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OPTIMAL REGULARITY OF LOWER DIMENSIONAL OBSTACLE PROBLEMS

Abstract. In this paper we prove that solutions to the "boundary obstacle problem" have the optimal regularity, $C^{1,1/2}$, in any space dimension. This bound depends only on the local L^2 norm of the solution. Main ingredients in the proof are the quasiconvexity of the solution and a monotonicity formula for an appropriate weighted average of the local energy of the normal derivative of the solution.

Dedicated to Nina Nikolaevna Uraltseva on the occasion of her 70th birthday

§1. Introduction

The problem of minimizing the Dirichlet integral, or more general integrals which correspond to second order operators, in domain Ω in $\mathbb{R}^n$ among all functions with prescribed boundary values $f(x)$ and constrained to remain above a prescribed obstacle $\varphi(x)$ when restricted to $(n-1)$-manifold $\mathcal{M}$ in Ω, is known as the thin obstacle problem or the n-dimensional Signorini problem.

When studying the regularity of the solution the best that one expects to attain is a $C^{1,1/2}$ on each side of the manifold $\mathcal{M}$ even with a smooth φ. This is indeed the case as it was shown by D. Richardson in [6] for $n = 2$ and $\mathcal{M}$ a hyperplane, for higher dimensions, though, i.e., when $n \geq 3$, only a $C^{1,a}$ has been achieved for some $0 < a \leq \frac{1}{2}$ (see [2]). More general situations were treated by N. N. Uraltseva in [8].

In the present paper, we show that $\frac{1}{2}$ can be obtained for any dimension by making use of the almost concavity of u in the normal direction and considering a monotonicity formula of an appropriate quantity. Also, the parabolic version of this monotonicity formula is given which renders

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the elliptic theory applicable if some control of time derivative is known. (This will be treated in a forthcoming work.)

This problem and variations of it appear in many applications. In fact, the book of Duvaut and Lions [4] presents a series of challenging problems, especially in the nonstationary case, related to semi-permeable membranes, regulating temperature or heat from the boundary, and to linear elasticity such as the (real) Signorini problem, i.e., the equilibrium of an elastic body resting on a rigid surface.

The interpretation of the thin obstacle problem as an obstacle problem on $\mathbb{R}^{n-1}$ where the $H^{1/2}$-norm is minimized also arises in mathematical finance when the Levi processes are considered. In this direction, L. Silvestre proves in [7], for the case of the obstacle problem for $\Delta^{\alpha}(\alpha < 1)$, the $C^{1,\beta}$ regularity for $\beta < \alpha$.

We want to take this opportunity to congratulate Nina Uraltseva in her 70th birthday for her scientific achievements and to thank her for the influence she had on all of us and for her valuable friendship.

§2. Setting and known results

In order to clarify the main ideas we shall take the $(n-1)$-manifold $\mathcal{M}$ to lie in the hyperplane $\mathbb{R}^{n-1}$, the domain Ω to be symmetric with respect to $\mathbb{R}^{n-1}$, the boundary values to be zero, and restrict to the minimization of the Direchlet integral. More precisely, given a bounded, (simply) connected domain $\Omega \subset \mathbb{R}^n$ with smooth boundary and symmetric about $\mathbb{R}^{n-1} \cap \Omega$ hyperplane, and a smooth function φ defined on $\mathbb{R}^{n-1} \cap \Omega$ with $\varphi\big|_{\partial\Omega} < 0$. We want to minimize the Dirichlet integral

$$I(u) = \int\limits_{\Omega} |\nabla u|^2 dx$$

among all functions u which belong to the closed convex set

$$\mathbb{K} = \{u \in H_0^1(\Omega) : u \geq \varphi \ \text{ on } \ \mathbb{R}^{n-1} \cap \Omega, \ \ \varphi|_{\partial\Omega} < 0\}.$$

In $\mathbb{K}$ there exists a unique solution which is harmonic on either side of the hyperplane $\mathbb{R}^{n-1} \cap \Omega$ and on the hyperplane whenever $u > \varphi$. It is globally Lipschitz continuous and the normal to the hyperplane derivative has a jump across the hyperplane, but on either side enjoys C^a regularity up to it. Therefore u is $C^{1,a}$ on either side of the hyperplane (see [2] and its references).

Furthermore, for simplicity we take the zero obstacle, i.e, $\varphi \equiv 0$, and study the problem in a local setting, e.g., in $B_1(0)$, with zero a free boundary point, which is best fitted in studying the free boundary regularity. Its solution, still called u, changes sign in $B_1(0)$, is nonnegative on $B_1'(0) := B_1(0) \cap \mathbb{R}^{n-1}$, and harmonic away from the coincidence set, i.e., $u(x',0) = 0$; also, on $\{(x',0) : u(x',0) > 0\}$, $u_{x_n} = 0$ and on $\{(x',0) : u(x',0) = 0\}$ $u_{x_n} \leq 0$.

For the sake of completeness, in Secs. 3 and 4, we redo the known regularity theory by giving some new proofs and estimating everything by the L^2 norm of the solution in $B_1(0)$. By standard regularity theory it is enough to prove the $C^{1/2}$ regularity of u_{x_n} when $x \in \mathbb{R}^n_+ := \{x = (x', x_n) : x_n \geq 0\}$ approaches a free boundary point, i.e., a boundary point of the coincidence set. In Sec. 5, we achieve this in the whole space case (which can be obtained from the local one by blowing up) where we have full convexity of the solution along tangential directions. In Sec. 6, we prove our main result. Finally, in the appendix we state a parabolic version of the monotonicity formula of Sec. 5.

§3. Lipschitz regularity and almost convexity

The existence of the solution in H^1 is immediate and in a more general setting has been proved in [5]. As mentioned above we redo the regularity by estimating everything by the L^2 norm of the solution.

Lemma 1. *Let u be a solution in B_1 to the zero thin obstacle problem. Then there exists constants C such that*

$$\| u \|_{H^1(B_{1/2})} \leq C \| u \|_{L^2(B_1)}$$

and

$$\| u \|_{L^\infty(B_{1/2})} \leq C \| u \|_{L^2(B_1)} .$$

Proof. First we will show that u^+ and u^- are subharmonic. Since our problem always admits positive perturbations our solution is superharmonic. Thus $\min(0,u)$ is also superharmonic. That u^+ is a subharmonic in obtain by perturbing u as follows: Let η be a cut-off function in B_1, i.e., $0 \leq \eta \leq 1$ and $\eta = 0$ near ∂B_1 and $\gamma_\delta(u)$ the delta linear approximation of the Heaviside function, i.e., $\gamma_\delta(u) = 0$ for $u \leq 0$, $\gamma_\delta(u) = \frac{1}{\delta}u$ for $0 < u < \delta$ and $\gamma_\delta(u) = 1$ for $u > \delta$. By choosing $0 < \varepsilon < \delta$, the function $u_\varepsilon := u - \varepsilon\eta\gamma_\delta(u)$ is admissible. Let, now, ε go to zero then we have that the primitive of $\gamma_\delta(u)$ is subharmonic. But the primitive of $\gamma_\delta(u)$ is a δ-approximation of u^+ and therefore by passing to the limit u^+ is subharmonic.

Now, since u^+ and u^- are subharmonic by standard energy inequality

$$\int_{B_{1/2}} |\nabla u^{\pm}|^2 dx \leq C \int_{B_1} (u^{\pm})^2 dx$$

whence

$$\int_{B_{1/2}} |\nabla u|^2 dx \leq C \int_{B_1} u^2 dx.$$

Again, since u^+ and u^- are subharamonic, we have

$$\sup_{B_{1/2}}(u^{\pm}) \leq C \int_{B_1} u^{\pm} dx \leq C\Big(\int_{B_1} |u^{\pm}|^2 dx\Big)^{\frac{1}{2}}.$$

Hence the second inequality is proved, too. □

Theorem 1. *Let u be a solution in B_1 of the zero thin obstacle problem. Then there exists constants C such that*

$$\text{(a)} \qquad u \,\|_{\mathrm{Lip}(B_{1/2})} \leq C \,\|\, u \,\|_{L^2(B_1)}$$

and

$$\text{(b)} \qquad \inf_{B_{1/2}} u_{\tau\tau} \geq -C \,\|\, u \,\|_{L^2(B_1)},$$

where τ is any direction parallel to the $\mathbb{R}^{n-1}$ hyperplane.

Proof. In order to prove (a) we enlarge the thin obstacle by taking the auxiliary function h such that

$$\begin{cases} \Delta h = 0 & \text{on} \quad B_1 \setminus \{x_n = 0\} \\ h = 0 & \text{on} \quad B_1' := B_1 \cap \{x_4 = 0\} \\ h = -m & \text{on} \quad \partial B_1, \end{cases}$$

where m is a positive constant with $-m < \min_{\partial B_1} u$. Now, h is defined in all of B_1 and u is also a solution of the obstacle problem with h as an obstacle. Notice that h is analytic in $B^+_{1/2}$ (and $B^-_{1/2}$) up to $x_n = 0$. Let $M = \sup_{B_{1/2}} |\nabla h|$ and define $v(x) := \max(M^2, |\nabla u|^2)$ for $x \in B_{1/2}$. Since $|\nabla u| \leq |\nabla h|$ (see [3]) v is subharmonic and

$$v \leq M^2 + \int_{B_{1/2}} |\nabla u|^2 dx.$$

Thus, by Lemma 1 (a) is proved.

To bound $u_{\tau\tau}$ by below we enlarge the obstacle to $h_\varepsilon(x', x_n) = -\frac{x_n^2}{\varepsilon}$. Then, by results of the thick obstacle problem (see Theorem 3 in [3]), $D^2(u_\varepsilon - h_\varepsilon) \geq 0$ on the coincidence set and along the free boundary with a modulus of continuity coming from the noncoincidence set where u_ε is the corresponding solution. In particular, $(u_\varepsilon)_{\tau\tau} \geq 0$ along the free boundary. We proceed now using the Berstein technique. Set, for $0 \leq \eta \leq 1$ smooth and vanishing near $(\partial B_1)^+$,

$$g(x) := \eta(x)(u_\varepsilon)_{\tau\tau}(x) - \lambda|\nabla u_\varepsilon(x)|^2$$

and we have, on the noncoincidence set,

$$\Delta g = \Delta\eta(u_\varepsilon)_{\tau\tau} + 2\nabla\eta \cdot \nabla(u_\varepsilon)_{\tau\tau} - 2\lambda(u_\varepsilon)_{x_i x_j}(u_\varepsilon)_{x_i x_j}.$$

At a minimum x_0

$$\nabla g = (\nabla\eta)(u_\varepsilon)_{\tau\tau} + \eta\nabla(u_\varepsilon)_{\tau\tau} - 2\lambda(u_\varepsilon)_{x_i}\nabla(u_\varepsilon)_{x_i}$$

and therefore

$$0 \leq \Big(\Delta\eta - 2\frac{|\nabla\eta|^2}{\eta}\Big)(u_\varepsilon)_{\tau\tau} + 4\lambda\frac{\nabla\eta}{\eta} \cdot (u_\varepsilon)_{x_i}\nabla(u_\varepsilon)_{x_i} - 2\lambda(u_\varepsilon)_{x_i x_j}(u_\varepsilon)_{x_i x_j}$$

or, setting $M := -(u_\varepsilon)_{\tau\tau}(x_0)$, we have

$$2\lambda\sum_{i,j=1}^{n}(u_\varepsilon)^2_{x_i x_j}(x_0) - C_0 M \leq 4\lambda C_1\big|\nabla(u_\varepsilon)_{x_i}(x_0)\big|\,\big|(u_\varepsilon)_{x_i}(x_0)\big|.$$

Now, choose λ large enough ($\lambda > C_0$) and by the fact that $M \leq |D^2 u_\varepsilon|$, ($M > 1$), we have

$$\lambda|D^2 u_\varepsilon|^2 \leq 2\lambda C_1|D^2 u_\varepsilon|\,|\nabla u_\varepsilon|$$

therefore

$$M \leq |D^2 u_\varepsilon| \leq 2C_1|\nabla u_\varepsilon|.$$

But $|\nabla u_\varepsilon|$ is subharmonic outside of $\{u_\varepsilon = h_\varepsilon\}$ and $\nabla u_\varepsilon = \nabla h_\varepsilon$ on $\{u_\varepsilon = h_\varepsilon\}$. Since the function h of part (a) being Lipschitz satisfies $h(x', x_n) > h_\varepsilon(x', x_n)$ if $x_n < C\varepsilon$, $\{u_\varepsilon = h_\varepsilon\} \subset \{(x', x_n) : |x_n| < C\varepsilon\}$. This, together with the fact, that $\max(u, \frac{-x_n^2}{\varepsilon})$ is an admissible function, we have

$$\int_{B_1}|\nabla_{u_\varepsilon}|^2 dx \leq \int_{B_1}|\nabla\max\Big(u, \frac{-x_n^2}{2}\Big)|^2 dx \leq \int_{B_1}|\nabla u|^2 dx + C,$$

i.e., a bound independent of ε. Now that $|\nabla u_\varepsilon|$ is bounded independently of ε in $B_{1/2}$ follows immediately since $\max(|\nabla u_\varepsilon|, |\nabla h_\varepsilon|)$ is subharmonic. Hence M is independent of ε and the proof is complete. $\square$

§4. The local $C^{1,\alpha}$ estimate

We start this section by pointing out that we already know some control of u by above. More precisely

Lemma 2. *Let u be our solution in $B_1(0)$ and $\| u \|_{L^2(B_1)} \leq 1$. Then for some universal constant in $B^+_{1/2}$*

(a) *$u(x', x_n) - C x_n^2$ is concave in x_n and $u(x', x_n) + C|x'|^2$ is convex in x'*

In particular

(b) $u_{x_n}(x', t) - u_{x_n}(x', s) \leq C(t - s)$

(c) $u(x', t) - u(x, 0) \leq C t^2$

and

(d) *if $u(x', t) \geq h$ then in the half ball*

$$HB'_\rho(x') := \{z' : |z' - x'| \leq \rho, \langle z' - x', \nabla_{x'} u\rangle \geq 0\}$$

we have $u(z', t) \geq h - C\rho^2$. □

Proof. (a) is a restatment of Theorem 1 and the rest is elementary. □

We will now use cylinders, i.e., $\Gamma'_r := B'_r \times [0, \frac{r}{2n}]$ instead to simplify the geometry.

Theorem 2. *Under our hypothesis $\| u \|_{L^2(\Gamma_1)} \leq 1$ and 0 a free boundary point, suppose that*

$$\inf_{\Gamma_r} u_{x_n} \geq -r^\alpha .$$

Then

$$\operatorname*{osc}_{\Gamma_{r/2}} u \leq C r^{1+\alpha} .$$

Proof. Since

$$\sup_{\Gamma_r} u_{x_n} \leq Cr$$

$$\sup_{\Gamma_r} |u_{x_n}| \leq C r^\alpha .$$

Therefore for fixed x'

$$\operatorname*{osc}_{0 \leq t \leq \frac{r}{2n}} u(x', t) \leq C r^{1+\alpha} .$$

If $\sup_{B'_{r/2}} u(x',0) \le Mr^{1+\alpha}$ the proof is complete. If not, there is, by Lemma 2(d), a half ball with center in $B'_{r/2}$ and radius r where

$$u(z',0) \ge Mr^{1+\alpha} - Cr^2 \ge \frac{1}{2}Mr^{1+\alpha}.$$

In particular,

$$u(z',t) \ge (\frac{1}{4}M)r^{1+\alpha}.$$

By Green's formula applied to u and the potential V given in remark (d) of Theorem 9 in [3], we have

$$u(0,t) - \fint_{B'_r} u(z',t)dz' = -\int_{B'_r} (\Delta_{z'}u)V dz'.$$

Since 0 is a free boundary point,

$$|u(0,t)| \le Cr^{1+\alpha}$$

and, consequently, the left hand side can be estimated, using the bound of $u(z',t)$, by

$$u(0,t) - \fint_{B'_r} u(z',t) \le -CMr^{1+\alpha}.$$

Integrating in t we get, on one hand,

$$\int_0^r \left[u(0,t) - \fint_{B_r} u(z',t)dz'\right]dt \le -CMr^{2+\alpha}.$$

On the other hand, since $-\Delta_{z'}u = u_{x_n x_n}$

$$\begin{aligned}\Big| -\int_0^r \int_{B'_r} (\Delta_{z'}u)V dz' dt\Big| &= \Big|\int_{B'_r} V\left[u_{x_n}(z',r) - u_{x_n}(z',0)\right]dz'\Big| \\ &\le Cr^{-n+2}r^n r^\alpha \le Cr^{2+\alpha}.\end{aligned}$$

A contradiction for M large. (Notice that M is independent of α for $\alpha \le 1$.) $\square$

Theorem 3 ($C^{1,\alpha}$ estimate)**.** *Let u be our solution in Γ_1 with $\| u \|_{L^2(\Gamma_1)} \leq 1$ and 0 a free boundary point. Then there exists a constant C and $\mu < 1$ such that*

$$\sup_{\Gamma_{4^{-k}}} |u| \leq C\Big(\frac{\mu}{4}\Big)^k.$$

Proof. We will really prove that

$$\sup_{\Gamma_{4^{-k}}} |u_{x_n}| \leq C\mu^k.$$

So assume by induction that the theorem is true for $k \leq k_0$, i.e.,

$$\sup_{\Gamma_{4^{-k_0}}} |u_{x_n}| \leq C\mu^{k_0}.$$

In particular,

$$\sup_{\frac{1}{2}\Gamma_{4^{-k_0}}} |u| \leq CM\Big(\frac{\mu}{4}\Big)^{k_0}.$$

We normalize to Γ_1 by setting

$$\overline{u}(x) = C^{-1}\Big(\frac{\mu}{4}\Big)^{k_0} \cdot u(4^{-k_0}x).$$

Therefore, we have

(i) $\sup |\overline{u}_{x_n}| \leq 1$,

(ii) $\sup |\overline{u}| \leq M$,

and

(iii) $\overline{u}_{x_n x_n} \leq C_0 C^{-1}(4\mu)^{k_0}$ and $\overline{u}_{\tau\tau} \geq -C_0 C^{-1}(4\mu)^{-k_0}$.

(We can choose k^* such that $C_0C^{-1} = C_1\mu^{k^*}$ for $\mu < 1$.)

The idea, now, is to locate a large region of points $(x', 0)$ where $|u_{x_n}|$ is small. If u were actually convex in the tangential direction and, since $(0, 0)$ is a free boundary point, $u_{x_n} \equiv 0$ in half the ball of B_1'. Then Poisson formula would imply that $|u_{x_n}| \leq \mu < 1$ in the interior and away from $x_n = 0$ but, by concavity, we would "shoot back" to $x_n = 0$ to have the desired result. Since we are not in that situation we must correct the argument.

Set

$$w := \overline{u} - C_1\mu^{k^*}(4k)^{-k_0}\big(|x'|^2 - (n-1)x_n^2\big).$$

The function w is harmonic away from the coincidence set, goes to zero near the origin and it is negative on the coincidence set. Therefore it must attain its nonnegative maximum on $\partial\Big(\Gamma_{1/2} \cup \Gamma^*_{1/2} - \{u(x', 0) = 0\}\Big)$ where $\Gamma^*_{1/2}$ is the reflection of $\Gamma_{1/2}$ with respect to $\{x_n = 0\}$. By symmetry it is enough to examine the situation only on one side.

There are two cases.

Case 1. The maximum is attained "far from $x_n = 0$," i.e., at $(x_0, \frac{1}{2})$). Then

$$\overline{u}\Big(x_0, \frac{1}{2}\Big) \geq -\Big(\frac{n-1}{16n^2}\Big) C_1 \mu^{k^*} (4\mu)^{-k_0}$$

and, by (d) of Lemma 2, there exists an $(n-1)$-dimensional half ball $HB'_{1/2}(x_0, \frac{1}{2})$ such that

$$\overline{u}\Big(x', \frac{1}{2}\Big) \geq -\frac{(n-1)}{8n^2} C_1 \mu^{k^*} (4\mu)^{-k_0} \quad \forall\, x' \in HB'_{1/2}(x_0, 0).$$

"Shooting back" by concavity in y

$$\overline{u}_{x_n}(x, \frac{1}{2}) \geq -\frac{(n+1)}{8n^2} C_1 \mu^{k^*} (4\mu)^{-k_0} \quad \forall\, x' \in HB'_{1/2}(x_0, 0).$$

Case 2. The maximum is attained "close to $x_n = 0$," i.e., on the lateral side of the cylinder $\Gamma_{1/2}$, say at (x'_0, t). By the choice of the width of the unit cylinder makes

$$|x'|^2 \geq (2n)^2 x_n^2$$

and hence

$$u(x'_0, t) \geq C_2 \mu^{k^*} (4\mu)^{-k_0}$$

from the half ball estimate

$$u(x', t) \geq \frac{1}{2} C_2 \mu^{k^*} (4\mu)^{-k_0} \quad \forall\; x' \in HB'_{1/2}(x'_0, t)$$

and by concavity estimate

$$u(x', 0) > 0 \quad \forall\; x' \in HB'_{1/2}(x'_0, 0).$$

Thus $u_{x_n} = 0$ in $HB'_{1/2}(x', 0)$.

In both cases, we have reached the following conclusion:

In a half ball $HB'_{1/2}$ centered at some point of $B'_{1/2}$ we have

$$u_{x_n}(z', 0) \geq -C_3 \mu^{k^*} (4\mu)^{-k_0}.$$

If we make sure that $\frac{1}{2} > \mu > \frac{1}{4}$, we have for an appropriate k^*

$$0 \geq u_{x_n} \geq -C_3 \left(\frac{1}{2}\right)^{k^*} \geq -\frac{1}{2}.$$

Since $|u_{x_n}| \leq 1$ in Γ_1 by Poisson representation formula we get

$$|u_{x_n}| < \theta < 1$$

in the interior domain

$$D := \left\{ |x'| \leq \frac{1}{2},\ x_n = \frac{1}{2} \right\}.$$

Again by convexity we shoot back to $x_n = 0$ to have

$$|u_{x_n}| \leq \theta + C_3 \mu^{k^*} (4\mu)^{-k_0} \leq \theta + C_3 2^{-k^*} =: \mu < 1$$

and the proof is complete. □

Remark. Notice that the conclusion of Theorem 3 implies the existence of an $0 < \alpha < 1$ such that $\sup_{\Gamma_r} |u_{x_n}| < Cr^{\alpha}$ and $\sup_{\Gamma_r} |u| \leq Cr^{1+\alpha}$. Hence we are under the hypothesis of Theorem 2.

Corollary. *Let u be our solution and $\| u \|_{L^2(B_1)} \leq 1$, then there exists C such that*

$$\| u \|_{C^{1,\alpha}(B_{1/4})} \leq C.$$

Proof. Let $d_F(x', t)$ denote the distance of (x', t) to the free boundary and $d(x_1, x_2)$ the distance between the points x_1 and x_2. Now take two fixed points $x_1 = (x_1', t_1)$ and $x_2 = (x_2', t_2)$. There are two cases to consider:

First case. If

$$d_F := \max_{i=1,2} d_F(x_i', t_i) \geq 4d((x_1', t_1), (x_2', t_2))$$

then, since u is harmonic in $B_{d_F}(x_i', t_i)$

$$\operatorname*{osc}_{\frac{1}{2}B_{d_F}(x_1', t_1)} u \leq C d_F^{1+\alpha}$$

and we use a priori estimates.

Second case. If

$$d_F := \max_{i=1,2} d_F(x_i', t_i) \leq 4d((x_1', t_1), (x_2', t_2))$$

then

$$|u(x_i', t_i)| \leq C d_F^{1+\alpha} \leq C d^{1+\alpha}$$

and

$$|\nabla u(x_i', t_i)| \leq C d_F^{\alpha} \leq C d^{\alpha}$$

□

§5. Convex case

In this section, we would like to treat first the case where $u_{\tau\tau} > 0$ $(u_{x_n x_n} < 0)$. This is the global situation since it can be obtained by a blow up process, i.e., it is the limit as $r_k \to 0$ of a sequence of solutions $u_k(x) = \frac{1}{r_k} u(r_k x)$ for $x \in B_{r_k}$ centered at a free boundary point taken for simplicity to be the origin. Notice that the second tangential derivatives yield the convexity of $\{(x', 0) : u_{x_n}(x', 0) < 0\}$ since it coincides with the coincidence set.

Our proof relies on the calculation of the first eigenvalue of the following eigenvalue problem.

Lemma 3. *Let ∇_θ denote the surface gradient on the unit sphere ∂B_1. Set*

$$\lambda_0 = \inf_{\substack{w \in H^{1/2}(\partial B_1^+) \\ w=0 \text{ on } (\partial B_1')^-}} \frac{\int_{\partial B_1^+} |\nabla_\theta w|^2 dS}{\int_{\partial B_1^+} w^2 dS},$$

where

$$B_1^+ = \{x = (x', x_n) \in B_1 : x_n > 0\}, \ B_1' = B_1 \cap \mathbb{R}^{n-1}$$

and

$$(\partial B_1')^- = \{x' = (x'', x_{n-1}) \in \mathbb{R}^{n-1} : x_{n-1} < 0\}.$$

Then

$$\lambda_0 = \frac{2n-3}{4}.$$

Proof. It is easy to see that the first eigenfunction of this problem is given by

$$w_0 = \varrho^{\frac{1}{2}} \cos\left(\frac{\psi}{2}\right),$$

where $\varrho = \sqrt{x_{n-1}^2 + x_n^2}$ and $\psi = \arctan(\frac{x_n}{x_{n-1}})$. Indeed, w_0 is the restriction to the upper unit hemisphere of a harmonic homogeneous function of degree $\frac{1}{2}$:

$$w(x) = \varrho^{1/2} h(\theta) \quad \text{for} \quad \theta \in S^1.$$

Thus $h(\theta)$ is an eigenfunction of the the Laplace–Beltrami operator on the sphere. More precisely, computing the spherical Laplacian we get

$$\Delta_\theta h = -\frac{1}{4} h + \frac{n-1}{2} h$$

or

$$\lambda_0 = \frac{2n-3}{4}.$$

Furthermore, h is nonnegative and, if we reflect it, it is an eigenfunction of S^1 with Dirichlet data on the slit domain, this makes h the first eigenfunction of the problem. $\square$

Now, our monotonicity result.

Lemma 4. *Let w be any continuous function on $\overline{B_r^+}$, harmonic in B_r^+, $w(0) = 0$, $w(x', 0) \leq 0$, $w(x', 0) \cdot w_{x_n}(x', 0) = 0$, $\forall\, x' \in B_r'$ and $\{x' \in B_r' : w(x', 0) < 0\}$ be nonempty and convex. Set*

$$\varphi(r) = \frac{1}{r} \int\limits_{B_r^+} \frac{|\nabla w|^2}{|x|^{n-2}} dx$$

then $\varphi(r) < +\infty$ and increasing for $r \in (0, R)$.

Proof. Since $\triangle w^2 = 2w\triangle w + 2|\nabla w|^2 = 2|\nabla w|^2$

$$\varphi(r) = \frac{1}{2r} \int\limits_{B_r^+} \frac{\triangle(w^2)}{|x|^{n-2}} dx.$$

Now, by a mollification argument, we obtain

$$\varphi'(r) = -\frac{1}{r^2} \int\limits_{B_r^+} \frac{\triangle(w^2)}{|x|^{n-2}} dx + \frac{1}{r^{n-1}} \int\limits_{\partial B_r^+} |\nabla w|^2 dS$$

$$= -\frac{1}{r^n} \int\limits_{\partial B_r^+} w w_r \, dS - \frac{n-2}{2r^{n+1}} \int\limits_{\partial B_r^+} w^2 dS + \frac{1}{r^{n-1}} \int\limits_{\partial B_r^+} |\nabla w|^2 dS,$$

where we have used the fact that $ww_{x_n} = 0$ on B'_r. Applying Schwartz inequality, we have

$$\varphi'(r) \geq -\Big(\int\limits_{\partial B_r^+} \frac{w^2}{2r^{n+1}} dS\Big)^{\frac{1}{2}} \Big(\int\limits_{\partial B_r^+} \frac{2w_r^2}{r^{n-1}} dS\Big)^{\frac{1}{2}} - \frac{n-2}{2r^{n+1}} \int\limits_{\partial B_r^+} w^2 dS$$

$$+\frac{1}{r^{n-1}} \int\limits_{\partial B_r^+} w_r^2 dS + \frac{1}{r^{n-1}} \int\limits_{\partial B_r^+} |\nabla_\tau w|^2 dS,$$

where ∇_τ is the tangential gradient on ∂B_r. Then

$$\varphi'(r) \geq -\frac{2n-3}{4r^{n+1}} \int\limits_{\partial B_r^+} w^2 dS + \frac{1}{r^{n+1}} \int\limits_{\partial B_r^+} |\nabla_\theta w|^2 dS.$$

By convexity of the set $\{w < 0\}$ on B'_r and Lemma 3, we see that

$$\varphi'(r) \geq 0 \qquad \forall\ 0 < r \leq R$$

and therefore $\varphi(r) \leq \varphi(R)\ \forall\ 0 < r \leq R$. □

Theorem 4. *Let u be a solution to our global zero thin obstacle problem then $u \in C^{1,1/2}$ up to the coincidence set on either side.*

Proof. As it was remarked before it is enough to prove that $u_{x_n}(x)$ goes to zero in $C^{1/2}$ fashion as x approaches a point in the noncoincidence set, which we take it to be the origin. Set $w = u_{x_n}$, then w satisfies all the hypotheses of Lemma 4, in particular, w vanishes at the origin. Therefore

$$\frac{1}{r^{n-1}} \int\limits_{B_r^+} |\nabla w|^2 dx < C.$$

Since w vanishes on at least half of the ball along the boundary by Poincaré inequality

$$⨍_{B_r^+} w^2 dx \leq Cr^2 ⨍_{B_r^+} |\nabla w|^2 \leq Cr.$$

Since w^2 is a subharmonic function across $x_n = 0$

$$w^2\big|_{B_{r/2}} \leq Cr.$$

Given now a generic point $x = (x', t)$ we have two cases:

(a)) If $d(x, \{u=0\} \cup \mathbb{R}^{n-1}) < 4t$ then in the ball $B_t(x)$ w is harmonic and $|w| \leq Ct^{1/2}$.

(b) If $d(x, \{u=0\} \cup \mathbb{R}^{n-1}) > 4t$ then in the ball $B_{d/2}(x)$ w is harmonic and $|w| \leq Cd^{1/2}$. □

§6. Main result

In this section, we prove our main result, i.e., Theorem 5. We will assume that we have C^α regularity for the normal derivative for some $0 < a \leq \frac{1}{2}$ and show how we can improve to reach $C^{1/2}$. First a lemma which uses the almost concavity and convexity in the normal and tangential direction, respectively.

Lemma 5. *Let u be a solution of the zero thin obstacle problem in the unit ball centered at the origin which is taken to be a free boundary point. Then there exists a $\delta > 0$, depending on the Hölder exponent α of the $C^{1,\alpha}$ regularity such that the convex hull of $\{x', 0) : u_{x_n} < -r^{\alpha+\delta}\}$ in B'_r, for $r < 1$ does not contain the origin, i.e.,*

$$(0,0) \notin \Gamma(\{u_{x_n} < -r^{\alpha+\delta}\}) \cap B'_r.$$

Proof. Let $(x', 0) \in \{u_{x_n} < -r^{\alpha+\delta}\}$ then since $u_{x_n x_n} < M$

$$u(x', h) \leq -r^{\alpha+\delta} h + \frac{M}{2} h^2.$$

Choose $h = \frac{r^{\alpha+m\delta}}{M}$ for some $m > 1$ then

$$u(x, h) \leq -\frac{r^{2\alpha+(m+1)\delta}}{2M}.$$

Moreover, if we restrict to $|x'| \leq \frac{r}{2M}$

$$u(x, h) + M|x|^2 < -\frac{r^{2\alpha+(m+1)\delta}}{4M}$$

provided that $\delta < \frac{2(1-\alpha)}{m+1}$.

On the other hand, by $C^{1,\alpha}$ regularity

$$u(0, h) \geq -c_0 h^{1+\alpha}$$

or, with $h = \frac{r^{\alpha+m\delta}}{M}$,

$$u(0, h) \geq -c_0 \frac{r^{(\alpha+m\delta)(1+\alpha)}}{M^{1+\alpha}}.$$

Therefore, if we choose $\delta > \frac{\alpha(1-\alpha)}{\alpha m-1}$ and $m > 1 + \frac{2}{\alpha}$, we have

$$(0,0) \notin \Gamma(\{u_{x_n} < -r^{\alpha+\delta}\}).$$

□

Remark. In Lemma 5, if $0 < \alpha_0 \leq \alpha \leq \frac{1}{2}$ then δ can be taken to be in $[\frac{\alpha_0}{4}, \frac{\alpha_0}{2}]$.

Lemma 6. *Let $\delta > 0$ and u be as in Lemma 3. Set $w \equiv u_{x_n}$ and, for $r < 1$,*

$$\varphi(r) = \frac{1}{r} \int_{B_r^+} \frac{|\nabla w|^2}{|x|^{n-2}} dx.$$

Then, there exists a universal constant C, such that

(i) *if* $2\alpha + \delta > 1$, $\varphi(r) \leq C$

and

(ii) *if* $2\alpha + \delta < 1$, $\varphi(r) \leq C\, r^{2\alpha+\delta-1}$.

Proof. Since $\triangle w^2 = w\triangle w + 2|\nabla w|^2 = 2|\nabla w|^2$ we have as in Lemma 2

$$\begin{aligned} &\varphi'(r) - \frac{1}{2r^2} \int_{B_r^+} \frac{\triangle w^2}{|x|^{n-2}} dx + \frac{1}{r^{n-1}} \int_{\partial B_r^+} |\nabla w|^2 dS \\ &\geq -\frac{2n-3}{4r^{n+1}} \int_{\partial B_r^+} w^2 dS + \frac{1}{r^{n-1}} \int_{\partial B_r^+} |\nabla_\tau w|^2 dS. \end{aligned}$$

Now, consider the truncated function $w_t := -(w + r^{\alpha+\delta})^-$ and observe that

$$\int_{\partial B_r^+} |\nabla_\tau w_t|^2 dS \leq \int_{\partial B_r^+} |\nabla_\tau w|^2 dS.$$

Then, we have

$$\varphi'(r) \geq -\frac{2n-3}{4r^{n-1}} \int_{\partial B_r^+} ((w - w_t) + w_t)^2 dS + \int_{\partial B_r^+} |\nabla_\tau w_t|^2 dS$$

and, by Lemma 1,

$$\varphi'(r) \geq -\frac{2n-3}{4r^{n-1}} \int_{\partial B_r^+} [(w - w_t)^2 + 2w_t(w - w_t)] dS$$

or

$$\varphi'(r) \geq -\frac{3(2n-3)}{4} r^{2a+2\delta-2}.$$

Therefore,

$$\varphi(1) - \varphi(r) \geq -\frac{3}{4}\left(\frac{2n-3}{2a+\delta-1}\right) + \frac{3}{4}\frac{2n-3}{2a+\delta-1} r^{2a+\delta-1}$$

and the proof of the lemma is complete. □

Finally, our main result.

Theorem 5. *Let u be the solution of the zero thin obstacle problem. Then u is $C^{1,1/2}$ on each side and up to the obstacle.*

Proof. Enough to prove that u_{x_n} is $C^{1/2}$. Let $w = u_{x_n}$ and w_t be as in Lemma 4. Since w_t vanishes on more than half of B'_r, by Poincaré inequality

$$\int_{B_r^+} w_t^2 dx \leq Cr^2 \int_{B_r^+} |\nabla w_t|^2 dx.$$

Therefore

$$\frac{1}{r^{n+1}} \int_{B_r^+} w_t^2 dx \leq \frac{C}{r^{n-1}} \int_{B_r^+} |\nabla w|^2 dx \leq C\varphi(r).$$

Since w_t^2 is subharmonic, for any $|x| \leq r$ we have

$$w_t^2(x) \leq \frac{n}{w_n s^n} \int_{B_s(x)} w_t^2 dx \leq \frac{n}{w_n s^n} \int_{B_r} w_t^2 dx \quad s < r - |x|$$

$$\leq C\left(\frac{r}{s}\right)^n \frac{n}{w_n r^n} \int_{B_r^+} w_t^2 dx \leq C\left(\frac{r}{s}\right)^n \varphi(r) r.$$

Now, by Lemma, 4 in the case where $2\alpha + \delta > 1$, i.e., $\varphi(r) \leq C$ we conclude directly, as in the convex case, that w is $C^{1/2}$. In the second case, where $2\alpha + \delta < 1$, i.e., $\varphi(r) \leq Cr^{2\alpha+\delta-1}$, we conclude that w is $C^{\alpha+\delta/2}$. In such a situation we repeat our theorem with α replaced by $\alpha_1 = \alpha + \frac{\delta}{2}$ to obtain $\alpha_1 + \frac{\delta}{2}$. So, by a finite number of repetition we have $\alpha_\kappa + \frac{\delta}{2} > \frac{1}{2}$, i.e., we fall under the first case. □

§7. Appendix

The purpose of this section, is to state an analogous monotonicity formula for the parabolic case in all of space.

Let w be any continuous function on $\overline{\mathbb{R}^n_+}\times[-1,0]$, caloric in $\mathbb{R}^n_+\times[-1,0]$ where $\mathbb{R}^n_+ := \{x = (x', x_n) : x_n > 0\}$ with

$$\int\limits_{B_R} w^2(x,-1)dx \le Ce^{\frac{|x|^2}{4+\varepsilon}}$$

for R large and some $\varepsilon > 0$. Assume $w(0) = 0$, $w(x',0) \le 0$, $w(x',0)\cdot w_{x_n}(x',0) = 0$ for every $x' \in \mathbb{R}^{n-1}$, and the set $\{x' \in \mathbb{R}^{n-1} : w(x',0) < 0\}$ is nonempty and convex. Let $G(x,t) = \frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x|^2}{4t}}$ and set

$$\varphi(t) := \frac{1}{t^{1/2}}\int\limits_{-t}^{0} |\nabla w|^2 G(x,-s)dxds$$

then $\varphi(t)$ is monotone increasing in t.

The proof of this result relies on the calculation of the first eigenvalue of the following eigenvalue problem

$$\lambda_0 = \inf_{\substack{w\in H^1(\mathbb{R}^n_+)\\ w=0 \text{ on } \mathbb{R}^{n-1}_-}} \frac{\int_{\mathbb{R}^n_+} |\nabla w(y,-1)|^2 e^{-\frac{|y|^4}{4}}dy}{\int_{\mathbb{R}^n_+} w^2(y,-1)e^{-\frac{|y|^2}{4}}dy},$$

where

$$\mathbb{R}^{n-1}_- := \{(x',0) : x' \in \mathbb{R}^{n-1}, x_{n-1} < 0\}.$$

A careful calculation shows that $\lambda_0 = \frac{1}{4}$. It can be shown that under some control of the time derivative the above can be applied to obtain the optimal regularity of the parabolic version of the thin obstacle problem, as the one treated in [1]. This and the proofs of the above is the content of a forthcoming paper.

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