \subset B^+_{2R}, \quad O_{\frac{R}{2}} \subset \Omega_R \subset O_{2R}, \tag{2.8}$$

where $O_R \equiv \psi^{-1}(B^+_R)$.

Now we assume $\Gamma \subset \partial\Omega$ satisfies Main Condition (see Sec. 1), $x_0 \in \Gamma$ and R_0, M, M', φ_{x_0} are fixed by (1.3), (1.4).Denote

$$R_1 = R_1(M, R_0) \equiv \overline{R}_1(M') = \frac{1}{400}(M'+1)^{-1} \tag{2.9}$$

then for any $R \leqslant R_1$ inequality (2.7)) and hence both relations (2.6) and (2.8) are satisfied.

In conclusion, we note that operators $\widehat{\nabla}$ and $\widehat{\Delta}$ satisfy relations similar to the corresponding relations for the usual operators ∇ and Δ:

$$\widehat{\Delta} v = \widehat{\nabla} \cdot (\widehat{\nabla} v), \quad (v \cdot \widehat{\nabla}) v = \widehat{\nabla} \cdot (v \otimes v) \ \text{ if } \ \widehat{\nabla} \cdot v = 0,$$
$$\widehat{\Delta}(\zeta v) = \zeta \widehat{\Delta} v + 2(\widehat{\nabla} v)\widehat{\nabla}\zeta + v\widehat{\Delta}\zeta,$$

and for $F \in W^1_s(\Omega; \mathbb{M}^{3\times 3})$, $w \in \overset{\circ}{W}{}^1_{s'}(\Omega; \mathbb{R}^3)$, $p \in W^1_s(\Omega)$

$$\int\limits_\Omega (\widehat{\nabla} \cdot F) \cdot w\, dx = -\int\limits_\Omega F : \widehat{\nabla} w\, dx, \qquad \int\limits_\Omega \widehat{\nabla} p \cdot w\, dx = -\int\limits_\Omega p \widehat{\nabla} \cdot w\, dx.$$

3. Global and local estimates for the perturbed Stokes system

In this section, we develop the theory of global and local coercive estimates for the system which we call the Perturbed Stokes System:

$$\frac{\partial v}{\partial t} - \widehat{\Delta} v + \widehat{\nabla} p = f, \quad \widehat{\nabla} \cdot v = 0,$$
$$v|_{x_3=0} = 0.$$

Here operators $\widehat{\Delta} \equiv \widehat{\Delta}_\varphi$, $\widehat{\nabla} \equiv \widehat{\nabla}_\varphi$ are defined by means of φ via relations (2.4), (2.5). We remind that φ is a function of class C^2 and we assume (2.1) holds.

The results of the this section naturally generalize and develop the results for the usual Stokes system obtained earlier by Seregin [16] and V. A. Solonnikov [19–21]. One of the basic tool of our approach is the result on solutions of the Stokes system with nonzero divergence obtained in [21], see also [5]:

Proposition 3.1. *Let r, $s \in (1, \infty)$ and assume functions f and g are* *such that*

$$
\begin{gathered}
f \in L_{s,r}(\Pi_1^+), \quad g \in W^{1,0}_{s,r}(\Pi_1^+), \quad \exists\ \frac{\partial g}{\partial t} \in L_r(-1,0; W_s^{-1}(\mathbb{R}^3_+)), \\
g(\cdot,0) = 0 \ \text{ in } W_s^{-1}(\mathbb{R}^3_+) \ \text{ and} \\
\int\limits_{\mathbb{R}^3_+} g(x,t)\, dx = 0, \quad \text{for a.e. } \ t \in (-1,0).
\end{gathered}
\tag{3.1}
$$

Then the problem

$$
\begin{gathered}
\frac{\partial v}{\partial t} - \Delta v + \nabla p = f, \quad \nabla \cdot v = g, \quad \text{in} \ \ \Pi_1^+ \\
v|_{x_3=0} = 0, \quad v|_{=-1} = 0,
\end{gathered}
\tag{3.2}
$$

has the unique solution $v \in W^{2,1}_{s,r}(\Pi_1^+)$, $\nabla p \in L_{s,r}(\Pi_1^+)$, *and, moreover, this solution satisfies the estimate*

$$
\begin{gathered}
\|v\|_{W^{2,1}_{s,r}(\Pi_1^+)} + \|\nabla p\|_{L_{s,r}(\Pi_1^+)} \leqslant \\
\leqslant C\left(\|f\|_{L_{s,r}(\Pi_1^+)} + \|\nabla g\|_{L_{s,r}(\Pi_1^+)} + \|\frac{\partial g}{\partial t}\|_{L_r(-1,0;W^1_s(\mathbb{R}^3_+))}\right).
\end{gathered}
\tag{3.3}
$$

First we study the initial boundary value problem for the Perturbed Stokes system in a half-space. For the future it is convenient for us to consider here the problem with nonzero divergence:

$$
\begin{gathered}
\frac{\partial v}{\partial t} - \widehat{\Delta} v + \widehat{\nabla} p = f, \quad \widehat{\nabla} \cdot v = g \ \text{ in } \ \Pi_1^+, \\
v|_{x_3=0} = 0, \quad v|_{t=-1} = 0.
\end{gathered}
\tag{3.4}
$$

Lemma 3.1. *Suppose* r, $s \in (1,\infty)$. *Assume* $\varphi \in C^2(\mathbb{R}^2) \cap W^2_\infty(\mathbb{R}^2)$ *and consider problem* (3.4) *where the coefficients of operators* $\widehat{\Delta}$, $\widehat{\nabla}$ *are defined by* φ *via relations* (2.4), (2.5). *There is an absolute constant* $\mu_* \in (0, \frac{1}{100})$ *such that if* φ *satisfies* (2.1) *with* $\mu \leqslant \mu_*$ *then for any* f *and* g *satisfying* (3.1) *problem* (3.4) *has a unique solution* $v \in W^{2,1}_{s,r}(\Pi_1^+)$, $\nabla p \in L_{s,r}(\Pi_1^+)$ *and estimate* (3.3) *holds with some absolute constant* C. *Note that as* $\mu_* \leqslant \frac{1}{100}$ *then* (2.6) *holds.*

For the case of $s = r$ the analogue of Lemma 3.1 for the usual Stokes system was originally obtained by Solonnikov, see [18, 19]. Many mathematicians contributed to the proof of this $L_{s,r}$-estimate in the case of the Stokes system, $r \neq s$, see, for example, [6, 12], and references there.

Proof of Lemma 3.1. Using Proposition 3.1, we define functions $v^{(0)} \in W^{2,1}_{s,r}(\Pi^+_1)$, $\nabla p^{(0)} \in L_{s,r}(\Pi^+_1)$ as a solution to the Stokes problem

$$\frac{\partial v^0}{\partial t} - \Delta v^0 + \nabla p^0 = f, \quad \nabla \cdot v^0 = g \quad \text{in} \quad \Pi^+_1,$$
$$v^0|_{x_3=0} = 0, \quad v^0|_{t=-1} = 0.$$

Then using Proposition 3.1 again we define iterations $v^{(k+1)} \in W^{2,1}_{s,r}(\Pi^+_1)$, $\nabla p^{(k+1)} \in L_{s,r}(\Pi^+_1)$, $k = 0, 1, \dots$ as solutions to the system

$$\begin{gathered}\frac{\partial v^{(k+1)}}{\partial t} - \Delta v^{(k+1)} + \nabla p^{(k+1)} = f - (\Delta - \widehat{\Delta}) v^{(k)} + (\nabla - \widehat{\nabla}) p^{(k)}, \\ \nabla \cdot v^{(k+1)} = (\nabla - \widehat{\nabla}) \cdot v^{(k)} + g \quad \text{in} \quad \Pi^+_1, \\ v^{(k+1)}|_{x_3=0} = 0, \quad v^{(k+1)}|_{t=-1} = 0.\end{gathered} \tag{3.5}$$

Existence of such functions is provided by the fact that

$$(\nabla - \widehat{\nabla}) \cdot v^{(k)} = \nabla \cdot h^{(k)}, \quad \text{where} \quad h^{(k)} \equiv \left(0, 0, \varphi_{,1} v^{(k)}_1 + \varphi_{,2} v^{(k)}_2\right) \tag{3.6}$$

and hence the data of problem (3.5) satisfy requirement (3.1). Moreover, taking into account (2.1), we obtain from (3.6)

$$\|(\nabla - \widehat{\nabla}) \cdot v^{(k)}\|_{W^{1,0}_{s,r}(\Pi^+_1)} \leqslant C\mu_* \|v^{(k)}\|_{W^{2,1}_{s,r}(\Pi^+_1)},$$

$$\begin{aligned}\|\frac{\partial}{\partial t}(\nabla - \widehat{\nabla}) \cdot v^{(k)}\|_{L_r(-1,0;W^{-1}_s(\mathbb{R}^3_+))} \leqslant \|\frac{\partial h^{(k)}}{\partial t}\|_{L_{s,r}(\Pi^+_1)} \leqslant \\ \leqslant C\mu_* \|\frac{\partial v^{(k)}}{\partial t}\|_{L_{s,r}(\Pi^+_1)}.\end{aligned}$$

Functions $v^{(k)}$ and $\nabla p^{(k)}$ satisfy estimates

$$\begin{gathered}\|v^{(k+1)}\|_{W^{2,1}_{s,r}(\Pi^+_1)} + \|\nabla p^{(k+1)}\|_{L_{s,r}(\Pi^+_1)} \leqslant \\ \leqslant C \left(\|f\|_{L_{s,r}(\Pi^+_1)} + \|\nabla g\|_{L_{s,r}(\Pi^+_1)} + \|\frac{\partial g}{\partial t}\|_{L_r(-1,0;W^1_s(\mathbb{R}^3_+))} \right) + \\ + C\mu_* \left(\|v^{(k-1)}\|_{W^{2,1}_{s,r}(\Pi^+_1)} + \|\nabla p^{(k-1)}\|_{L_{s,r}(\Pi^+_1)} \right).\end{gathered}$$

and

$$\begin{gathered}\|v^{(k+1)} - v^{(m+1)}\|_{W^{2,1}_{s,r}(\Pi^+_1)} + \|\nabla p^{(k+1)} - \nabla p^{(m+1)}\|_{L_{s,r}(\Pi^+_1)} \leqslant \\ \leqslant C\mu_* \left(\|v^{(k)} - v^{(m)}\|_{W^{2,1}_{s,r}(\Pi^+_1)} + \|\nabla p^{(k)} - \nabla p^{(m)}\|_{L_{s,r}(\Pi^+_1)} \right).\end{gathered}$$

Choosing $\mu_* = \frac{1}{2C+100}$ we can prove convergences $v^{(k)} \to v$ in $W^{2,1}_{s,r}(\Pi_1^+)$ and $\nabla p^{(k)} \to \nabla p$ in $L_{s,r}(\Pi_1^+)$, where functions v and ∇p obviously satisfy system (3.4) and estimate (3.3). Lemma 3.1 is proved.

Besides (3.4) we consider the Perturbed Stokes system (with zero divergence) in a half-cylinder $Q^+ \equiv Q^+(1)$:

$$\begin{gathered} \frac{\partial v}{\partial t} - \widehat{\Delta} v + \widehat{\nabla} p = f, \quad \widehat{\nabla} \cdot v = 0 \ \text{ in } \ Q^+, \\ v|_{x_3=0} = 0. \end{gathered} \tag{3.7}$$

For solutions of this system we will need the following result:

Lemma 3.2. *Suppose m, r, $s \in (1,\infty)$, $m \geqslant s$ and μ_* is fixed in Lemma 3.1. Assume $\varphi \in C^2(\overline{K}_1)$ satisfies (2.1) with $\mu \leqslant \mu_*$. Then for any functions $v \in W^{2,1}_{s,r}(Q^+)$, $p \in W^{1,0}_{s,r}(Q^+)$, $f \in L_{m,r}(Q^+)$ satisfying system (3.7) we have $v \in W^{2,1}_{m,r}(Q^+(\frac12))$, $\nabla p \in L_{m,r}(Q^+(\frac12))$ and the following local estimate holds:*

$$\begin{gathered} \|v\|_{W^{2,1}_{m,r}(Q^+(\frac12))} + \|\nabla p\|_{L_{m,r}(Q^+(\frac12))} \leqslant \\ \leqslant C\left(\|f\|_{L_{m,r}(Q^+)} + \|\nabla v\|_{L_{s,r}(Q^+)} + \|p-p_0\|_{L_{s,r}(Q^+)}\right). \end{gathered} \tag{3.8}$$

with some absolute constant C and arbitrary function $p_0 = p_0(t)$, $p_0 \in L_r(-1,0)$.

The analogue of Lemma 3.2 for the usual Stokes system was originally obtained by Seregin in [16].

Proof of Lemma 3.2.

Step 1. First we shall prove estimate (3.8) in the case of $m = s$. Choosing the cut-off function $\zeta \in C^\infty(\overline{Q})$ in such a way that $\zeta \equiv 1$ on $Q(\frac34)$ and ζ vanishes near $\partial' Q$, we denote

$$u = \zeta v, \quad q = \zeta(p - p_0). \tag{3.9}$$

These functions satisfy the system

$$\begin{gathered} \frac{\partial u}{\partial t} - \widehat{\Delta} u + \widehat{\nabla} q = f_0, \quad \widehat{\nabla} \cdot u = g_0 \ \text{ in } \ \Pi_1^+, \\ u|_{x_3=0} = 0, \quad u|_{t=-1} = 0, \end{gathered} \tag{3.10}$$

where

$$f_0 = \zeta f - 2(\widehat{\nabla} v)\widehat{\nabla}\zeta - v\widehat{\Delta}\zeta + (p-p_0)\widehat{\nabla}\zeta, \quad g_0 = v \cdot \widehat{\nabla}\zeta. \tag{3.11}$$

From Lemma 3.1 we obtain the estimate

$$\|u\|_{W^{2,1}_{s,r}(\Pi_1^+)} + \|\nabla q\|_{L_{s,r}(\Pi_1^+)} \leqslant \tag{3.12}$$
$$\leqslant C\left(\|f_0\|_{L_{s,r}(Q^+)} + \|\nabla g_0\|_{L_{s,r}(Q^+)} + \|\frac{\partial g_0}{\partial t}\|_{L_r(-1,0;W_s^{-1}(B^+))}\right).$$

Note that

$$\|f_0\|_{L_{s,r}(Q^+)} \leqslant$$
$$\leqslant C\left(\|f\|_{L_{s,r}(Q^+)} + \|\nabla v\|_{L_{s,r}(Q^+)} + \|p-p_0\|_{L_{s,r}(Q^+)}\right), \tag{3.13}$$
$$\|\nabla g_0\|_{L_{s,r}(Q^+)} \leqslant C\|v\|_{W^{1,0}_{s,r}(Q^+)}. \tag{3.14}$$

Moreover, as

$$\frac{\partial g_0}{\partial t} = \frac{\partial v}{\partial t}\cdot \widehat{\nabla}\zeta + v\cdot\frac{\partial}{\partial t}\widehat{\nabla}\zeta \in L_{s,r}(Q^+),$$

we have

$$\|\frac{\partial g_0}{\partial t}\|_{W_s^{-1}(B^+)} = \sup_{\|w\|_{W^1_{s'}}\leqslant 1}\Big|\int\limits_{B^+}\frac{\partial g_0}{\partial t}\cdot w\; dx\Big| \leqslant$$
$$\leqslant C\left(\|\frac{\partial v}{\partial t}\|_{W_s^{-1}(B^+)} + \|v\|_{L_s(B^+)}\right)$$

Representing $\frac{\partial v}{\partial t}$ from the equation

$$\frac{\partial v}{\partial t} = f + \widehat{\Delta}v - \widehat{\nabla}p \quad \text{a.e. in} \quad Q^+,$$

and taking into account relations

$$\|\widehat{\Delta}v\|_{W_s^{-1}(B^+)} = \sup_{\|w\|_{W^1_{s'}}\leqslant 1}\Big|\int\limits_{B^+}\widehat{\nabla}v : \widehat{\nabla}w\; dx\Big| \leqslant C\|\widehat{\nabla}v\|_{L_s(B^+)},$$

$$\|\widehat{\nabla}p\|_{W_s^{-1}(B^+)} =$$
$$= \sup_{\|w\|_{W^1_{s'}}\leqslant 1}\Big|\int\limits_{B^+}(p-p_0)\widehat{\nabla}\cdot w\; dx\Big| \leqslant C\|p-p_0\|_{L_s(B^+)},$$

we conclude

$$\|\frac{\partial g_0}{\partial t}\|_{L_r(-1,0;W_s^{-1}(B^+))} \leqslant$$
$$\leqslant C(\|f\|_{L_{s,r}(B^+)} + \|v\|_{W^{1,0}_{s,r}(Q^+)} + \|p-p_0\|_{L_{s,r}(Q^+)}). \tag{3.15}$$

Taking into account the condition $v|_{x_3} = 0$ and the Poincaré inequality $\|v\|_{W^{1,0}_{s,r}(Q^+)} \leqslant C\|\nabla v\|_{L_{s,r}(Q^+)}$ from (3.12)–(3.15) we obtain (3.8) in the case of $m = s$.

Step 2. To prove (3.8) in the case of $m > s$ we denote by $s_1 = s$ and by $s_2 = \min\{m, \frac{3s}{3-s}\}$ if $s < 3$ and $s_2 = m$ otherwise. Again we introduce the cut-off function ζ with the same properties as above and assume additionally $\operatorname{supp}\zeta \subset Q_{\frac{3}{4}}$. Define functions (u, q) by formulas (3.9). These functions satisfy system (3.10). The data of problem (3.10) satisfy the following estimates:

$$\begin{gathered}\|f_0\|_{L_{s_2,r}(Q^+)} \leqslant \\ \leqslant C\left(\|f\|_{L_{s_2,r}(Q^+)} + \|v\|_{W^{1,0}_{s_2,r}(Q^+_{\frac{3}{4}})} + \|p - p_0\|_{L_{s_2,r}(Q^+_{\frac{3}{4}})}\right).\end{gathered}$$

Due to the embedding theorem $W^1_{s_1}(B^+_{\frac{3}{4}}) \hookrightarrow L_{s_2}(B^+_{\frac{3}{4}})$ we obtain the inequality

$$\|f_0\|_{L_{s_2,r}(Q^+)} \leqslant C\left(\|f\|_{L_{s_2,r}(Q^+)} + \|v\|_{W^{2,1}_{s_1,r}(Q^+_{\frac{3}{4}})} + \|\nabla p\|_{L_{s_1,r}(Q^+_{\frac{3}{4}})}\right).$$

Similarly, using the same embedding theorem we obtain

$$\|g_0\|_{W^{1,0}_{s_2,r}(Q^+)} \leqslant C\|v\|_{W^{1,0}_{s_2,r}(Q^+_{\frac{3}{4}})} \leqslant C\|v\|_{W^{2,1}_{s_1,r}(Q^+_{\frac{3}{4}})}.$$

Moreover, we have $\int g_0(x,t)\,dx = 0$ for a.e. $t \in (-1, 0)$ and $g_0|_{t=-1} = 0$. Finally,

$$\|\frac{\partial g_0}{\partial t}\|_{W^{-1}_{s_2}(B^+)} \leqslant C\|\frac{\partial v}{\partial t}\|_{W^{-1}_{s_2}(B^+_{\frac{3}{4}})} \leqslant C\|\frac{\partial v}{\partial t}\|_{L_{s_1}(B^+_{\frac{3}{4}})},$$

and

$$\|\frac{\partial g_0}{\partial t}\|_{L_r(-1,0;W^{-1}_{s_2}(B^+))} \leqslant C\|v\|_{W^{2,1}_{s_1,r}(Q^+_{\frac{3}{4}})}.$$

So, the data of problem (3.10) satisfy all conditions of Lemma 3.1. Hence we conclude that $u \in W^{2,1}_{s_2,r}(\Pi^+_1)$, $\nabla q \in L_{s_2,r}(\Pi^+_1)$ and $(u, \nabla q)$ satisfy the estimate

$$\begin{gathered}\|u\|_{W^{2,1}_{s_2,r}(\Pi^+_1)} + \|\nabla q\|_{L_{s_2,r}(\Pi^+_1)} \leqslant \\ \leqslant C\left(\|f\|_{L_{s_2,r}(Q^+)} + \|v\|_{W^{2,1}_{s_1,r}(Q^+_{\frac{3}{4}})} + \|\nabla p\|_{L_{s_1,r}(Q^+_{\frac{3}{4}})}\right),\end{gathered} \tag{3.16}$$

where $s_2 > s_1$. The last inequality together with the estimate

$$\|v\|_{W^{2,1}_{s_1,r}(Q^+_{\frac34})} + \|\nabla p\|_{L_{s_1,r}(Q^+_{\frac34})} \leqslant$$
$$\leqslant C\|f\|_{L_{m,r}(Q^+)} + C\left(\|\nabla v\|_{L_{s_1,r}(Q^+)} + \|p-[p]_{B^+}\|_{L_{s_1,r}(Q^+)}\right)$$

(which follows from Step 1) implies (3.8) in the case of $s_2 = m$.

If $s_2 < m$ we define $s_3 = \min\{m, \frac{3s_2}{3-s_2}\}$ if $s_2 < 3$ and $s_3 = m$ otherwise, take a new cut-off function ζ_1 with the support supp $\zeta_1 \subset B(\frac34)$, define new functions (u, q) by formulas (3.9) with ζ_1 instead of ζ. Repeating the arguments above we obtain for these new functions (u, q) the estimate of type (3.16) with the numbers s_3 and s_2 instead of s_2 and s_1, respectively, and with the norms at the RHS of (3.16) calculated over some smaller half-cylinder containing $Q^+(\frac12)$. Repeating, if necessary, the same arguments again we obtain the result by the obvious boot-strap. Lemma 3.2 is proved.

Lemma 3.3. *Suppose m, r, $s \in (1,\infty)$, $m \geqslant s$ and μ_* is fixed in Lemma 3.1. Assume $R \leqslant 1$ and $\varphi \in C^2(\overline{K}_R)$ satisfies (2.1) with $\mu \leqslant \mu_*$. Then for any functions $v \in W^{2,1}_{s,r}(Q^+(R))$, $p \in W^{1,0}_{s,r}(Q^+(R))$, $f \in L_{m,r}(Q^+(R))$ satisfying the Perturbed Stokes system (3.7) in $Q^+(R)$ we have $v \in W^{2,1}_{m,r}(Q^+(\frac R2))$, $\nabla p \in L_{m,r}(Q^+(\frac R2))$ and the following local estimate holds:*

$$\begin{aligned}\frac{1}{R^2}\|v\|_{L_{m,r}(Q^+(\frac R2))} + \frac1R\|\nabla v\|_{L_{m,r}(Q^+(\frac R2))} + [v]_{W^{2,1}_{m,r}(Q^+(\frac R2))} + \\ +\|\nabla p\|_{L_{m,r}(Q^+(\frac R2))} \leqslant C\|f\|_{L_{m,r}(Q^+(R))} + \\ +\frac{C}{R^{1+\alpha}}\left(\|\nabla v\|_{L_{s,r}(Q^+(R))} + \|p-p_0\|_{L_{s,r}(Q^+(R))}\right),\end{aligned} \tag{3.17}$$

where $\alpha = \frac3s - \frac3m$ and C is an absolute constants which do not depend on R.

Proof of Lemma 3.3. For $(y,\tau) \in Q^+$ we define the functions

$$\begin{aligned} v^R(y,\tau) = Rv(Ry, R^2\tau), \quad p^R(y,\tau) = R^2p(Ry,R^2\tau), \\ f^R(y,\tau) = R^3 f(Ry,R^2\tau), \quad p_0^R(\tau) = R^2p_0(R^2\tau),\end{aligned} \tag{3.18}$$

and for $(y_1, y_2) \in K_1$ define also

$$\varphi_R(y_1,y_2) = \frac1R\varphi(Ry_1,Ry_2). \tag{3.19}$$

These functions satisfy the system

$$\begin{gathered}\frac{\partial v^R}{\partial \tau} - \widehat{\Delta}_{\varphi_R} v^R + \widehat{\nabla}_{\varphi_R} p^R = f^R, \quad \widehat{\nabla}_{\varphi_R} \cdot v = 0 \ \text{ in } \ Q^+, \\ v^R|_{x_3=0} = 0, \quad v^R|_{t=-1} = 0,\end{gathered} \tag{3.20}$$

where operators $\widehat{\Delta}_{\varphi_R}$ and $\widehat{\nabla}_{\varphi_R}$ are defined by relations (2.4), (2.5) with functions φ_R instead of φ. Using Taylor formula and (2.1) it is not difficult to show that

$$\|\varphi_R\|_{C^2(\overline{K}_1)} \leqslant \mu R \leqslant \mu_*,$$

as $R \leqslant 1$. Hence we conclude that (v^R, p^R, φ_R) satisfy all conditions of Lemma 3.2. Applying this lemma we obtain $v^R \in W^{2,1}_{m,r}(Q^+(\frac{1}{2}))$, $\nabla p^R \in L_{m,r}(Q^+(\frac{1}{2}))$ and also the following estimate holds

$$\begin{gathered}\|v^R\|_{W^{2,1}_{m,r}(Q^+(\frac{1}{2}))} + \|\nabla p^R\|_{L_{m,r}(Q^+(\frac{1}{2}))} \leqslant \\ \leqslant C\|f^R\|_{L_{m,r}(Q^+)} + C(\|\nabla v^R\|_{L_{s,r}(Q^+)} + \|p^R - p^R_0\|_{L_{s,r}(Q^+)}).\end{gathered} \tag{3.21}$$

where C is some absolute constant. Now (3.17) follows from (3.21) by change of variables. Lemma 3.3 is proved.

In conclusion of this section, we present one useful proposition which follows from results of this section.

Proposition 3.2. *Suppose m, r, $s \in (1, \infty)$, and $m \geqslant r$. Let $\varphi \in C^2(\overline{K}_1)$ be such that*

$$\varphi(0) = 0, \quad \nabla\varphi(0) = 0, \quad \sup_{|x'|\leqslant 1} |\nabla^2\varphi| \leqslant M.$$

For $\tau \in (0, 1]$, we introduce sets

$$\Omega_\varphi(\tau) = \{x \in B(\tau) : x_3 > \varphi(x_1, x_2)\}, \quad \omega_\varphi(\tau) = \Omega_\varphi(\tau) \times (-\tau^2, 0).$$

Denote by Γ the set

$$\Gamma = \{x \in B(1) : x_3 = \varphi(x_1, x_2)\}$$

and consider the Stokes system in the domain $\omega_\varphi(1)$:

$$\begin{gathered}\frac{\partial v}{\partial t} - \Delta v + \nabla p = 0, \quad \nabla \cdot v = 0, \ \ in \ \ \omega_\varphi(1), \\ v|_{\Gamma\times(-1,0)} = 0.\end{gathered} \tag{3.22}$$

Moreover, assume Γ satisfies the Main Condition from Sec. 1 and constants M, R_0 are fixed by (1.3), (1.4). Then for any $0<\tau_1<\tau_2<1$ there is a constant C depending only on m, r, s, M, R_0, τ_1, and τ_2, such that for any $f\in L_{m,r}(\omega_\varphi(1))$, $v\in W^{2,1}_{s,r}(\omega_\varphi(1))$, $p\in W^{1,0}_{s,r}(\omega_\varphi(1))$ satisfying (3.22), we have $v\in W^{2,1}_{m,r}(\omega_\varphi(\tau_1))$, $p\in W^{1,0}_{m,r}(\omega_\varphi(\tau_1))$ and the following estimate holds:

$$\begin{aligned}\|v\|_{W^{2,1}_{m,r}(\omega_\varphi(\tau_1))}+\|\nabla p\|_{L_{m,r}(\omega_\varphi(\tau_1))}\leqslant C\|f\|_{L_{m,r}(\omega_\varphi(\tau_2))}+\\ +C\left(\|\nabla v\|_{L_{s,r}(\omega_\varphi(\tau_2))}+\|p-[p]_{\Omega_\varphi(\tau_2)}\|_{L_{s,r}(\omega_\varphi(\tau_2))}\right).\end{aligned}\tag{3.23}$$

Proposition 3.2 easily follows from Lemma 3.2 above, so here we only briefly sketch its proof. Note that our assumptions on the set Γ guarantees existence of numbers N_0 and ρ_0 such that for any $\rho\leqslant\rho_0$ there is a finite covering of $\omega(\tau_1)$ by the sets of type $\omega_\varphi(z_i,\rho)\equiv\omega_\varphi(\tau_1)\cap Q(z_i,\rho)$, where points $z_i=(x_i,t_i)$ are of two types: either $x_i\in\Gamma$ or $Q(z_i,\rho)\subset\omega_\varphi(\tau_2)$. Moreover, for any $\rho\leqslant\rho_0$ the total number of intersections of different $\omega(z_i,2\rho)$ at any point $z\in\omega_\varphi(\tau_1)$ does not increase N_0. Now assume $R_*\equiv\{\frac{\mu_*}{2M'},R_1\}$, where μ_* is fixed in Lemma 3.1, M' is fixed in (1.4) and R_1 is fixed in (2.9). Choosing $\rho\leqslant R_*$ (in accordance with τ_1 and τ_2) we introduce for this ρ the finite covering of $\omega_\varphi(\tau_1)$ by sets $\omega(z_i,\rho)$ we described above. For those z_i for which $\omega(z_i,\rho)\equiv Q(z_i,\rho)\subset\omega_\varphi(\tau_2)$ the estimate of type (3.23) is well-known, see [16] and references there. In the other case, if x_i belongs to the boundary Γ, we pass to the local coordinates $\{y_i\}_{i=1}^3$ and consider a diffeomorphism (2.2) of a set $\Omega(x_i,2\rho)$ onto some subdomain of $\mathbb{R}^3_+$ (thanks to the inequality $\rho\leqslant R_1$ this subdomain contains the half-ball B^+_ρ, see (2.8)). Using the scaling transform of type (3.18), (3.19) we reduce the problem in a small half-cylinder Q^+_ρ to the corresponding problem in Q^+. Note that thanks to conditions (1.3), (1.4) with the help of Taylor formula for the rescaled function φ_ρ one can show the estimate $\|\varphi_\rho\|_{C^2(K_1)}\leqslant M'\rho$. As $\rho\leqslant\frac{\mu_*}{2M'}$ we conclude that for the rescaled problem in Q^+ all conditions of our Lemma 3.2 hold and hence the estimate of type (3.23) is valid. This estimate provides the corresponding estimate for original solutions in the sets $\omega(z_i,\rho)$, when $x_i\in\Gamma$. The total estimate (3.23) now follows by the usual procedure of gluing together of the local estimates in each $\omega(z_i,\rho)$.

4. Proof of Theorem 1.1

We consider the Perturbed Navier–Stokes system in $Q^+ \equiv Q^+(1)$:

$$\begin{gathered}\frac{\partial v}{\partial t} + \widehat{\nabla}_\varphi \cdot (v \otimes v) - \widehat{\Delta}_\varphi v + \widehat{\nabla}_\varphi p = 0, \quad \widehat{\nabla}_\varphi \cdot v = 0, \quad \text{in } Q^+ \\ v|_{x_3=0} = 0,\end{gathered} \tag{4.1}$$

(here operators $\widehat{\nabla}_\varphi$ and $\widehat{\Delta}_\varphi$ are defined by (2.4), (2.5)). We introduce the functional:

$$Y_\theta(v,p) = \Big(\fint_{Q^+(\theta)} |v|^3 \, dxdt\Big)^{1/3} + \theta\Big(\fint_{Q^+(\theta)} |p - [p]_{B^+(\theta)}|^{3/2} \, dxdt\Big)^{2/3}.$$

We assume that functions (v, p, φ) have the following properties:

$$v \in L_{2,\infty} \cap W^{1,0}_2 \cap W^{2,1}_{\frac{9}{8},\frac{3}{2}}(Q^+), \quad p \in L_{\frac{3}{2}} \cap W^{1,0}_{\frac{9}{8},\frac{3}{2}}(Q^+), \tag{4.2}$$

$$\varphi \in C^2(\overline{K}), \quad \varphi(0) = 0, \quad \nabla\varphi(0) = 0, \quad \|\varphi\|_{C^2(\overline{K})} \leqslant \mu, \tag{4.3}$$

(v, p, φ) satisfy (4.1) in the sense of distributions and also satisfy the local energy inequality

$$\begin{gathered}\int_{B^+} \zeta(x,t)|v(x,t)|^2 \, dx + 2\int_{-1}^{t}\int_{B^+} \zeta|\widehat{\nabla}_\varphi v|^2 \, dxdt' \leqslant \\ \leqslant \int_{-1}^{t}\int_{B^+} \Big\{ |v|^2 \Big(\frac{\partial\zeta}{\partial t} + \widehat{\Delta}_\varphi \zeta\Big) + v \cdot \widehat{\nabla}_\varphi \zeta \left(|v|^2 + 2p\right)\Big\} \, dxdt'.\end{gathered} \tag{4.4}$$

for a.e. $t \in (-1, 0)$ and all nonnegative functions $\zeta \in C_0^\infty(Q)$.

In this section, we shall call any set of functions (v, p, φ) satisfying (4.1)–(4.4) a *boundary suitable weak solution of the Perturbed Navier–Stokes system in* Q^+.

Theorem 4.1. *Suppose* (4.3) *holds. Assume* $\mu \leqslant \frac{\mu_*}{2}$ *where* μ_* *is defined in* Lemma 3.1. *There exist an absolute constant* $\varepsilon_* > 0$ *such that for any boundary suitable weak solution* (v, p, φ) *of the perturbed Navier–Stokes system in* Q^+ *if*

$$Y_1(v,p) < \varepsilon_*,$$

then v is Hölder continuous on $\overline{Q}^+(\frac{1}{2})$ and

$$\sup_{\overline{Q}^+(1/2)} |v| \leqslant C,$$

where the constant C depends only on ε_*.

Lemma 4.1. *Assume all conditions of Theorem 4.1 hold. Then for any $\theta \in (0, \frac{1}{2})$ there exist constants $\varepsilon_* > 0$ and $C_* > 0$ depending only on θ such that for any boundary suitable weak solution (v, p, φ) of the perturbed Navier–Stokes system in Q^+ if*

$$Y_1(v, p) \leqslant \varepsilon_*,$$

then

$$Y_\theta(v, p) \leqslant C_* \theta^{1/3} Y_1(v, p). \tag{4.5}$$

Here C_* is some absolute constant.

Proof of Lemma 4.1.

Step 1. Arguing by contradiction, we assume there is a number $\theta \in (0, \frac{1}{2})$, the sequence $\varepsilon_h \to 0$ and functions (v^h, p^h, φ^h) which are the boundary suitable weak solutions satisfying (4.1)–(4.4) (with the operators $\widehat{\nabla}_\varphi$, $\widehat{\Delta}_\varphi$ changed by the operators $\widehat{\nabla}_h \equiv \widehat{\nabla}_{\varphi^h}$, $\widehat{\Delta}_h \equiv \widehat{\Delta}_{\varphi^h}$), and also satisfying relations

$$\begin{gathered} Y_1(v^h, p^h) = \varepsilon_h \to 0, \quad Y_\theta(v^h, p^h) \geqslant C_* \theta^{\frac{1}{3}} \varepsilon_h, \\ \|\varphi^h\|_{C^2(\overline{K})} \leqslant \mu, \quad \varphi^h(0) = 0, \quad \nabla \varphi^h(0) = 0. \end{gathered} \tag{4.6}$$

We introduce the new functions

$$u^h = \frac{1}{\varepsilon_h} v^h, \quad q^h = \frac{1}{\varepsilon_h}(p^h - [p^h]_{B^+}).$$

These functions meet relations

$$Y_1(u^h, q^h) = 1, \quad Y_\tau(u^h, q^h) \geqslant C_* \tau^{\frac{1}{3}}, \tag{4.7}$$

the system

$$\left.\begin{aligned} \frac{\partial u^h}{\partial t} + \varepsilon_h \widehat{\nabla}_h \cdot (u^h \otimes u^h) - \widehat{\Delta}_h u^h + \widehat{\nabla}_h q^h = 0, \\ \widehat{\nabla}_h \cdot u^h = 0, \end{aligned}\right\} \quad \text{a.e. in} \quad Q^+ \tag{4.8}$$

$$u^h|_{x_3=0} = 0,$$

and also the local energy inequality

$$\int\limits_{B+} \zeta(x,t)|u^h(x,t)|^2\, dx + 2\int\limits_{-1}^{t}\int\limits_{B+} \zeta|\widehat{\nabla}_h u^h|^2\, dxdt' \leqslant \tag{4.9}$$

$$\leqslant \int\limits_{-1}^{t}\int\limits_{B+} \left\{ |u^h|^2 \left(\frac{\partial \zeta}{\partial t} + \widehat{\Delta}_h \zeta\right) + u^h \cdot \widehat{\nabla}\zeta \left(\varepsilon_h |u^h|^2 + 2q^h\right)\right\}\, dxdt'$$

holds for a.e. $t \in (-1,0)$ and all nonnegative $\zeta \in C_0^\infty(Q)$.

From (4.7) we obtain the estimate

$$\|u^h\|_{L_3(Q+)} + \|q^h\|_{L_{\frac{3}{2}}(Q+)} \leqslant 1, \tag{4.10}$$

and from (4.9) taking into account (4.10) we get

$$\|u^h\|_{L_{2,\infty}(Q+(\frac{3}{4}))} + \|u^h\|_{W_2^{1,0}(Q+(\frac{3}{4}))} \leqslant C \tag{4.11}$$

with some absolute constant C. From (4.11) and usual multiplicative inequality we conclude that

$$\|u^h\|_{L_{\frac{10}{3}}(Q+(\frac{3}{4}))} \leqslant C, \tag{4.12}$$

and Eqs. (4.8) provide the estimate of the negative norm of the time derivative of u^h:

$$\|\frac{\partial u^h}{\partial t}\|_{L_{\frac{3}{2}}(-(\frac{3}{4})^2,0;W_{\frac{3}{2}}^{-1}(B+(\frac{3}{4})))} \leqslant C. \tag{4.13}$$

From estimates (4.10)–(4.13) existence of subsequences (u^h, q^h) follows such that

$$u^h \rightharpoonup u^0 \quad \text{in} \quad W_2^{1,0}\Big(Q^+\Big(\frac{3}{4}\Big)\Big) \cap L_3(Q^+), \tag{4.14}$$

$$u^h \to u^0 \quad \text{in} \quad L_3\Big(Q^+\Big(\frac{3}{4}\Big)\Big), \tag{4.15}$$

$$p^h \rightharpoonup p^0 \quad \text{in} \quad L_{\frac{3}{2}}(Q^+). \tag{4.16}$$

Moreover, from (4.6), we obtain for some subsequence

$$\varphi^h \overset{*}{\rightharpoonup} \varphi^0 \quad \text{in} \quad W_\infty^2(K_1), \quad \varphi^h \to \varphi^0 \quad \text{in} \quad C^1(\overline{K}_1). \tag{4.17}$$

Hence $\varphi^0 \in C^2(\overline{K})$ satisfies the relation

$$\|\varphi^0\|_{C^2(\overline{K}_1)} \leqslant \mu.$$

Convergences (4.14)–(4.16) allow us to pass to the limit in Eqs. (4.8). So we obtain that (u^0, q^0, φ^0) satisfy the system

$$\begin{gathered}\frac{\partial u^0}{\partial t} - \widehat{\Delta} u^0 + \widehat{\nabla} q^0 = 0, \quad \widehat{\nabla} \cdot u^0 = 0, \quad \text{in } Q^+\Big(\frac{3}{4}\Big) \\ u^0|_{x_3=0} = 0,\end{gathered} \tag{4.18}$$

with the operators $\widehat{\nabla} \equiv \widehat{\nabla}_{\varphi^0}$ and $\widehat{\Delta} \equiv \widehat{\Delta}_{\varphi^0}$. Moreover, from the second relation in (4.7), we obtain the estimate

$$\liminf_{h\to\infty} Y_\theta(u^h, q^h) \geqslant C_* \theta^{\frac{1}{3}}. \tag{4.19}$$

On the other hand, below we will show that

$$\limsup_{h\to\infty} Y_\theta(u^h, q^h) \leqslant C_{**} \theta^{\frac{1}{3}}, \tag{4.20}$$

for some constant C_{**} depending only on μ_*. Taking in (4.19) a constant $C_* > C_{**}$ we obtain a contradiction between (4.19) and (4.20). This contradiction proves the lemma.

Step 2. To prove (4.20), we split Y_θ onto two parts:

$$Y_\theta(u^h, q^h) = Y_\theta^1(u^h) + Y_\theta^2(q^h),$$

where

$$Y_\theta^1(u^h) \equiv \Big(\fint_{Q^+(\theta)} |u^h|^3 \, dxdt \Big)^{\frac{1}{3}}, \quad Y_\theta^2(q^h) \equiv \theta \Big(\fint_{Q^+(\theta)} |q^h|^{\frac{3}{2}} \, dxdt \Big)^{\frac{2}{3}}$$

The strong convergence (4.15) allows us to pass to the limit in Y_θ^1, $\theta \in (0, \frac{3}{4})$:

$$\lim_{h\to\infty} Y_\theta^1(u^h) = Y_\theta^1(u^0). \tag{4.21}$$

As u^0 is a solution of the perturbed Stokes system (4.18) in $Q^+(\frac{3}{4})$ from Lemma 3.2, we obtain the estimate

$$\|u^0\|_{W^{2,1}_{9,\frac{3}{2}}(Q^+(\frac{1}{2}))} \leqslant C \left(\|\nabla u^0\|_{L_{\frac{3}{2}}(Q^+(\frac{3}{4}))} + \|q^0\|_{L_{\frac{3}{2}}(Q^+)} \right). \tag{4.22}$$

Thanks to estimates (4.10) and (4.11) and the lower semicontinuity of the corresponding norms with respect to the weak convergences (4.14), (4.16), the RHS of (4.22) can be estimated by some absolute constant C.

On the other hand, the imbedding of functional spaces

$$\begin{gathered} W^{2,1}_{9,\frac{3}{2}}(Q^+(\frac{1}{2})) \hookrightarrow C^{\alpha,\frac{\alpha}{2}}(\overline{Q}^+(\frac{1}{2})), \\ \|u^0\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q}^+(\frac{1}{2}))} \leqslant C\|u^0\|_{W^{2,1}_{9,\frac{3}{2}}(Q^+(\frac{1}{2}))} \end{gathered} \tag{4.23}$$

holds with $\alpha = 2 - \frac{3}{9} - \frac{4}{3} = \frac{1}{3}$ (see [10, 1]). As $u^0|_{x_3=0} = 0$, relation (4.22) together with inequality (4.23), implies the estimate

$$Y^1_\theta(u^0) \leqslant C\theta^{\frac{1}{3}}\|u^0\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q}^+(\frac{1}{2}))} \leqslant C_0\theta^{\frac{1}{3}}, \tag{4.24}$$

where a constant C_0 depends only on μ_*.

Step 3. Combining (4.21) and (4.24), we obtain the estimate

$$\limsup_{h\to\infty} Y_\theta(u^h, q^h) \leqslant C_0\theta^{\frac{1}{3}} + \limsup_{h\to\infty} Y^2_\theta(q^h). \tag{4.25}$$

So, to arrive at (4.20), we need to estimate the second term in the RHS of (4.25). For this purpose we define

$$f^h = \begin{cases} -\varepsilon_h(u^h \cdot \widehat{\nabla}_h)u^h, & \text{in } \ Q^+\Big(\dfrac{3}{4}\Big), \\ 0, & \text{in } \ \Pi^+\Big(\dfrac{3}{4}\Big) \setminus Q^+\Big(\dfrac{3}{4}\Big). \end{cases}$$

Moreover, extend the functions φ^h (and hence the coefficients of the operators $\widehat{\nabla}_h$, $\widehat{\Delta}_h$) from the disc $K_1 \equiv \{|x'| \leqslant 1\}$ onto $\mathbb{R}^2$ in such a way that

$$\|\varphi^h\|_{C^2(\mathbb{R}^2)} \leqslant 2\mu \leqslant \mu_*,$$

and define functions $\overline{u}^h \in W^{2,1}_{\frac{9}{8},\frac{3}{2}}(\Pi^+(\frac{3}{4}))$, $\overline{q}^h \in W^{1,0}_{\frac{9}{8},\frac{3}{2}}(\Pi^+(\frac{3}{4}))$ as a solution to the problem

$$\begin{gathered} \frac{\partial \overline{u}^h}{\partial t} - \widehat{\Delta}_h\overline{u}^h + \widehat{\nabla}_h\overline{q}^h = f^h, \quad \widehat{\nabla}_h \cdot \overline{u}^h = 0 \ \text{ in } \quad \Pi^+(\frac{3}{4}), \\ \overline{u}^h|_{t=-(\frac{3}{4})^2} = 0, \quad \overline{u}^h|_{x_3=0} = 0. \end{gathered}$$

From Lemma 3.1, we obtain the estimate

$$\begin{aligned} &\|\overline{u}^h\|_{W^{2,1}_{\frac{9}{8},\frac{3}{2}}(\Pi^+_{\frac{3}{4}})} + \|\nabla \overline{q}^h\|_{L_{\frac{9}{8},\frac{3}{2}}(\Pi^+_{\frac{3}{4}})} \leqslant \\ &\leqslant C\|f^h\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\frac{3}{4}))} \to 0, \quad \text{as} \quad h \to \infty. \end{aligned} \tag{4.26}$$

Now consider the functions $\widetilde{u}^h \equiv u^h - \overline{u}^h$, $\widetilde{q}^h = q^h - \overline{q}^h$. These functions satisfy the homogeneous system

$$\frac{\partial \widetilde{u}^h}{\partial t} - \widehat{\Delta}_h \widetilde{u}^h + \widehat{\nabla}_h \widetilde{q}^h = 0, \quad \widehat{\nabla}_h \cdot \widetilde{u}^h = 0 \quad \text{in} \quad Q^+(\frac{3}{4}),$$
$$\widetilde{u}^h|_{x_3=0} = 0.$$

From Lemma 3.2, we obtain an estimate

$$\begin{aligned} &\|\nabla \widetilde{q}^h\|_{L_{9,\frac{3}{2}}(Q^+(\frac{1}{2}))} \leqslant \\ &\qquad \leqslant C\left(\|\nabla \widetilde{u}^h\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\frac{3}{4}))} + \|\widetilde{q}^h - [\widetilde{q}^h]_{B^+(\frac{3}{4})}\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\frac{3}{4}))}\right). \end{aligned}$$

The RHS of the last inequality can be controlled by the folowing majorants which do not depend on h:

$$\|\nabla \widetilde{u}^h\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\frac{3}{4}))} \leqslant \|\nabla u^h\|_{L_2(Q^+(\frac{3}{4}))} + \|\nabla \overline{u}^h\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\frac{3}{4}))} \leqslant C,$$

$$\begin{aligned} &\|\widetilde{q}^h - [\widetilde{q}^h]_{B^+(\frac{3}{4})}\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\frac{3}{4}))} \leqslant \\ &\leqslant \|q^h - [q^h]_{B^+(\frac{3}{4})}\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\frac{3}{4}))} + \|\overline{q}^h - [\overline{q}^h]_{B^+(\frac{3}{4})}\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\frac{3}{4}))} \leqslant \\ &\leqslant C\left(\|q^h\|_{L_{\frac{3}{2}}(Q^+)} + \|\nabla \overline{q}^h\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\frac{3}{4}))}\right) \leqslant \\ &\leqslant C\left(1 + \|f^h\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\frac{3}{4}))}\right) \leqslant C. \end{aligned}$$

Hence, by the Poincaré and Hölder inequalities

$$Y^2_\theta(\widetilde{q}^h) \leqslant C\theta^{\frac{1}{3}}\|\nabla \widetilde{q}^h\|_{L_{9,\frac{3}{2}}(Q^+(\frac{1}{2}))} \leqslant C\theta^{\frac{1}{3}}.$$

The last estimate together with (4.25), (4.26) implies (4.20) which contradicts to (4.19) and hence proves Lemma 4.1.

Lemma 4.2. *Assume all conditions of Theorem 4.1 holds. There exist an absolute constants $\varepsilon_* > 0$ and $\theta_* \in (0, \frac{1}{2})$ such that for any boundary suitable weak solution (v, p, φ) of the perturbed Navier–Stokes system in Q^+ if*

$$Y_1(v, p) < \varepsilon_*,$$

then for any $k \in \mathbb{N}$

$$Y_{\theta_*^{k-1}}(v, p) < \varepsilon_* \quad \text{and} \quad Y_{\theta_*^k}(v, p) \leqslant \theta_*^{1/6} Y_{\theta_*^{k-1}}(v, p). \tag{4.27}$$

The proof of Lemma 4.2 is essentially identical to the proof of Proposition 2.5 in [4].

To proceed, we define the functional

$$Y_{z_0,R}(v, p) = \Big(\fint_{Q^+(x_0,R)} |v|^3 \, dxdt \Big)^{1/3} + \\ + R \Big(\fint_{Q^+(z_0,R)} |p - [p]_{B^+(x_0,R)}|^{3/2} \, dxdt \Big)^{2/3}.$$

Lemma 4.3. *Suppose $R \leqslant 1$, $\varphi_{x_0} \in C^2(\overline{K}_R)$ and (2.1) holds. Let $\mu \leqslant \frac{\mu_*}{2}$, where μ_* is fixed in Lemma 3.1. Assume θ_*, ε_* are fixed as in Lemma 4.2. Let (v, p, φ_{x_0}) be a boundary suitable weak solution of the Perturbed Navier–Stokes system (2.3) in a half-cylinder $Q^+(z_0, R)$ such that if for some $R \leqslant 1$*

$$R \, Y_{z_0,R}(v, p) < \varepsilon_*.$$

Then for any $k \in \mathbb{N}$

$$Y_{z_0,\theta_*^k R}(v, p) \leqslant \theta_*^{\frac{k}{6}} \varepsilon_*. \tag{4.28}$$

The proof of Lemma 4.3 is a direct consequence of Lemma 4.2 and the scaling transform

$$\begin{aligned} v^R(y, \tau) &= Rv(x_0 + Ry, t_0 + R^2\tau), \\ p^R(y, \tau) &= R^2 p(x_0 + Ry, t_0 + R^2\tau), \\ \varphi^R(y_1, y_2) &= \frac{1}{R}\varphi_{x_0}(Ry_1, Ry_2). \end{aligned} \tag{4.29}$$

Note that for φ^R, using (2.1) and Taylor formula, we obtain the estimate $\|\varphi^R\|_{C^2(\overline{K}_1)} \leqslant \mu R \leqslant \frac{\mu_*}{2}$, as $R \leqslant 1$ and hence for (v^R, p^R, φ^R) all conditions of Lemma 4.2 holds.

Now we turn to the proof of Theorem 4.1.

Proof of Theorem 4.1. We introduce a functional $\overline{Y}_{z_0,R}(v,p)$ which is slightly different from the previous two:

$$\overline{Y}_{z_0,R}(v,p) = \Big(\fint_{\omega(x_0,R)} |v-(v)_{\omega(z_0,R)}|^3\ dxdt\Big)^{1/3} + \\ + R\Big(\fint_{\omega(z_0,R)} |p-[p]_{\Omega(x_0,R)}|^{3/2}\ dxdt\Big)^{2/3}.$$

(We remind that we denote by $\omega(z_0,R) \equiv \Omega(x_0,R)\times(t_0-R^2,t_0)$, where $\Omega(x_0,R) \equiv \Omega\cap B(x_0,R)$, and in Theorem 4.1 the role of Ω plays the set B^+.)

From [4], we know that there are constants $\overline{\varepsilon}_1>0$ and $\theta_1>0$ such that if $\overline{Y}_{0,1}(v,p)<\overline{\varepsilon}_1$ then for all $z_0\in Q^+(\frac34)$, $R>0$ such that $Q(z_0,R)\subset Q^+$ the following inequality holds:

$$\overline{Y}_{z_0,\theta_1^k R}(v,p) \leqslant \theta_1^{\frac{k}{6}}\overline{\varepsilon}_1.$$

On the other hand, due to Lemma 4.3 this inequality also true for all z_0 on the plane part of the lateral boundary of the half-cylinder $Q^+(\frac34)$ if we assume $\varepsilon_*\leqslant\overline{\varepsilon}_1$ and $\theta_*\leqslant\theta_1$. Note that the proof of Lemma 4.2 allows such a choice.

Combining internal and boundary estimates in a standard way, we obtain the inequality

$$\overline{Y}_{z_0,\rho}(v,p)\leqslant C\rho^{\frac16},\quad \text{for all}\quad z_0\in\overline{Q}^+(\frac12)\quad\text{and all}\quad \rho\leqslant\frac{1}{16}.$$

Here C is some absolute constant. This estimate implies the Hölder continuity of v on $\overline{Q}^+(\frac12)$. Theorem 4.1 is proved.

Proof of Theorem 1.1. Redenote the constant ε_* from Theorem 4.1 by $\overline{\varepsilon}_*$ and set $\varepsilon_*=\frac{1}{4\sqrt2}\overline{\varepsilon}_*$. Assume (1.6) holds and $\rho_*<R_1$, where R_1 is defined in (2.9). Consider local coordinates (1.2) and transformation (2.2). Define $O(x_0,\rho)=\psi_{x_0}^{-1}(B_\rho^+)$. To simplify notations we denote $R=\frac{\rho_*}{2}$. As $O(x_0,R)\subset\Omega(x_0,\rho_*)$ from (1.6), we conclude

$$\frac{1}{R^2}\int\limits_{t_0-R^2}^{t_0}\int\limits_{O(x_0,R)}\Big(|v|^3+|p|^{\frac32}\Big)\ dydt<4\varepsilon_*.$$

As we have

$$\int\limits_{O(x_0,R)} |p-[p]_{O(x_0,R)}|^{\frac{3}{2}}\ dy \leqslant \sqrt{2} \int\limits_{O(x_0,R)} |p|^{\frac{3}{2}}\ dy,$$

we conclude

$$\frac{1}{R^2}\int\limits_{t_0-R^2}^{t_0}\int\limits_{O(x_0,R)} \Big(|v|^3 + |p-[p]_{O(x_0,R)}|^{\frac{3}{2}}\Big)\ dydt < \overline{\varepsilon}_*. \tag{4.30}$$

The change of variables (2.2) leads to

$$\frac{1}{R^2}\int\limits_{t_0-R^2}^{t_0}\int\limits_{B^+(x_0,R)} \Big(|v|^3 + |p-[p]_{B^+(x_0,R)}|^{\frac{3}{2}}\Big)\ dxdt < \overline{\varepsilon}_*,$$

(to simplify notations we denote functions (v,p) associated with new variables by the same symbols). Note also that functions (v,p,φ) associated with x-variable are a boundary suitable weak solution of the Perturbed Navier–Stokes system in $Q^+(z_0,R)$. Consider new functions (v^R,p^R,φ^R) defined for $(y,\tau)\in Q^+$ and $y'\in K_1$ by relations (4.29). Then (v^R,p^R,φ^R) is a boundary suitable weak solution of the Perturbed Navier–Stokes system (4.1) in Q^+ such that

$$Y_1(v^R,p^R) < \varepsilon_0.$$

Moreover, from (1.4) using Taylor formula, we obtain the estimate

$$\|\varphi^R\|_{C^2(\overline{K}_1)} \leqslant R\|\varphi\|_{C^2(\overline{K}_R)} \leqslant M'R \tag{4.31}$$

and hence if we define

$$R_* = \min\Big\{\frac{\mu_*}{2M'}, R_1\Big\} \tag{4.32}$$

(where μ_* is defined in Lemma 3.1, M' is defined in (1.4)) and assume $\rho_*\leqslant R_*$ then for functions (v^R,p^R,φ^R) all conditions of Theorem 4.1 hold. Applying this theorem we conclude that v^R is Hölder continuous on $\overline{Q}^+(\frac{1}{2})$ and hence v is Hölder continuous on $\overline{O}(x_0,\frac{R}{2})\times[t_0-\frac{R^2}{4},t_0]$. As $\Omega(x_0,\frac{\rho_*}{8})\subset O(x_0,\frac{R}{2})$ Theorem 1.1 is proved.

5. Proof of Theorem 1.2

First we consider a boundary suitable weak solution (v, p, φ) of the Perturbed Navier–Stokes system in a half-cylinder Q^+. We assume this solution satisfies relations (4.1)–(4.4). For $R \leqslant 1$ we introduce four principal functionals

$$
\begin{aligned}
C(R) &\equiv \Big(\frac{1}{R^2}\int\limits_{Q^+(R)} |v|^3\, dydt\Big)^{1/3}\\
D(R) &\equiv \Big(\frac{1}{R^2}\int\limits_{Q^+(R)} |p-[p]_{B^+(R)}|^{3/2}\, dydt\Big)^{2/3},\\
E(R) &\equiv \Big(\frac{1}{R}\int\limits_{Q^+(R)} |\nabla v|^2\, dydt\Big)^{1/2},\\
A(R) &\equiv \Big(\frac{1}{R}\sup_{t\in(-R^2,0)}\int\limits_{B^+(R)} |v|^2\, dy\Big)^{1/2},
\end{aligned}
\tag{5.1}
$$

and one axillary functionals

$$
D_0(R) = \frac{1}{R}\Big(\int\limits_{-R^2}^{0}\Big(\int\limits_{B^+(R)} |\nabla p|^{\frac{9}{8}}\, dy\Big)^{\frac{8}{9}\frac{3}{2}}\, dt\Big)^{2/3}. \tag{5.2}
$$

We note that due to the embedding $W^1_{\frac{9}{8}} \hookrightarrow L_{\frac{3}{2}}$ the estimate

$$
D(R) \leqslant cD_0(R)
$$

holds with some constant c independent of R.

Theorem 5.1. *Suppose $\varphi \in C^2(\overline{K}_1)$ and (4.3) holds. Assume $\mu \leqslant \frac{\mu_*}{2}$, where μ_* is defined in Lemma 3.1. There exists an absolute constant $\varepsilon_0 > 0$ such that for any boundary suitable weak solution (v, p, φ) of the Perturbed Navier–Stokes system in Q^+ if*

$$
\sup_{\rho\leqslant 1} E(\rho) < \varepsilon_0, \tag{5.3}
$$

then

$$
\liminf_{r\to 0}[C(r) + D(r)] < \varepsilon_*,
$$

where the number ε_* is fixed in Theorem 4.1.

Proof of Theorem 5.1.

Step 1. Let $\theta \in (0, \frac{1}{2})$ be a number whose precise value will be fixed later and assume $\rho \leqslant 1$ is arbitrary. Interpolation inequality provides the estimate
$$C(\theta\rho) \leqslant cA^{\frac{1}{2}}(\theta\rho)E^{\frac{1}{2}}(\theta\rho).$$
Since
$$A(\theta\rho) \leqslant \theta^{-\frac{1}{2}}A(\frac{3\rho}{4}), \quad E(\theta\rho) \leqslant \theta^{-\frac{1}{2}}E(\rho),$$
we obtain
$$C(\theta\rho) \leqslant c\theta^{-\frac{1}{2}}A^{\frac{1}{2}}(\frac{3\rho}{4})E^{\frac{1}{2}}(\rho). \tag{5.4}$$

Step 2. Choosing in (4.4) the test function φ in the appropriate way we arrive at the following Caccioppoli inequality:
$$\sup_{t\in(-(\frac{3\rho}{4})^2,0)} \int\limits_{B^+(\frac{3\rho}{4})} |v(x,t)|^2\, dx + \int\limits_{Q^+(\frac{3\rho}{4})} |\nabla v|^2\, dz \leqslant$$
$$\leqslant \frac{C}{\rho^2}\int\limits_{Q^+(\rho)} |v|^2\, dz + \frac{C}{\rho}\int\limits_{Q^+(\rho)} |v|^3\, dz +$$
$$+\frac{C}{\rho}\Big(\int\limits_{Q^+(\rho)} |v|^3\, dz\Big)^{\frac{1}{3}}\Big(\int\limits_{Q^+(\rho)} |p-[p]_{B^+(\rho)}|^{3/2}\, dz\Big)^{\frac{2}{3}}.$$
Dividing both sides of this inequality by ρ we obtain
$$A\Big(\frac{3\rho}{4}\Big) + E\Big(\frac{3\rho}{4}\Big) \leqslant c\Big[C(\rho) + C^{\frac{3}{2}}(\rho) + D(\rho)\Big]. \tag{5.5}$$

Step 3. Splitting of p. Extend the function φ (and hence the coefficients of the operators $\widehat{\nabla}$, $\widehat{\Delta}$) from the disc $K_1 \equiv \{|x'| \leqslant 1\}$ onto $\mathbb{R}^2$ in such a way that
$$\|\varphi\|_{C^2(\mathbb{R}^2)} \leqslant 2\mu \leqslant \mu_*,$$
and define
$$\widehat{f} = \begin{cases} (v\cdot\widehat{\nabla})v, & \text{in } \ Q^+\Big(\dfrac{3\rho}{4}\Big) \\ \quad 0, & \text{in } \ \Pi_1^+ \setminus Q^+\Big(\dfrac{3\rho}{4}\Big). \end{cases}$$

Let $(\overline{v},\overline{p})$ be a uniquely defined solution to the problem

$$\frac{\partial\overline{v}}{\partial t}-\widehat{\Delta}\overline{v}+\widehat{\nabla}\overline{p}=\widehat{f},\quad \widehat{\nabla}\cdot\overline{v}=0\ \text{ in }\ \Pi_1^+$$
$$\overline{v}|_{t=-1}=0,\quad \overline{v}|_{x_3=0}=0.$$

From Lemma 3.1, we conclude that a solution to this problem exists and the following estimate holds:

$$\begin{aligned}\|\overline{v}\|_{W^{2,1}_{\frac{9}{8},\frac{3}{2}}(\Pi_1^+)}+\|\nabla\overline{p}\|_{L_{\frac{9}{8},\frac{3}{2}}(\Pi_1^+)}&\leqslant c\|\widehat{f}\|_{L_{\frac{9}{8},\frac{3}{2}}(\Pi_1^+)}\leqslant\\ &\leqslant c\ \|(v\cdot\widehat{\nabla})v\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\frac{3\rho}{4}))}.\end{aligned}\tag{5.6}$$

The constant c in the last inequality does not depend on ρ. Then we define

$$\widetilde{v}=v-\overline{v},\quad \widetilde{p}=p-\overline{p}.\tag{5.7}$$

It is easy to see that $(\widetilde{v},\widetilde{p})$ satisfy the following relations:

$$\begin{gathered}\frac{\partial\widetilde{v}}{\partial t}-\widehat{\Delta}\widetilde{v}+\widehat{\nabla}\widetilde{p}=0,\quad \widehat{\nabla}\cdot\widetilde{v}=0\ \text{ in }\ Q^+(\frac{3\rho}{4})\\ \widetilde{v}|_{x_3=0}=0.\end{gathered}\tag{5.8}$$

Define two more axillary functionals:

$$D_1(R)=\frac{1}{R}\Big(\int\limits_{t_0-R^2}^{t_0}\Big(\int\limits_{B(x_0,R)}|\nabla\overline{p}|^{\frac{9}{8}}\,dx\Big)^{\frac{8}{9}\frac{3}{2}}\,dt\Big)^{2/3},$$
$$D_2(R)=\frac{1}{R}\Big(\int\limits_{t_0-R^2}^{t_0}\Big(\int\limits_{B(x_0,R)}|\nabla\widetilde{p}|^{\frac{9}{8}}\,dx\Big)^{\frac{8}{9}\frac{3}{2}}\,dt\Big)^{2/3},$$

Step 4. From the embedding theorem $W^1_{\frac{9}{8}}(B^+)\hookrightarrow L_{\frac{3}{2}}(B^+)$, we obtain

$$D(\theta\rho)\leqslant cD_0(\theta\rho)\leqslant c(D_1(\theta\rho)+D_2(\theta\rho))\tag{5.9}$$

Step 5. Inequality (5.6) provides the estimate of $\overline{p}$:

$$\|\nabla\overline{p}\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\frac{3\rho}{4}))}\leqslant c\ \|(v\cdot\widehat{\nabla})v\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\frac{3\rho}{4}))}.$$

By Hölder and the interpolation inequalities, we obtain:

$$\|(v\cdot\widehat{\nabla})v\|_{L_{\frac{9}{8}}(B^+(\frac{3\rho}{4}))}\leqslant\|\widehat{\nabla}v\|_{L_2(B^+(\frac{3\rho}{4}))}\|v\|_{L_{\frac{18}{7}}(B^+(\frac{3\rho}{4}))}$$

$$\|v\|_{L_{\frac{18}{7}}(B^+(\frac{3\rho}{4}))} \leqslant c\,\|v\|^{\frac{2}{3}}_{L_2(B^+(\frac{3\rho}{4}))}\|\nabla v\|^{\frac{1}{3}}_{L_2(B^+(\frac{3\rho}{4}))}.$$

Hence

$$\|(v\cdot\widehat{\nabla})v\|_{L_{\frac{9}{8}}(B^+(\frac{3\rho}{4}))} \leqslant c\,\|v\|^{\frac{2}{3}}_{L_2(B^+(\frac{3\rho}{4}))}\|\nabla v\|^{\frac{4}{3}}_{L_2(B^+(\frac{3\rho}{4}))}, \tag{5.10}$$

and, therefore,

$$\|\nabla\overline{p}\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\frac{3\rho}{4}))} \leqslant c\|v\|^{\frac{2}{3}}_{L_{2,\infty}(Q^+(\frac{3\rho}{4}))}\|\nabla v\|^{\frac{4}{3}}_{L_2(Q^+(\frac{3\rho}{4}))}.$$

Dividing both sides of this relation by ρ we arrive the inequality:

$$D_1(\frac{3\rho}{4}) \leqslant cA^{\frac{2}{3}}\Big(\frac{3\rho}{4}\Big)E^{\frac{4}{3}}(\rho). \tag{5.11}$$

Hence

$$D_1(\theta\rho) \leqslant c\theta^{-1}D_1\Big(\frac{3\rho}{4}\Big) \leqslant c\theta^{-1}A^{\frac{2}{3}}\Big(\frac{3\rho}{4}\Big)E^{\frac{4}{3}}(\rho).$$

Step 6. Estimate of $\widetilde{p}$ applying Hölder inequality, we obtain

$$\|\nabla\widetilde{p}\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\theta\rho))} \leqslant (\theta\rho)^{\frac{7}{3}}\|\nabla\widetilde{p}\|_{L_{9,\frac{3}{2}}(Q^+(\frac{\rho}{2}))}.$$

From Lemma 3.3, we conclude

$$\|\nabla\widetilde{p}\|_{L_{9,\frac{3}{2}}(Q^+(\frac{\rho}{2}))} \leqslant$$

$$\leqslant c\rho^{-\frac{10}{3}}\left[\|\nabla\widetilde{v}\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\frac{3\rho}{4}))} + \|\widetilde{p}-[\widetilde{p}]_{B^+(\rho)}\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\rho))}\right].$$

From the definition of $\widetilde{v}$ and $\widetilde{p}$ (see (5.7)), we obtain inequalities

$$\|\nabla\widetilde{v}\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\frac{3\rho}{4}))} \leqslant \|\nabla v\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\frac{3\rho}{4}))} + \|\nabla\overline{v}\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\frac{3\rho}{4}))}$$

and

$$\|\widetilde{p}-[\widetilde{p}]_{B^+(\rho)}\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\rho))} \leqslant$$

$$\leqslant \|p-[p]_{B^+(\rho)}\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\frac{3\rho}{4}))} + \|\overline{p}-[\overline{p}]_{B^+(\rho)}\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\rho))}.$$

With the help of Hölder inequality, we derive estimates

$$\|\nabla v\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\frac{3\rho}{4}))} \leqslant c\rho^{\frac{3}{2}}\|\nabla v\|_{L_2(Q^+(\frac{3\rho}{4}))} \leqslant c\rho^2E(\rho)$$

and

$$\|p-[p]_{B^+(\frac{3\rho}{4})}\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\rho))} \leqslant c\rho^{\frac{2}{3}}\|p-[p]_{B^+(\rho)}\|_{L_{\frac{3}{2}}(Q^+(\rho))} \leqslant c\rho^2 D(\rho).$$

To estimate $\left\|\nabla \overline{v}\right\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\frac{3\rho}{4}))}$ we use Hölder inequlity, the embedding $W^{2,1}_{\frac{9}{8},\frac{3}{2}}(\Pi_1^+) \hookrightarrow W^{1,0}_{\frac{9}{5},\frac{3}{2}}(\Pi_1^+)$, (5.6) and (5.10). We obtain

$$\left\|\nabla \overline{v}\right\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\frac{3\rho}{4}))} \leqslant c\rho\left\|\nabla \overline{v}\right\|_{L_{\frac{9}{5},\frac{3}{2}}(\Pi_1^+)} \leqslant c\rho\left\|\nabla \overline{v}\right\|_{W^{2,1}_{\frac{9}{8},\frac{3}{2}}(\Pi_1^+)} \leqslant$$

$$\leqslant c\rho\left\|(v\cdot\widehat{\nabla})v\right\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\frac{3\rho}{4}))} \leqslant c\rho^2 A^{\frac{2}{3}}\left(\frac{3\rho}{4}\right)E^{\frac{4}{3}}(\rho).$$

The term $\left\|\overline{p}-[\overline{p}]_{B^+(\rho)}\right\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\rho))}$ we estimate with the help of Poincaré inequlaity and (5.6). We obtain

$$\left\|\overline{p}-[\overline{p}]_{B^+(\rho)}\right\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\rho))} \leqslant c\rho\left\|\nabla\overline{p}\right\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\rho))} \leqslant$$

$$\leqslant c\rho\left\|(v\cdot\widehat{\nabla})v\right\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\frac{3\rho}{4}))} \leqslant c\rho^2 A^{\frac{2}{3}}\left(\frac{3\rho}{4}\right)E^{\frac{4}{3}}(\rho).$$

Gathering all these estimates together, we obtain the relation

$$D_2(\theta\rho) = \frac{1}{\theta\rho}\|\nabla\widetilde{p}\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\theta\rho))} \leqslant$$

$$\leqslant c_1\theta^{\frac{4}{3}}\left(E(\rho)+D(\rho)+cA^{\frac{2}{3}}(\frac{3\rho}{4})E^{\frac{4}{3}}(\rho)\right).$$

Step 7. So we have

$$C(\theta\rho)+D(\theta\rho) \leqslant c\ \theta^{-\frac{1}{2}}A^{\frac{1}{2}}(\frac{3\rho}{4})E^{\frac{1}{2}}(\rho)+c\theta^{-1}\ A^{\frac{2}{3}}(\frac{3\rho}{4})E^{\frac{4}{3}}(\rho)+$$

$$+c\ \theta^{\frac{4}{3}}\ \left(E(\rho)+D(\rho)+cA^{\frac{2}{3}}(\frac{3\rho}{4})E^{\frac{4}{3}}(\rho)\right).$$

Each term in the RHS of the last relation can be estimated with the help of the elementary inequality $ab \leqslant \varepsilon a^p + C(\varepsilon,p)b^{p'}$, $p=2$ or $\frac{3}{2}$. For example,

$$c\ \theta^{-\frac{1}{2}}A^{\frac{1}{2}}(\frac{3\rho}{4})E^{\frac{1}{2}}(\rho) = c\theta^{-\frac{1}{2}}\left[C^{\frac{1}{2}}(\rho)+C^{\frac{3}{4}}(\rho)+D^{\frac{1}{2}}(\rho)\right]E^{\frac{1}{2}}(\rho) \leqslant$$

$$\leqslant \frac{1}{8}[C(\rho)+D(\rho)]+c\theta^{-1}E(\rho)[1+c\theta^{-1}E(\rho)].$$

Similarly,

$$c\theta^{-1}\ A^{\frac{2}{3}}(\frac{3\rho}{4})E^{\frac{4}{3}}(\rho)\leqslant c\theta^{-1}\ \left[C^{\frac{2}{3}}(\rho)+C(\rho)+D^{\frac{2}{3}}(\rho)\right]\ E^{\frac{4}{3}}(\rho)\leqslant$$

$$\leqslant \frac{1}{8}[C(\rho)+D(\rho)]+c^3\theta^{-3}E^4(\rho)+c\theta^{-1}E^{\frac{4}{3}}(\rho)C(\rho)$$

Hence we obtain the relation

$$C(\theta\rho)+D(\theta\rho)\leqslant\left(\frac{1}{4}+c_1\theta^{\frac{4}{3}}+c_2\theta^{-1}E^{\frac{4}{3}}(\rho)\right)[C(\rho)+D(\rho)]+$$

$$+c_3\theta^{-3}E^4(\rho)+c_4(\theta^{-3}+\theta^4)E(\rho)[1+c_5\theta^{-1}E(\rho)]+c_6\theta^{\frac{4}{3}}E(\rho), \quad (5.12)$$

which holds for all $\rho\leqslant 1$. Here c_i, $i=1,\dots,6$ are some absolute constant.

Step 8. Choice of θ and ε_0. First we fix $\theta\in(0,\frac{1}{2})$ in such a way that

$$c_1\theta^{\frac{4}{3}}\leqslant\frac{1}{8}.$$

Next (keeping in mind relation (5.12) and the inequality $E(\rho)<\varepsilon_0$ which holds for all $\rho<1$) we define $\varepsilon_0>0$ in such a way that the following two inequalities hold:

$$c_2\theta^{-1}\varepsilon_0^{\frac{4}{3}}\leqslant\frac{1}{8},$$

and

$$c_3\theta^{-3}\varepsilon_0^4+c_4(\theta^{-3}+\theta^4)\varepsilon_0[1+c_5\theta^{-1}\varepsilon_0]+c_6\theta^{\frac{4}{3}}\varepsilon_0\leqslant\frac{\varepsilon_*}{4}.$$

From (5.12), we obtain the estimate

$$C(\theta\rho)+D(\theta\rho)\leqslant\frac{1}{2}[C(\rho)+D(\rho)]+\frac{\varepsilon_*}{4}, \quad (5.13)$$

which holds for all $\rho<1$. Iterating this estimate we obtain the result. Theorem 5.1 is proved.

Proof of Theorem 1.2. Without loss of generality we can assume $z_0=(x_0,t_0)=(0,0)$ and system of coordinates $\{y_i\}_{i=1}^3$ is chosen in such a way that (1.2)–(1.4) hold.

Redenote the constant ε_0 from Theorem 5.1 by $\overline{\varepsilon}_0$ and set $\varepsilon_0 = \frac{1}{8}\overline{\varepsilon}_0^2$ and assume (1.7). There exist $\rho_0 < R_*$, where R_* is defined in Theorem 1.1 (we remind that $R_* \equiv \min\{\frac{\mu_*}{2M'}, R_1\}$, see (4.32)) such that

$$\sup_{\rho \leqslant \rho_0} \frac{1}{\rho} \int\limits_{\Omega_\rho \times (-\rho^2, 0)} |\nabla v|^2 \, dydt < \varepsilon_0.$$

Consider transformation (2.2). To simplify notations we denote $R = \frac{\rho_0}{2}$. From (2.8), we obtain $\psi^{-1}(B_\rho^+) \subset \Omega_{2\rho}$ for all $\rho \leqslant R$. Taking into account relations (2.6) after the change of variables (2.2), we conclude

$$\sup_{\rho \leqslant R} \frac{1}{\rho} \int\limits_{Q^+(\rho)} |\nabla v|^2 \, dxdt < 8\varepsilon_0 \equiv \overline{\varepsilon}_0^2, \tag{5.14}$$

(we keep the same notations for functions (v, p) associated with new variables). Moreover, functions (v, p, φ) are the boundary suitable weal solution of the Perturbed Navier–Stokes system in $Q^+(R)$. Now we introduce the rescaled functions (v^R, p^R, φ^R) by formulas (4.29). These functions are the boundary suitable weak solution to the Perturbed Navier–Stokes system in Q^+. Note also, that for φ^R estimate (4.31) holds and $\|\varphi^R\|_{C^2(\overline{K}_1)} \leqslant \frac{\mu_*}{2}$ for all $R \leqslant R_*$. Moreover, from (5.14), we obtain

$$\sup_{\rho \leqslant 1} \frac{1}{\rho} \int\limits_{Q^+(\rho)} |\nabla v^R|^2 \, dz < \overline{\varepsilon}_0^2.$$

So, we conclude that for functions (v^R, p^R, φ^R) all conditions of Theorem 5.1 are satisfied. Applying this theorem we obtain

$$\liminf_{r \to 0} \frac{1}{r^2} \int\limits_{Q^+(r)} \left(|v^R|^3 + |p^R - [p^R]_{B_r^+}|^{3/2} \right) \, dz = 0.$$

Going back to the functions $v(x,t) = \frac{1}{R} v^R(\frac{x}{R}, \frac{t}{R^2})$ and $p(x,t) = \frac{1}{R^2} p^R(\frac{x}{R}, \frac{t}{R^2})$, we obtain the convergence

$$\liminf_{r \to 0} \frac{1}{r^2 R^2} \int\limits_{Q^+(rR)} \left(|v|^3 + |p - [p]_{B_{rR}^+}|^{3/2} \right) \, dz = 0.$$

Taking into account (2.6) after the back change of variables $y = \psi^{-1}(x)$ (see (2.2)), we conclude

$$\liminf_{\rho\to 0} \frac{1}{\rho^2} \int\limits_{-(\frac{\rho}{8})^2}^{0} \int\limits_{O(\frac{\rho}{8})} \left(|v|^3 + |p - [p]_{O(\frac{\rho}{8})}|^{\frac{3}{2}} \right) \, dydt = 0,$$

from which follows existence of $\rho_* \leqslant R_*$ such that (4.30) holds. The result follows from the proof of Theorem 1.1. Theorem 1.2 is proved.

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