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STEADY-STATE SOLUTIONS TO THE EQUATIONS OF MOTION OF SECOND-GRADE FLUIDS WITH GENERAL NAVIER-TYPE SLIP BOUNDARY CONDITIONS IN HÖLDER SPACES

Abstract. We consider a boundary-value problem for the stationary flow of an incompressible second-grade fluid in a bounded domain. The boundary condition allows for no-slip, Navier-type slip and free slip on different parts of the boundary.

We first establish the well-posedness of a linear auxiliary problem by means of a fixed-point argument in which it is decomposed into a Stokes-type problem and two transport equations. Then we use the method of successive approximations to prove the unique solvability in Hölder spaces of the nonlinear problem with a sufficiently small body force.

Dedicated to Professor Vsevolod A. Solonnikov on his seventieth birthday

1. Introduction

It is well-known that non-Newtonian fluids such as molten polymers do not satisfy the no-slip boundary condition [4]. A variety of constitutive relations and slip boundary conditions have been proposed to model such flows, but the wellposedness of the associated (initial-)boundary-value problems has been established in very few cases only. In this paper we prove the existence and uniqueness of a steady-state solution to the problem of the motion of a so-called *second-grade* fluid (or fluid of *grade two*) subject to a general Navier-type slip boundary condition.

The Cauchy stress tensor for the second-grade fluid model is given by

$$\widetilde{\mathbf{T}} = -\tilde{p}\,\mathbf{I} + \mu\,\mathbf{A}_1 + \tilde{\alpha}_1\,\mathbf{A}_2 + \tilde{\alpha}_2\,\mathbf{A}_1^2, \tag{1.1}$$

where $\tilde{p}$ is the pressure, μ is the viscosity, $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are the normal stress moduli, and $\mathbf{A}_1$ and $\mathbf{A}_2$ are the Rivlin-Ericksen tensors defined by

$$\mathbf{A}_1 = \nabla\mathbf{v} + (\nabla\mathbf{v})^T, \quad \mathbf{A}_2 = \frac{d}{dt}\mathbf{A}_1 + \mathbf{A}_1\nabla\mathbf{v} + (\nabla\mathbf{v})^T\,\mathbf{A}_1.$$

Here $\mathbf{v}$ is the velocity and d/dt denotes the material time derivative. Dunn and Fosdick [5] showed that *all* kinematically admissible flows of a fluid modelled by (1.1) will satisfy the Clausius-Duhem inequality if and only if $\mu \geq 0$ and $\tilde{\alpha}_1 + \tilde{\alpha}_2 = 0$. In addition, the assumption that the specific Helmholtz free energy is a minimum when the fluid is at rest imposes the restriction $\tilde{\alpha}_1 \geq 0$. These considerations are discussed in detail by Dunn and Rajagopal [6]. However, throughout this paper we assume only that

$$\mu > 0, \quad \tilde{\alpha}_1 \neq 0. \tag{1.2}$$

Furthermore, we assume that the fluid is incompressible, with constant density ϱ, and define $p = \tilde{p}/\varrho$, $\nu = \mu/\varrho$, $\alpha_1 = \tilde{\alpha}_1/\varrho$, $\alpha_2 = \tilde{\alpha}_2/\varrho$. Then

$$\begin{aligned}\mathbf{T} := \varrho^{-1}\,\widetilde{\mathbf{T}} = -p\,\mathbf{I} + \nu\,\mathbf{A}_1 + \alpha_1\,(\,\mathbf{v}\,\cdot\,\nabla\,\mathbf{A}_1 +\,\mathbf{A}_1\,\mathbf{W} -\,\mathbf{W}\,\mathbf{A}_1) \\ +(\alpha_1+\alpha_2)\,\mathbf{A}_1^2 + \alpha_1\frac{\partial}{\partial t}\,\mathbf{A}_1,\end{aligned} \tag{1.3}$$

where $\mathbf{W} = \left(\nabla\,\mathbf{v} - (\nabla\,\mathbf{v})^T\right)/2$. Thus the equations of motion for steady flows become

$$\mathbf{v}\,\cdot\,\nabla\,\mathbf{v} - \alpha_1\,\mathbf{v}\,\cdot\,\nabla\triangle\,\mathbf{v} - \nu\triangle\,\mathbf{v} + \nabla p - \mathbf{N}(\,\mathbf{v}) = \mathbf{f} \ \text{ in } \ \Omega, \tag{1.4}$$

$$\nabla\,\cdot\,\mathbf{v} = 0 \ \text{ in } \ \Omega, \tag{1.5}$$

where Ω is the flow region, a bounded domain in $\mathbb{R}^3$, $\mathbf{f}$ is the body force per unit mass,

$$\mathbf{N}(\,\mathbf{v}) = \alpha_1(\nabla\,\mathbf{v})^T : \nabla\,\mathbf{A}_1 + \alpha_1\nabla\cdot(\,\mathbf{A}_1\,\mathbf{W} -\,\mathbf{W}\,\mathbf{A}_1) + (\alpha_1+\alpha_2)\nabla\cdot\left(\,\mathbf{A}_1^2\right).$$

The solvability of the corresponding no-slip problem has been studied by Coscia and Galdi [3], Galdi [7], and Bresch and Lemoine [2] (in [2] it is assumed that $\alpha_1 + \alpha_2 = 0$) in Sobolev spaces, and by Mogilevskiĭ and Solonnikov [12] in Hölder spaces. Instead, we impose the impermeability boundary condition

$$\mathbf{v}\,\cdot\,\mathbf{n} = 0 \ \text{ on } \ \Gamma, \tag{1.6}$$

where Γ is the boundary of Ω and $\mathbf{n}$ is the *interior* unit normal vector on Γ, and the slip boundary condition

$$(1-k)[\,\mathbf{T}\,\mathbf{n}]_\tau = \sigma k\,\mathbf{v} \ \text{ on } \ \Gamma, \tag{1.7}$$

where $k : \Gamma \to [0,1]$ is a given dimensionless function, $\mathbf{u}_\tau$ denotes the tangential component of any vector field $\mathbf{u}$ defined on Γ, i.e., $\mathbf{u}_\tau = \mathbf{u} -$

$(\mathbf{u}\cdot\mathbf{n})\mathbf{n}$, and σ is a given positive constant which has the dimension of velocity. We can assume without loss of generality that $\sigma = 1$, because (1.7) holds if and only if $(1-\hat{k})[\mathbf{T}\,\mathbf{n}]_\tau - \hat{\sigma}\hat{k}\,\mathbf{v} = 0$ on Γ, where $\hat{\sigma} > 0$ is arbitrary and $\hat{k} = \sigma k/(\hat{\sigma}(1-k)+\sigma k)$. Then, by (1.3), condition (1.7) becomes

$$(1-k)\left(\nu\,[\mathbf{A}_1\,\mathbf{n}]_\tau + \alpha_1\,[(\mathbf{v}\cdot\nabla\,\mathbf{A}_1)\,\mathbf{n}]_\tau + \mathbf{N}_\Gamma(\mathbf{v})\right) = \sigma k\,\mathbf{v} \quad \text{on} \quad \Gamma, \tag{1.8}$$

where

$$\mathbf{N}_\Gamma(\mathbf{v}) = \left[(\alpha_1(\mathbf{A}_1\,\mathbf{W} - \mathbf{W}\,\mathbf{A}_1) + (\alpha_1+\alpha_2)\,\mathbf{A}_1^2)\;\mathbf{n}\right]_\tau .$$

Remark 1. (a) In this paper we only consider the case when the motion of the solid surface at Γ can be described by a single rigid body motion. In general, for example when the surface undergoes tangential stretching or shrinking, or when separate axially symmetric components of the surface rotate about their axes of symmetry, the slip condition must be formulated as

$$(1-k)[\mathbf{T}\,\mathbf{n}]_\tau = \sigma k(\mathbf{v}-\mathbf{b}) \quad \text{on} \quad \Gamma,$$

where $\mathbf{b}$ is a given vector field on Γ such that $\mathbf{b}\cdot\mathbf{n} = 0$ and denotes the tangential velocity of the solid surface.

(b) Condition (1.7) is equivalent to

$$[\mathbf{T}\,\mathbf{n}]_\tau = \phi\,\mathbf{v} \quad \text{on} \quad \Gamma, \tag{1.9}$$

where $\phi = \sigma k/(1-k) : \Gamma \to [0,\infty]$. One may think of ϕ as a "friction coefficient," because condition (1.9) (or (1.7)) reduces to the no-slip condition where $\phi = \infty$ ($k = 1$), Navier's slip condition where $\phi \in (0,\infty)$ ($k \in (0,1)$), and the free slip condition where $\phi = 0$ ($k = 0$).

(c) The local-in-time solvability of the initial-boundary-value problem for Newtonian fluids with the boundary conditions (1.6), (1.7) has been obtained in the case of compressible heat-conducting fluids, incompressible fluids with constant density and incompressible nonhomogeneous fluids by Tani [15, 16], Tani *et al.* [17, 9, 10], and Itoh and Tani [8], respectively. In these studies, k is a function of both position and time. The initial-boundary-value problem for incompressible second-grade fluids with $\alpha_1+\alpha_2 = 0$, $\alpha_1 > 0$, and the boundary conditions (1.6), (1.9) was studied by Le Roux [11] in Sobolev spaces in the case when ϕ is bounded (i.e., k is bounded away from 1) and is a function of both the position and the slip velocity.

In this paper, we adapt the approach of Mogilevskiĭ and Solonnikov [12] to study problem (1.4)–(1.6) in Hölder spaces. For this we need the following notation. Let m be a nonnegative integer and $\lambda \in (0,1)$, and suppose that the boundary Γ has at least $C^{m+\lambda}$ regularity (we write $\Gamma \in C^{m+\lambda}$). Then by $C^{m+\lambda}(\overline{\Omega})$ we denote the Banach space of functions u on $\overline{\Omega}$ that are differentiable up to order m and have finite norm

$$\|u\|_{\Omega}^{(m+\lambda)} = \|u\|_{\Omega}^{(m)} + < u >_{\Omega}^{(m+\lambda)},$$

where

$$\|u\|_{\Omega}^{(m)} = \sum_{|\beta| \le m} \sup_{x \in \Omega} \left|D^{\beta} u(x)\right|,$$

$$< u >_{\Omega}^{(m+\lambda)} = \sum_{|\beta| = m} \sup_{x,y \in \Omega, x \neq y} \frac{\left|D^{\beta} u(x) - D^{\beta} u(y)\right|}{|x-y|^{\lambda}}.$$

The Hölder space $C^{m+\lambda}(\Gamma)$ and its norm $\|\cdot\|_{\Gamma}^{(m+\lambda)}$ are defined in a standard way by virtue of the partition of unity.

For functions $k : \Gamma \to [0,1]$ we define $\Gamma_1 = \{x \in \Gamma \,|\, k(x) = 1\}$. For functions $p : \Omega \to \mathbb{R}$ we define $p_{\mathrm{av}} = |\Omega|^{-1} \int_{\Omega} p\, dx$ and $C_*^{m+\lambda}(\overline{\Omega}) = \{p \in C^{m+\lambda}(\overline{\Omega}) \,|\, p_{\mathrm{av}} = 0\}$. By $C(\Omega)$, $C(m,\lambda)$, etc. we denote generic positive constants which depend on the quantities in brackets. Fixed constants are denoted by other symbols: C_1, δ, etc.

The outline of the paper is as follows. In Sec. 2, we prove the unique solvability of an auxiliary linear problem and derive *a priori* estimates of its solution. Then, in Sec. 3, we use the results of Sec. 2 and a fixed point argument to obtain our main result:

Theorem 1. *Let $m \in \mathbb{N}$, $m \geq 2$, $\lambda \in (0,1)$, and assume that Ω is a bounded domain in $\mathbb{R}^3$ with boundary $\Gamma \in C^{m+3+\lambda}$ and $k \in C^{m+2+\lambda}(\Gamma)$ satisfies the inequality $0 \leq k(x) \leq 1$ on Γ. If $\mathbf{f} \in C^{m+\lambda}(\overline{\Omega})$ and*

$$\|\mathbf{f}\|_{\Omega}^{(m+\lambda)} \leq \eta$$

for a sufficiently small number $\eta > 0$, then problem (1.4)–(1.7) has a unique solution $(\mathbf{v}, p) \in C^{m+1+\lambda}(\overline{\Omega}) \times C_*^{m+\lambda}(\overline{\Omega})$. This solution satisfies the inequality

$$\|\mathbf{v}\|_{\Omega}^{(m+1+\lambda)} + \|p\|_{\Omega}^{(m+\lambda)} \leq C \|\mathbf{f}\|_{\Omega}^{(m-1+\lambda)}$$

with a constant C dependent on $\Omega, \eta, k, \nu, \alpha_1, \alpha_2$.

2. Auxiliary linear problem

In this section, we consider the following boundary-value problem: Find $\mathbf{v}$ and p such that

$$-\nu\triangle\,\mathbf{v}-\alpha_1\,\mathbf{w}\,\cdot\,\nabla\triangle\,\mathbf{v}+\nabla p=\mathbf{f}\quad\text{in}\quad\Omega, \tag{2.1}$$

$$\nabla\,\cdot\,\mathbf{v}=0\quad\text{in}\quad\Omega, \tag{2.2}$$

$$\mathbf{v}\,\cdot\,\mathbf{n}=0\quad\text{on}\quad\Gamma, \tag{2.3}$$

$$\begin{aligned}&\nu(1-k)\,[A_1(\mathbf{v})\,\mathbf{n}]_\tau-k\,\mathbf{v}+\alpha_1(1-k)\\&\times[(\mathbf{w}\,\cdot\,\nabla A_1(\mathbf{v}))\,\mathbf{n}]_\tau=(1-k)\,\mathbf{b}_1\quad\text{on}\quad\Gamma.\end{aligned} \tag{2.4}$$

Here $\mathbf{w}$ is a given vector field in Ω such that $\mathbf{w}\,\cdot\,\mathbf{n}=0$ on Γ, $\mathbf{b}_1$ is a given vector field on Γ such that $\mathbf{b}_1\,\cdot\,\mathbf{n}=0$, and $k:\Gamma\to[0,1]$ is a given function.

First we remind some lemmas. Theorem 3.1 in [13] implies the following result: (cf. [1])

Lemma 1. *Suppose that Ω is a bounded domain in $\mathbb{R}^3$ with boundary $\Gamma\in C^{j+1}$, $j\in\mathbb{N}\cup\{0\}$, $\nu>0$, $\alpha_1\in\mathbb{R}$, $\mathbf{w}\in C^{j+1}(\overline{\Omega})$, $\mathbf{w}\,\cdot\,\mathbf{n}=0$ on Γ, and $\mathbf{h}\in C^{j+\lambda}(\overline{\Omega})$, $\lambda\in(0,1)$. Then there exists a constant $\delta=\delta(j,\lambda)>0$ such that we have: if $\delta|\alpha_1|\cdot\|\nabla\,\mathbf{w}\|_\Omega^{(j)}<\nu$ then the transport equation*

$$\nu\,\mathbf{z}+\alpha_1\,\mathbf{w}\,\cdot\,\nabla\,\mathbf{z}=\mathbf{h}\quad\textit{in}\quad\Omega$$

has a unique solution $\mathbf{z}\in C^{j+\lambda}(\overline{\Omega})$. Moreover,

$$\|\,\mathbf{z}\|_\Omega^{(j+\lambda)}\le C_1\|\,\mathbf{h}\|_\Omega^{(j+\lambda)},\quad C_1:=\left(\nu-\delta|\alpha_1|\cdot\|\nabla\,\mathbf{w}\|_\Omega^{(j)}\right)^{-1}. \tag{2.5}$$

A similar result holds for transport equations on Γ:

Lemma 2. *Let Γ be a compact two-dimensional manifold of class C^{j+1}, $j\in\mathbb{N}\cup\{0\}$, and let Γ' be a submanifold of Γ such that $\partial\Gamma'\in C^{j+1}$. Furthermore, let $\nu>0$, $\alpha_1\in\mathbb{R}$, $\mathbf{w}'\in C^{j+1}(\overline{\Gamma'})$, $\mathbf{w}'=0$ on $\partial\Gamma'$, and $\mathbf{h}'\in C^{j+\lambda}(\overline{\Gamma'})$. Then there exists a constant $\delta'=\delta'(j,\lambda)>0$ such that we have: if $\delta'|\alpha_1|\cdot\|\nabla_\Gamma\,\mathbf{w}'\|_{\Gamma'}^{(j)}<\nu$, where ∇_Γ denotes the surface gradient operator, then the transport equation*

$$\nu\,\mathbf{z}'+\alpha_1\,\mathbf{w}'\,\cdot\,\nabla_\Gamma\,\mathbf{z}'=\mathbf{h}'\quad\textit{in}\quad\Gamma'$$

has a unique solution $\mathbf{z}'\in C^{j+\lambda}(\Gamma')$. Moreover,

$$\|\,\mathbf{z}'\|_{\Gamma'}^{(j+\lambda)}\le C_2\|\,\mathbf{h}'\|_{\Gamma'}^{(j+\lambda)},\quad C_2:=\left(\nu-\delta'|\alpha_1|\cdot\|\nabla_\Gamma\,\mathbf{w}'\|_{\Gamma'}^{(j)}\right)^{-1}. \tag{2.6}$$

Furthermore, using the regularizer method due to Solonnikov [14] and following the line of checking the complementing conditions in [15, 16], we can prove the following result for the Stokes equations with general Navier slip boundary conditions, since we can:

Lemma 3. *Let Ω be a bounded domain in $\mathbb{R}^3$ with boundary $\Gamma \in C^{j+3+\lambda}$, $j \in \mathbb{N} \cup \{0\}$, $\lambda \in (0,1)$. Suppose that $\mathbf{z} \in C^{j+\lambda}(\overline{\Omega})$, $g \in C^{j+1+\lambda}(\overline{\Omega})$, $b \in C^{j+2+\lambda}(\Gamma)$, $\int_\Omega g\,dx = -\int_\Gamma b\,ds$, $k \in C^{j+1+\lambda}(\Gamma)$, $\mathbf{b} \in C^{j+1+\lambda}(\Gamma)$ and $\mathbf{b}|_{\Gamma_1} \in C^{j+2+\lambda}(\Gamma_1)$. Then there exist a unique vector field $\mathbf{v} \in C^{j+2+\lambda}(\overline{\Omega})$ and a function $q \in C_*^{j+1+\lambda}(\overline{\Omega})$ such that*

$$\triangle \mathbf{v} + \nabla q = \mathbf{z} \quad in \ \Omega, \tag{2.7}$$

$$\nabla \cdot \mathbf{v} = g \quad in \ \Omega, \tag{2.8}$$

$$\mathbf{v} \cdot \mathbf{n} = b \quad on \ \Gamma, \tag{2.9}$$

$$\nu(1-k)[\mathbf{A}_1(\mathbf{v})\,\mathbf{n}]_\tau - k\,\mathbf{v} = \mathbf{b} \quad on \ \Gamma. \tag{2.10}$$

Moreover, there is a constant $C_3 = C_3(\Omega, j, \lambda, \nu, k)$ such that

$$\|\mathbf{v}\|_\Omega^{(j+2+\lambda)} + \|q\|_\Omega^{(j+1+\lambda)} \le C_3\Big(\|\mathbf{z}\|_\Omega^{(j+\lambda)} + \|g\|_\Omega^{(j+1+\lambda)} + \|b\|_\Gamma^{(j+2+\lambda)} + \|\mathbf{b}\|_{\Gamma\setminus\Gamma_1}^{(j+1+\lambda)} + \|\mathbf{b}\|_{\Gamma_1}^{(j+2+\lambda)}\Big). \tag{2.11}$$

In particular, if $\mathbf{b} = (1-k)\,\mathbf{b}_1 - k\,\mathbf{b}_2$ for some $\mathbf{b}_1 \in C^{j+1+\lambda}(\Gamma)$ and $\mathbf{b}_2 \in C^{j+2+\lambda}(\Gamma)$, then there is a constant $C_4 = C_4(\Omega, j, \lambda, \nu, k)$ such that

$$\|\mathbf{v}\|_\Omega^{(j+2+\lambda)} + \|q\|_\Omega^{(j+1+\lambda)} \le C_4\Big(\|\mathbf{z}\|_\Omega^{(j+\lambda)} + \|g\|_\Omega^{(j+1+\lambda)} + \|\mathbf{b}_1\|_{\Gamma\setminus\Gamma_1}^{(j+1+\lambda)} + \|\mathbf{b}_2\|_\Gamma^{(j+2+\lambda)} + \|b\|_\Gamma^{(j+2+\lambda)}\Big). \tag{2.12}$$

In the following Theorem we derive an *a priori* estimate for solutions of problem (2.1)–(2.4). Since the problem is linear, this also proves the uniqueness of a solution.

Theorem 2. *Let $m \in \mathbb{N}$, $m \ge 2$, $\lambda \in (0,1)$ and let Ω be a bounded domain in $\mathbb{R}^3$ with boundary $\Gamma \in C^{m+3+\lambda}$. Assume that $\mathbf{f} \in C^{m+\lambda}(\overline{\Omega})$, $k \in C^{m+2+\lambda}(\Gamma)$, $\mathbf{w} \in C^{m+2+\lambda}(\overline{\Omega})$, $\mathbf{w} \cdot \mathbf{n} = 0$ on Γ, $\mathbf{w} = 0$ on $\partial\Gamma_1$, $\mathbf{b}_1 \in C^{m+1+\lambda}(\Gamma)$ and $\mathbf{b}_1 \cdot \mathbf{n} = 0$ on Γ. Furthermore, suppose that $(\mathbf{v}, p)$ is a solution of problem (2.1)–(2.4) such that $\mathbf{v} \in C^{m+2+\lambda}(\overline{\Omega})$ and $p \in C_*^{m+1+\lambda}(\overline{\Omega})$. Then there exist positive constants C_5 and C_6, which depend only on Ω, m, λ, ν, $|\alpha_1|$ and k, such that the following holds: if*

$$\|\mathbf{w}\|_\Omega^{(m+2+\lambda)} \le C_5 \tag{2.13}$$

then

$$\|\mathbf{v}\|_{\Omega}^{(m+2+\lambda)} + \|p\|_{\Omega}^{(m+1+\lambda)} \le C_6 \left(\|\mathbf{f}\|_{\Omega}^{(m+\lambda)} + \|\mathbf{b}_1\|_{\Gamma}^{(m+1+\lambda)} \right). \quad (2.14)$$

Proof. Suppose that

$$\delta(m+1,\lambda)|\alpha_1| \cdot \|\nabla \mathbf{w}\|_{\Omega}^{(m+1)} < \nu, \quad (2.15)$$

where $\delta(m+1,\lambda)$ is the constant in Lemma 1 with $j = m+1$. Then there exists a unique solution $q^* \in C^{m+1+\lambda}(\overline{\Omega})$ to the transport equation

$$\nu q^* + \alpha_1 \mathbf{w} \cdot \nabla q^* = -p \quad \text{in } \Omega. \quad (2.16)$$

Moreover,

$$\|q^*\|_{\Omega}^{(m+1+\lambda)} \le C_1{}^{(m+1)} \|p\|_{\Omega}^{(m+1+\lambda)},$$

$$C_1{}^{(m+1)} := \left(\nu - \delta(m+1,\lambda)|\alpha_1| \cdot \|\nabla \mathbf{w}\|_{\Omega}^{(m+1)} \right)^{-1}.$$

Let $q = q^* - q^*_{\mathrm{av}}$. Then, by (2.16),

$$\nabla p = -\nu \nabla q - \alpha_1 (\nabla \mathbf{w})^T \nabla q - \alpha_1 (\mathbf{w} \cdot \nabla) \nabla q \quad \text{in } \Omega. \quad (2.17)$$

Thus, equation (2.1) is equivalent to

$$-\nu(\triangle \mathbf{v} + \nabla q) - \alpha_1 (\mathbf{w} \cdot \nabla)(\triangle \mathbf{v} + \nabla q) = \mathbf{f} + \alpha_1 (\nabla \mathbf{w})^T \nabla q \quad \text{in } \Omega, \quad (2.18)$$

which we can rewrite as a transport equation in $\mathbf{z} := \triangle \mathbf{v} + \nabla q \in C^{m+\lambda}(\overline{\Omega})$:

$$\nu \mathbf{z} + \alpha_1 \mathbf{w} \cdot \nabla \mathbf{z} = -\mathbf{f} - \alpha_1 (\nabla \mathbf{w})^T \nabla q \quad \text{in } \Omega. \quad (2.19)$$

Thus, by inequality (2.15) (since $\delta(m,\lambda) \le \delta(m+1,\lambda)$) and Lemma 1 with $j = m$,

$$\|\mathbf{z}\|_{\Omega}^{(m+\lambda)} \le C_1{}^{(m)} \left(\|\mathbf{f}\|_{\Omega}^{(m+\lambda)} + C_7 \|\nabla \mathbf{w}\|_{\Omega}^{(m+\lambda)} \|\nabla q\|_{\Omega}^{(m+\lambda)} \right), \quad (2.20)$$

where $C_1{}^{(m)} = \left(\nu - \delta(m,\lambda)|\alpha_1| \cdot \|\nabla \mathbf{w}\|_{\Omega}^{(m)} \right)^{-1}$ and $C_7 = C_7(m,\lambda)$.

Next, by direct computation we see that

$$[(\mathbf{w} \cdot \nabla \mathbf{A}_1(\mathbf{v}))\mathbf{n}]_{\tau} = \mathbf{w} \cdot \nabla_{\Gamma} ([\mathbf{A}_1(\mathbf{v})\mathbf{n}]_{\tau}) + \mathbf{B}(\mathbf{w}, \mathbf{A}_1(\mathbf{v})) \quad \text{on } \Gamma,$$

where

$$\mathbf{B}(\mathbf{w}, \mathbf{A}_1) := (\mathbf{n} \cdot \mathbf{A}_1 \mathbf{n}) \mathbf{w} \cdot \nabla \mathbf{n} + ((\mathbf{A}_1 \mathbf{n}) \cdot (\mathbf{w} \cdot \nabla \mathbf{n}))\, \mathbf{n} - [\mathbf{A}_1 (\mathbf{w} \cdot \nabla \mathbf{n})]_{\tau}.$$

Thus

$$(1-k)\,[(\mathbf{w}\cdot\nabla\mathbf{A}_1)\,\mathbf{n}]_\tau = \mathbf{w}\cdot\nabla_\Gamma\,((1-k)[\mathbf{A}_1\,\mathbf{n}]_\tau)$$
$$-(\mathbf{w}\cdot\nabla_\Gamma(1-k))\,[\mathbf{A}_1\,\mathbf{n}]_\tau + (1-k)\,\mathbf{B}(\mathbf{w},\,\mathbf{A}_1) \quad \text{on} \quad \Gamma.$$

Using this identity, we can write the slip boundary condition (2.4) as

$$\nu(1-k)[\mathbf{A}_1\,\mathbf{n}]_\tau - k\,\mathbf{v} + \frac{\alpha_1}{\nu}\,\mathbf{w}\cdot\nabla_\Gamma\,(\nu(1-k)[\mathbf{A}_1\,\mathbf{n}]_\tau - k\,\mathbf{v}) = \mathbf{h}(\mathbf{v}) \quad \text{on} \quad \Gamma, \tag{2.21}$$

where

$$\mathbf{h}(\mathbf{v}) := (1-k)\,\mathbf{b}_1 + \alpha_1\,(\mathbf{w}\cdot\nabla_\Gamma(1-k))\,[\mathbf{A}_1(\mathbf{v})\,\mathbf{n}]_\tau$$
$$-\frac{\alpha_1}{\nu}\,\mathbf{w}\cdot\nabla_\Gamma(k\,\mathbf{v}) - \alpha_1(1-k)\,\mathbf{B}(\mathbf{w},\,\mathbf{A}_1(\mathbf{v})).$$

Setting $\mathbf{b} = \nu(1-k)[\mathbf{A}_1\,\mathbf{n}]_\tau - k\,\mathbf{v} \in C^{m+1+\lambda}(\Gamma)$, we can rewrite equation (2.21) as the transport equation

$$\mathbf{b} + \frac{\alpha_1}{\nu}\,\mathbf{w}\cdot\nabla_\Gamma\,\mathbf{b} = \mathbf{h}(\mathbf{v}) \quad \text{on} \quad \Gamma. \tag{2.22}$$

Now, $\mathbf{h}(\mathbf{v}) \in C^{m+1+\lambda}(\Gamma)$ by virtue of our hypotheses. Moreover, by boundary conditions (2.3), (2.4), $\mathbf{v} = 0$ on Γ_1 and therefore, since $\mathbf{w}\cdot\mathbf{n} = 0$, $\mathbf{h}(\mathbf{v}) = 0$ on Γ_1. Let $\delta'(\Gamma_1, m+1, \lambda)$ denote the constant in Lemma 2 with $j = m+1$ and $\Gamma' = \Gamma_1$ ($\partial\Gamma_1 \in C^{m+2}$ by the implicit function theorem and the regularity of k). Taking $\delta'(\Gamma_1, m+1, \lambda)$ smaller if necessary, we may assume that

$$\delta'(\Gamma_1, m+1, \lambda)\|\nabla_\Gamma\,\mathbf{w}\|_{\Gamma_1}^{(m+1)} \le \delta(m+1, \lambda)\|\nabla\,\mathbf{w}\|_\Omega^{(m+1)} \quad \text{for all} \quad \mathbf{w} \in C^{m+2+\lambda}(\overline{\Omega}). \tag{2.23}$$

Thus, it follows from inequality (2.15) and Lemma 2 that $\mathbf{b} = 0$ on Γ_1.

Similarly, we set $\Gamma' = \Gamma\backslash\Gamma_1$ and denote the corresponding constant in Lemma 2 with $j = m+1$ by $\delta'(\Gamma', m+1, \lambda)$. Then, as in (2.23), we may assume that

$$\delta'(\Gamma', m+1, \lambda)\|\nabla_\Gamma\,\mathbf{w}\|_{\Gamma'}^{(m+1)} \le \delta(m+1, \lambda)\|\nabla\,\mathbf{w}\|_\Omega^{(m+1)} \quad \text{for all} \quad \mathbf{w} \in C^{m+2+\lambda}(\overline{\Omega}). \tag{2.24}$$

Thus, again by inequality (2.15), we can apply Lemma 2 with $\mathbf{w}' = \mathbf{w}|_{\Gamma'}$, $\mathbf{h}' = \mathbf{h}(\mathbf{v})|_{\Gamma'}$ and $\mathbf{z}' = \mathbf{b}|_{\Gamma'}$. Moreover, (2.24) implies that ${C_2}^{(m+1)} \le$

${C_1}^{(m+1)}$, where ${C_1}^{(m+1)}$ and ${C_2}^{(m+1)}$ denote the corresponding constants in estimates (2.5) and (2.6), respectively. Hence, applying inequality (2.6) and simplifying the resulting terms, we obtain

$$\|\mathbf{b}\|_{\Gamma\setminus\Gamma_1}^{(m+1+\lambda)} \leq \nu {C_1}^{(m+1)}\|\mathbf{h}(\mathbf{v})\|_{\Gamma\setminus\Gamma_1}^{(m+1+\lambda)}$$

$$\leq \nu {C_1}^{(m+1)} C_8\Big(\big(1+\nu^{-1}\big)|\alpha_1|\cdot\|k\|_{\Gamma\setminus\Gamma_1}^{(m+2+\lambda)}\|\mathbf{w}\|_{\Omega}^{(m+1+\lambda)}\|\mathbf{v}\|_{\Omega}^{(m+2+\lambda)}$$

$$+\|k\|_{\Gamma\setminus\Gamma_1}^{(m+1+\lambda)}\|\mathbf{b}_1\|_{\Gamma\setminus\Gamma_1}^{(m+1+\lambda)}\Big) \tag{2.25}$$

for some constant $C_8 = C_8(\Omega, m, \lambda)$.

Now, by definition of $\mathbf{z}$ and $\mathbf{b}$, $(\mathbf{v}, q)$ is the solution of the Stokes problem (2.7)–(2.10) with $g \equiv 0$ and $b_3 \equiv 0$. Hence, combining estimates (2.11), (2.20), and (2.25), we arrive at the inequality

$$\|\mathbf{v}\|_{\Omega}^{(m+2+\lambda)} + \|q\|_{\Omega}^{(m+1+\lambda)}$$

$$\leq\; {C_1}^{(m+1)} C_3 C_8(\nu+1)|\alpha_1|\cdot\|k\|_{\Gamma}^{(m+2+\lambda)}\|\mathbf{w}\|_{\Omega}^{(m+1+\lambda)}\|\mathbf{v}\|_{\Omega}^{(m+2+\lambda)}$$

$$+{C_1}^{(m)} C_3\|\mathbf{f}\|_{\Omega}^{(m+\lambda)} + {C_1}^{(m)} C_3 C_7\|\nabla\mathbf{w}\|_{\Omega}^{(m+\lambda)}\|\nabla q\|_{\Omega}^{(m+\lambda)}$$

$$+\nu {C_1}^{(m+1)} C_3 C_8\|k\|_{\Gamma}^{(m+1+\lambda)}\|\mathbf{b}_1\|_{\Gamma}^{(m+1+\lambda)}. \tag{2.26}$$

Thus, if

$${C_1}^{(m+1)} C_3 C_8(\nu+1)|\alpha_1|\cdot\|k\|_{\Gamma}^{(m+2+\lambda)}\|\mathbf{w}\|_{\Omega}^{(m+1+\lambda)} \leq \frac{1}{2},$$

$${C_1}^{(m)} C_3 C_7\|\nabla\mathbf{w}\|_{\Omega}^{(m+\lambda)} \leq \frac{1}{2},$$

or, equivalently,

$$2C_3C_8(\nu+1)|\alpha_1|\cdot\|k\|_{\Gamma}^{(m+2+\lambda)}\|\mathbf{w}\|_{\Omega}^{(m+1+\lambda)}$$

$$+\delta(m+1,\lambda)|\alpha_1|\cdot\|\nabla\mathbf{w}\|_{\Omega}^{(m+1)} \leq \nu, \tag{2.27}$$

$$2C_3C_7\|\nabla\mathbf{w}\|_{\Omega}^{(m+\lambda)} + \delta(m,\lambda)|\alpha_1|\cdot\|\nabla|\mathbf{w}\|_{\Omega}^{(m)} \leq \nu, \tag{2.28}$$

then

$$\|\mathbf{v}\|_{\Omega}^{(m+2+\lambda)} + \|q\|_{\Omega}^{(m+1+\lambda)} \leq 2{C_1}^{(m)} C_3\|\mathbf{f}\|_{\Omega}^{(m+\lambda)}$$

$$+2\nu C_1{}^{(m+1)} C_3 C_8 \|k\|_\Gamma^{(m+1+\lambda)} \|\mathbf{b}_1\|_\Gamma^{(m+1+\lambda)}. \tag{2.29}$$

A sufficient condition for (2.27), (2.28) (and so also for (2.15)) is

$$C_9\Big((\nu+1)|\alpha_1|\cdot\|k\|_\Gamma^{(m+2+\lambda)}\|\mathbf{w}\|_\Omega^{(m+1+\lambda)}$$

$$+|\alpha_1|\cdot\|\nabla\mathbf{w}\|_\Omega^{(m+1)} + \|\mathbf{w}\|_\Omega^{(m+\lambda)}\Big) \le \nu, \tag{2.30}$$

where $C_9 = C_9(\Omega, m, \lambda, \nu, k) := 2\max\{C_3C_8, \delta(m+1,\lambda), C_3C_7\}$. Moreover, if we let

$$C_5 = C_5(\Omega, m, \lambda, \nu, k) := \frac{\nu}{3C_9\max\{(\nu+1)|\alpha_1|\cdot\|k\|_\Gamma^{(m+2+\lambda)}, |\alpha_1|, 1\}},$$

then assumption (2.13) ensures that (2.30) holds. Furthermore, it follows from (2.30) that $C_1{}^{(m+1)} \le 2/\nu$. Hence, if (2.30) holds, inequality (2.29) implies that

$$\|\mathbf{v}\|_\Omega^{(m+2+\lambda)} + \|q\|_\Omega^{(m+1+\lambda)} \le C_{10}\Big(\|\mathbf{f}\|_\Omega^{(m+\lambda)} + \|\mathbf{b}_1\|_\Gamma^{(m+1+\lambda)}\Big), \tag{2.31}$$

where $C_{10} = C_{10}(\Omega, m, \lambda, \nu, k) := 4C_3(\nu^{-1} + C_8)$.

We can estimate $\|\nabla p\|_\Omega^{(m-1+\lambda)}$ by means of equation (2.17) and inequalities (2.13), (2.31). To estimate $\|p\|_\Omega^{(m+1+\lambda)}$, we use Lemma 3 again. First note that

$$\triangle(\mathbf{w}\cdot\nabla\mathbf{v}) = \triangle\mathbf{w}\cdot\nabla\mathbf{v} + \mathbf{w}\cdot\nabla\triangle\mathbf{v} + \mathbf{M}(\mathbf{w}, \mathbf{v}), \tag{2.32}$$

where $\mathbf{M}(\mathbf{w}, \mathbf{v}) = (M_1, M_2, M_3)$, $M_i = \sum_{j,\,k=1}^3 w_{k,j} v_{i,jk}$ $(i = 1, 2, 3)$. Hence, setting $\mathbf{y} = \nu\mathbf{v} + \alpha_1\mathbf{w}\cdot\nabla\mathbf{v} \in C^{m+1+\lambda}(\overline{\Omega})$, we can rewrite equation (2.1) as

$$-\triangle\mathbf{y} + \nabla p = \mathbf{f} - \alpha_1\triangle\mathbf{w}\cdot\nabla\mathbf{v} - \alpha_1\mathbf{M}(\mathbf{w}, \mathbf{v}) \quad \text{in } \Omega. \tag{2.33}$$

Moreover, since $\nabla\cdot(\mathbf{w}\cdot\nabla\mathbf{v}) = (\nabla\mathbf{w})^T : \nabla\mathbf{v} + \mathbf{w}\cdot\nabla(\nabla\cdot\mathbf{v})$ and $\nabla\cdot\mathbf{v} = 0$,

$$\nabla\cdot\mathbf{y} = \alpha_1(\nabla\mathbf{w})^T : \nabla\mathbf{v} \quad \text{in } \Omega. \tag{2.34}$$

Furthermore, since $(\mathbf{w}\cdot\nabla\mathbf{v})\cdot\mathbf{n} = \mathbf{w}\cdot\nabla_\Gamma(\mathbf{v}\cdot\mathbf{n}) - (\mathbf{w}\cdot\nabla_\Gamma\mathbf{n})\cdot\mathbf{v}$ and $\mathbf{w}\cdot\mathbf{n} = \mathbf{v}\cdot\mathbf{n} = 0$,

$$\mathbf{y}\cdot\mathbf{n} = -\alpha_1(\mathbf{w}\cdot\nabla_\Gamma\mathbf{n})\cdot\mathbf{v} \quad \text{on } \Gamma. \tag{2.35}$$

Lastly, since

$$\mathbf{A}_1(\mathbf{w}\cdot\nabla\mathbf{v}) = (\mathbf{w}\cdot\nabla)\,\mathbf{A}_1(\mathbf{v}) + (\nabla\mathbf{v})(\nabla\mathbf{w}) + (\nabla\mathbf{w})^T(\nabla\mathbf{v})^T,$$

it follows from the slip boundary condition (2.4) that

$$\begin{aligned}
&(1-k)\,[\,\mathbf{A}_1(\mathbf{y})\,\mathbf{n}]_\tau - \frac{1}{\nu}\;k\,\mathbf{y}_\tau \\
&= \nu(1-k)\,[\,\mathbf{A}_1(\mathbf{v})\,\mathbf{n}]_\tau + \alpha_1(1-k)\,[(\mathbf{w}\cdot\nabla\,\mathbf{A}_1(\mathbf{v}))\,\mathbf{n}]_\tau - k\,\mathbf{v} \\
&+ \alpha_1(1-k)\left[(\nabla\mathbf{v})(\nabla\mathbf{w})\,\mathbf{n} + (\nabla\mathbf{w})^T(\nabla\mathbf{v})^T\,\mathbf{n}\right]_\tau - \frac{1}{\nu}\;\alpha_1 k[\mathbf{w}\cdot\nabla\mathbf{v}]_\tau \\
&= (1-k)\big(\mathbf{b}_1 + \alpha_1\left[(\nabla\mathbf{v})(\nabla\mathbf{w})\,\mathbf{n} + (\nabla\mathbf{w})^T(\nabla\mathbf{v})^T\,\mathbf{n}\right]_\tau\big) \\
&\qquad -\frac{1}{\nu}\;\alpha_1 k[\mathbf{w}\cdot\nabla\mathbf{v}]_\tau \quad \text{on} \quad \Gamma. \qquad (2.36)
\end{aligned}$$

Hence, $(\mathbf{y},p)$ is a solution of the Stokes problem of type (2.7)–(2.10). Note that $[\mathbf{w}\cdot\nabla\mathbf{v}]_\tau = 0$ on Γ_1 because $\mathbf{w}\cdot\mathbf{n} = 0$ and $\mathbf{v} = 0$ on Γ_1. Thus, we can apply estimates (2.12) and (2.31) to obtain

$$\begin{aligned}
\|p\|_\Omega^{(m+1+\lambda)} &\le C_4 C(\Omega,m,\lambda)\Big(\|\,\mathbf{f}\|_\Omega^{(m+\lambda)} + (1+\nu)|\alpha_1| \\
&\quad \cdot\|\,\mathbf{w}\|_\Omega^{(m+2+\lambda)}\|\,\mathbf{v}\|_\Omega^{(m+2+\lambda)} + \nu\|\,\mathbf{b}_1\|_\Gamma^{(m+1+\lambda)}\Big) \\
&\le C_4 C(\Omega,m,\lambda)\max(1,\nu)(1+(1+\nu)|\alpha_1|C_5 C_{10}) \\
&\quad \cdot\Big(\|\,\mathbf{f}\|_\Omega^{(m+\lambda)} + \|\,\mathbf{b}_1\|_\Gamma^{(m+1+\lambda)}\Big) \\
&\le C_{11}\Big(\|\,\mathbf{f}\|_\Omega^{(m+\lambda)} + \|\,\mathbf{b}_1\|_\Gamma^{(m+1+\lambda)}\Big),
\end{aligned}$$

where $C_{11} = C_{11}(\Omega,m,\lambda,\nu,|\alpha_1|,k)$. Finally, we set $C_6 = C_{10} + C_{11}$. □

We will now show that problem (2.1)–(2.4) has a solution.

Theorem 3. *Let m, λ, Ω, Γ, $\mathbf{f}$, k, $\mathbf{w}$, and $\mathbf{b}_1$ be as in Theorem 2 and assume that inequality (2.13) holds. Then problem (2.1)–(2.4) has a unique solution $(\mathbf{v},q)\in C^{m+2+\lambda}(\overline{\Omega})\times C_*^{m+1+\lambda}(\overline{\Omega})$.*

Proof. We start by showing that the boundary-value problem

$$\begin{aligned}
-\nu(\triangle\mathbf{v}+\nabla q) - \alpha_1\,\mathbf{w}\cdot\nabla(\triangle\mathbf{v}+\nabla q)& \\
= \mathbf{f} + (\nabla\mathbf{w})^T\nabla q \qquad & \text{in} \quad \Omega, \qquad (2.37) \\
\nabla\cdot\mathbf{v} = 0 \qquad & \text{in} \quad \Omega, \qquad (2.38) \\
\mathbf{v}\cdot\mathbf{n} = 0 \qquad & \text{on} \quad \Gamma, \qquad (2.39)
\end{aligned}$$

$$\nu(1-k)[\mathbf{A}_1(\mathbf{v})\mathbf{n}]_\tau - k\mathbf{v} + \alpha_1(1-k)$$

$$\cdot[(\mathbf{w}\cdot\nabla\mathbf{A}_1(\mathbf{v}))\mathbf{n}]_\tau = (1-k)\mathbf{b}_1 \quad \text{on} \quad \Gamma \tag{2.40}$$

has a solution $(\mathbf{v}, p)$ in $C^{m+2+\lambda}(\overline{\Omega}) \times C_*^{m+1+\lambda}(\overline{\Omega})$. To this end, let $(\mathbf{u}, r) \in \mathcal{X}$ be given, where

$$\mathcal{X} := \Big\{(\mathbf{u}, r) \in C^{m+2+\lambda}(\overline{\Omega}) \times C_*^{m+1+\lambda}(\overline{\Omega}) \;\Big|\; \nabla\cdot\mathbf{u} = 0 \quad \text{in} \quad \Omega,$$

$$\mathbf{u}\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma, \;\; \mathbf{u} = 0 \quad \text{on} \quad \Gamma_1\Big\}.$$

As shown in the proof of Theorem 2, inequality (2.13) implies inequality (2.15). Thus, as in (2.19), (2.20), it follows from Lemma 1 that there exists a unique solution $\mathbf{z} \in C^{m+\lambda}(\overline{\Omega})$ to the transport equation

$$\nu\mathbf{z} + \alpha_1\mathbf{w}\cdot\nabla\mathbf{z} = -\mathbf{f} - (\nabla\mathbf{w})^T\nabla r \quad \text{in} \quad \Omega, \tag{2.41}$$

and

$$\|\mathbf{z}\|_\Omega^{(m+\lambda)} \le C_1{}^{(m)}\Big(\|\mathbf{f}\|_\Omega^{(m+\lambda)} + C_7\|\nabla\mathbf{w}\|_\Omega^{(m+\lambda)}\|\nabla r\|_\Omega^{(m+\lambda)}\Big). \tag{2.42}$$

Furthermore, as in equation (2.22), it follows from Lemma 2 that the transport equation

$$\mathbf{b} + \frac{\alpha_1}{\nu}\mathbf{w}\cdot\nabla_\Gamma\mathbf{b} = \mathbf{h}(\mathbf{u}) \quad \text{on} \quad \Gamma, \tag{2.43}$$

where $\mathbf{h}(\mathbf{u})$ is defined as in (2.21) with $\mathbf{v}$ replaced by $\mathbf{u}$, has a unique solution $\mathbf{b} \in C^{m+1+\lambda}(\Gamma)$. As in the proof of Theorem 2, it follows that $\mathbf{b} = 0$ on Γ_1 because $\mathbf{u} = 0$ on Γ_1. Moreover, the analogue of estimate (2.25) holds:

$$\|\mathbf{b}\|_{\Gamma\setminus\Gamma_1}^{(m+1+\lambda)} \le \nu C_1{}^{(m+1)}C_8\Big(\big(1+\nu^{-1}\big)|\alpha_1| \tag{2.44}$$

$$\cdot\|k\|_{\Gamma\setminus\Gamma_1}^{(m+2+\lambda)}\|\mathbf{w}\|_\Omega^{(m+1+\lambda)}\|\mathbf{u}\|_\Omega^{(m+2+\lambda)} + \|k\|_{\Gamma\setminus\Gamma_1}^{(m+1+\lambda)}\|\mathbf{b}_1\|_{\Gamma\setminus\Gamma_1}^{(m+1+\lambda)}\Big).$$

Now, by Lemma 3, the Stokes problem

$$\triangle\mathbf{v} + \nabla q = \mathbf{z} \quad \text{in} \quad \Omega, \tag{2.45}$$

$$\nabla\cdot\mathbf{v} = 0 \quad \text{in} \quad \Omega, \tag{2.46}$$

$$\mathbf{v}\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma, \tag{2.47}$$

$$\nu(1-k)[\mathbf{A}_1(\mathbf{v})\mathbf{n}]_\tau - k\mathbf{v} = \mathbf{b} \quad \text{on} \quad \Gamma \tag{2.48}$$

has a unique solution $(\mathbf{v}, q) \in C^{m+2+\lambda}(\overline{\Omega}) \times C_*^{m+1+\lambda}(\overline{\Omega})$. Thus, we can define a mapping $\Psi : \mathcal{X} \to \mathcal{X}$ by $\Psi(\mathbf{u}, r) = (\mathbf{v}, q)$. Moreover, combining estimates (2.11), (2.42), and (2.44), we obtain the analogue of estimate (2.26):

$$\|\mathbf{v}\|_\Omega^{(m+2+\lambda)} + \|q\|_\Omega^{(m+1+\lambda)} \leq C_1{}^{(m+1)} C_3 C_8 (\nu+1) |\alpha_1|]$$

$$\cdot \|k\|_\Gamma^{(m+2+\lambda)} \|\mathbf{w}\|_\Omega^{(m+1+\lambda)} \|\mathbf{u}\|_\Omega^{(m+2+\lambda)} + C_1{}^{(m)} C_3 \|\mathbf{f}\|_\Omega^{(m+\lambda)}$$

$$+ C_1{}^{(m)} C_3 C_7 \|\nabla \mathbf{w}\|_\Omega^{(m+\lambda)} \|\nabla r\|_\Omega^{(m+\lambda)}$$

$$+ \nu C_1{}^{(m+1)} C_3 C_8 \|k\|_\Gamma^{(m+1+\lambda)} \|\mathbf{b}_1\|_\Gamma^{(m+1+\lambda)}.$$

Hence, applying inequalities (2.27) and (2.28) and proceeding as in the derivation of estimate (2.31), we find that

$$\|\mathbf{v}\|_\Omega^{(m+2+\lambda)} + \|q\|_\Omega^{(m+1+\lambda)} \leq \frac{1}{2}\Big(\|\mathbf{u}\|_\Omega^{(m+2+\lambda)} + \|r\|_\Omega^{(m+1+\lambda)}\Big)$$

$$+ \frac{C_{10}}{2}\Big(\|\mathbf{f}\|_\Omega^{(m+\lambda)} + \|\mathbf{b}_1\|_\Gamma^{(m+1+\lambda)}\Big).$$

Let $R \geq C_{10}\big[\|\mathbf{f}\|_\Omega^{(m+\lambda)} + \|\mathbf{b}_1\|_\Gamma^{(m+1+\lambda)}\big]$. Then

$$\|\mathbf{v}\|_\Omega^{(m+2+\lambda)} + \|q\|_\Omega^{(m+1+\lambda)} \leq R \ \text{ if } \ \|\mathbf{u}\|_\Omega^{(m+2+\lambda)} + \|r\|_\Omega^{(m+1+\lambda)} \leq R,$$

i.e., Ψ maps the ball

$$B(\mathcal{X}, R) = \{(\mathbf{u}, r) \in \mathcal{X} \mid \|\mathbf{u}\|_\Omega^{(m+2+\lambda)} + \|r\|_\Omega^{(m+1+\lambda)} \leq R\}$$

into itself.

Let $(\mathbf{u}^i, r^i) \in \mathcal{X}$ and $(\mathbf{v}^i, q^i) = \Psi(\mathbf{u}^i, r^i)$, $i = 1, 2$. Then it follows from the linearity properties of equations (2.41), (2.43), problem (2.45)–(2.48) and the associated *a priori* estimates that

$$\|\mathbf{v}^1 - \mathbf{v}^2\|_\Omega^{(m+2+\lambda)} + \|q^1 - q^2\|_\Omega^{(m+1+\lambda)}$$

$$\leq \frac{1}{2}\Big(\|\mathbf{u}^1 - \mathbf{u}^2\|_\Omega^{(m+2+\lambda)} + \|r^1 - r^2\|_\Omega^{(m+1+\lambda)}\Big).$$

Hence, Ψ is a contraction mapping on $B(\mathcal{X}, R)$ and has a unique fixed point $(\mathbf{v}, q)$ in $B(\mathcal{X}, R)$. By definition of Ψ, $(\mathbf{v}, q)$ is a solution of problem (2.37)–(2.40).

Set $p = -\nu q - \mathbf{w} \cdot \nabla q \in C^{m+\lambda}(\overline{\Omega})$. Then p and q satisfy relation (2.17), and equation (2.37) implies that $(\mathbf{v}, p)$ satisfies equation (2.1). The uniqueness of $(\mathbf{v}, p)$ follows from Theorem 2. Thus, it remains to show that $p \in C_*^{m+1+\lambda}(\overline{\Omega})$. We will do this by considering $(\mathbf{w} \cdot \nabla \mathbf{v}, p)$ as a solution of the Stokes problem of type (2.7)–(2.10). We use identity (2.32) to rewrite equation (2.1) as

$$-\alpha_1 \triangle(\mathbf{w} \cdot \nabla \mathbf{v}) + \nabla p = \mathbf{f} + \nu \triangle \mathbf{v} - \alpha_1 \triangle \mathbf{w} \cdot \nabla \mathbf{v} - \alpha_1 \mathbf{M}(\mathbf{w}, \mathbf{v}) =: \mathbf{z}^* \quad \text{in} \quad \Omega.$$

Further, as in (2.34), (2.35) we have

$$\nabla \cdot (\mathbf{w} \cdot \nabla \mathbf{v}) = (\nabla \mathbf{w})^T : \nabla \mathbf{v} =: g^* \quad \text{in} \quad \Omega,$$

$$(\mathbf{w} \cdot \nabla \mathbf{v}) \cdot \mathbf{n} = -(\mathbf{w} \cdot \nabla_\Gamma \mathbf{n}) \cdot \mathbf{v} =: b^* \quad \text{on} \quad \Gamma.$$

Lastly, using boundary condition (2.4) as in (2.36), we get

$$\alpha_1(1-k)[\mathbf{A}_1(\mathbf{w} \cdot \nabla \mathbf{v})\mathbf{n}]_\tau - \frac{1}{\nu}\,\alpha_1 k[\mathbf{w} \cdot \nabla \mathbf{v}]_\tau = \alpha_1(1-k)[(\mathbf{w} \cdot \nabla \mathbf{A}_1(\mathbf{v}))\mathbf{n}]_\tau$$

$$+\alpha_1(1-k)[(\nabla \mathbf{v})(\nabla \mathbf{w})\mathbf{n} + (\nabla \mathbf{w})^T(\nabla \mathbf{v})^T \mathbf{n}]_\tau - \frac{1}{\nu}\,\alpha_1 k[\mathbf{w} \cdot \nabla \mathbf{v}]_\tau$$

$$= (1-k)\big(\mathbf{b}_1 + \alpha_1[(\nabla \mathbf{v})(\nabla \mathbf{w})\mathbf{n} + (\nabla \mathbf{w})^T(\nabla \mathbf{v})^T \mathbf{n}]_\tau - \nu[\mathbf{A}_1(\mathbf{v})\mathbf{n}]_\tau\big)$$

$$-k\left(\frac{1}{\nu}\alpha_1[\mathbf{w} \cdot \nabla \mathbf{v}]_\tau - \mathbf{v}\right) =: (1-k)\,\mathbf{b}_1^* - k\,\mathbf{b}_2^* \quad \text{on} \quad \Gamma.$$

Since $\mathbf{z}^*$, g^*, $\mathbf{b}_1^*$, $\mathbf{b}_2^*$, and b^* have the regularity properties required by Lemma 3, we conclude that $\mathbf{w} \cdot \nabla \mathbf{v} \in C^{m+2+\lambda}(\overline{\Omega})$ and $p \in C_*^{m+1+\lambda}(\overline{\Omega})$. □

3. Solution of the nonlinear problem

We will now construct a solution of problem (1.4)–(1.7) as the limit of a sequence of solutions to the linear problems of type (2.1)–(2.4).

Proof of Theorem 1. Let m be an integer appeared in Theorem 1. We start by deriving some auxiliary estimates.

First, if $\mathbf{w} \in C^{m+2+\lambda}(\overline{\Omega})$, then

$$\|\mathbf{w} \cdot \nabla \mathbf{w}\|_\Omega^{(m+\lambda)} + \|\mathbf{N}(\mathbf{w})\|_\Omega^{(m+\lambda)} + \|\mathbf{N}_\Gamma(\mathbf{w})\|_\Gamma^{(m+1+\lambda)}$$

$$\le C(\Omega, m, \lambda)\Big(\|\mathbf{w}\|_\Omega^{(m+\lambda)}\|\mathbf{w}\|_\Omega^{(m+1+\lambda)} + (|\alpha_1| + |\alpha_1 + \alpha_2|)\left(\|\mathbf{w}\|_\Omega^{(m+2+\lambda)}\right)^2$$

$$+(|\alpha_1| + |\alpha_1+\alpha_2|)\|\mathbf{w}\|_\Omega^{(m+1+\lambda)}\|\mathbf{w}\|_\Omega^{(m+2+\lambda)}\Big)$$

$$\le C_{12}(1+|\alpha_1|+|\alpha_1+\alpha_2|)\left(\|\mathbf{w}\|_\Omega^{(m+2+\lambda)}\right)^2, \tag{3.1}$$

where $C_{12} = C_{12}(\Omega, m, \lambda)$.

Second, for $\mathbf{v}$, $\mathbf{z} \in C^{m+2+\lambda}(\overline{\Omega})$, we define

$$\begin{aligned}\mathbf{N}'(\mathbf{v},\mathbf{z}) :=& \alpha_1(\nabla\mathbf{v})^T : \nabla\mathbf{A}_1(\mathbf{z}) + \alpha_1\nabla\cdot\big(\mathbf{A}_1(\mathbf{v})\mathbf{W}(\mathbf{z})\\ &- \mathbf{W}(\mathbf{v})\mathbf{A}_1(\mathbf{z})\big) + (\alpha_1+\alpha_2)\nabla\cdot\big(\mathbf{A}_1(\mathbf{v})\mathbf{A}_1(\mathbf{z})\big),\\ \mathbf{N}'_\Gamma(\mathbf{v},\mathbf{z}) :=& \big[\big(\alpha_1(\mathbf{A}_1(\mathbf{v})\mathbf{W}(\mathbf{z}) - \mathbf{W}(\mathbf{v})\mathbf{A}_1(\mathbf{z}))\\ &+(\alpha_1+\alpha_2)\mathbf{A}_1(\mathbf{v})\mathbf{A}_1(\mathbf{z})\big)\mathbf{n}\big]_\tau .\end{aligned}$$

Thus, $\mathbf{N}'(\mathbf{v},\mathbf{v}) = \mathbf{N}(\mathbf{v})$ and $\mathbf{N}'_\Gamma(\mathbf{v},\mathbf{v}) = \mathbf{N}_\Gamma(\mathbf{v})$. Moreover, following the derivation of inequality (3.1), we see that

$$\begin{aligned}&\|\mathbf{v}\cdot\nabla\mathbf{z}\|_\Omega^{(m+\lambda)} + \|\mathbf{N}'(\mathbf{v},\mathbf{z})\|_\Omega^{(m+\lambda)} + \|\mathbf{N}'_\Gamma(\mathbf{v},\mathbf{z})\|_\Gamma^{(m+1+\lambda)}\\ &\quad\le C_{12}(1+|\alpha_1|+|\alpha_1+\alpha_2|)\|\mathbf{v}\|_\Omega^{(m+2+\lambda)}\|\mathbf{z}\|_\Omega^{(m+2+\lambda)}.\end{aligned} \tag{3.2}$$

Third, for any $\mathbf{v} \in C^{m+2+\lambda}(\overline{\Omega})$, $\mathbf{z} \in C^{m+\lambda}(\overline{\Omega})$, there is a constant $C_{13} = C_{13}(\Omega, m, \lambda)$ such that

$$\begin{aligned}&\|\mathbf{z}\cdot\nabla\triangle\mathbf{v}\|_\Omega^{(m-1+\lambda)} + \|[(\mathbf{z}\cdot\nabla\mathbf{A}_1(\mathbf{v}))\mathbf{n}]_\tau\|_\Gamma^{(m+\lambda)}\\ &\quad\le C_{13}\|\mathbf{z}\|_\Omega^{(m+\lambda)}\|\mathbf{v}\|_\Omega^{(m+2+\lambda)}.\end{aligned} \tag{3.3}$$

Now, for fixed $\varepsilon \in (0,1)$, set

$$\eta = \min\left\{C_5, \varepsilon\big[2C_6C_{12}(1+|\alpha_1|+|\alpha_1+\alpha_2|) + C_6C_{13}|\alpha_1|\big]^{-1}\right\} \tag{3.4}$$

and assume that

$$\begin{aligned}\|\mathbf{f}\|_\Omega^{(m+\lambda)} \le C_{14} &= C_{14}(\Omega, m, \lambda, \nu, |\alpha_1|, |\alpha_1+\alpha_2|, k)\\ &:= \min\left(\frac{\eta}{C_3}, \frac{\eta}{2C_6}\right).\end{aligned} \tag{3.5}$$

Then, by Lemma 3, the Stokes problem

$$\begin{aligned}-\nu\triangle\mathbf{v}^0 + \nabla p^0 = \mathbf{f} &\quad \text{in } \Omega,\\ \nabla\cdot\mathbf{v}^0 = 0 &\quad \text{in } \Omega,\\ \mathbf{v}^0\cdot\mathbf{n} = 0 &\quad \text{on } \Gamma,\\ \nu(1-k)[\mathbf{A}_1(\mathbf{v}^0)\mathbf{n}]_\tau = k\,\mathbf{v}^0 &\quad \text{on } \Gamma\end{aligned}$$

has a unique solution $(\mathbf{v}^0, p^0) \in C^{m+2+\lambda}(\overline{\Omega}) \times C_*^{m+1+\lambda}(\overline{\Omega})$, and

$$\|\mathbf{v}^0\|_\Omega^{(m+2+\lambda)} + \|p^0\|_\Omega^{(m+1+\lambda)} \le C_3 \|\mathbf{f}\|_\Omega^{(m+\lambda)} \le \eta.$$

For $i \in \mathbb{N}$, assume that $\mathbf{v}^{i-1} \in C^{m+2+\lambda}(\overline{\Omega})$, $\mathbf{v}^{i-1} \cdot \mathbf{n} = 0$ on Γ and $\|\mathbf{v}^{i-1}\|_\Omega^{(m+2+\lambda)} \le \eta$. Then

$$\mathbf{v}^{i-1} \cdot \nabla \mathbf{v}^{i-1} \in C^{m+1+\lambda}(\overline{\Omega}), \quad \mathbf{N}(\mathbf{v}^{i-1}) \in C^{m+\lambda}(\overline{\Omega}),$$

$$\mathbf{N}_\Gamma(\mathbf{v}^{i-1}) \in C^{m+1+\lambda}(\Gamma), \quad \mathbf{N}_\Gamma(\mathbf{v}^{i-1}) \cdot \mathbf{n} = 0 \text{ on } \Gamma$$

and

$$\|\mathbf{v}^{i-1}\|_\Omega^{(m+2+\lambda)} \le C_5.$$

Hence, by Theorems 2 and 3, there exists a unique pair $(\mathbf{v}^i, p^i) \in C^{m+2+\lambda}(\overline{\Omega}) \times C_*^{m+1+\lambda}(\overline{\Omega})$ such that

$$\begin{aligned}
-\nu \triangle \mathbf{v}^i - \alpha_1 \mathbf{v}^{i-1} \cdot \nabla \triangle \mathbf{v}^i + \nabla p^i = \mathbf{f} - \mathbf{v}^{i-1} \cdot \nabla \mathbf{v}^{i-1} + \mathbf{N}(\mathbf{v}^{i-1}) \quad & \text{in } \Omega, \\
\nabla \cdot \mathbf{v}^i = 0 \quad & \text{in } \Omega, \\
\mathbf{v}^i \cdot \mathbf{n} = 0 \quad & \text{on } \Gamma,
\end{aligned}$$

$$\begin{aligned}
\nu(1-k)[\mathbf{A}_1(\mathbf{v}^i)\mathbf{n}]_\tau - k\mathbf{v}^i + \alpha_1(1-k)[(\mathbf{v}^{i-1} \cdot \nabla \mathbf{A}_1(\mathbf{v}^i))\mathbf{n}]_\tau & \\
= -(1-k)\mathbf{N}_\Gamma(\mathbf{v}^{i-1}) \quad & \text{on } \Gamma.
\end{aligned}$$

Moreover, by inequalities (2.14) and (3.1)–(3.5),

$$\begin{aligned}
& \|\mathbf{v}^i\|_\Omega^{(m+2+\lambda)} + \|p^i\|_\Omega^{(m+1+\lambda)} \\
& \le C_6\Big(\|\mathbf{f}\|_\Omega^{(m+\lambda)} + \|\mathbf{v}^{i-1} \cdot \nabla \mathbf{v}^{i-1} - \mathbf{N}(\mathbf{v}^{i-1})\|_\Omega^{(m+\lambda)} \\
& + \|\mathbf{N}_\Gamma(\mathbf{v}^{i-1})\|_\Gamma^{(m+1+\lambda)}\Big) \le C_6\|\mathbf{f}\|_\Omega^{(m+\lambda)} + C_6 C_{12}(1 + |\alpha_1| + |\alpha_1 + \alpha_2|) \\
& \times \left(\|\mathbf{v}^{i-1}\|_\Omega^{(m+2+\lambda)}\right)^2 \le \beta\eta, \qquad (3.6)
\end{aligned}$$

where $\beta := (1+\varepsilon)/2 < 1$. Thus, by induction, the sequence $\{(\mathbf{v}^i, p^i)\}_{i=1}^\infty$ is well-defined and uniformly bounded in $C^{m+2+\lambda}(\overline{\Omega}) \times C_*^{m+1+\lambda}(\overline{\Omega})$.

Next we will show that $\{(\mathbf{v}^i, p^i)\}_{i=1}^{\infty}$ is a Cauchy sequence in $C^{m+1+\lambda}(\overline{\Omega}) \times C_*^{m+\lambda}(\overline{\Omega})$. Set $\mathbf{z}^i = \mathbf{v}^i - \mathbf{v}^{i-1}$ and $s^i = p^i - p^{i-1}$, $i = 1, 2, 3, \dots$ Then, for $i \geq 2$,

$$\begin{aligned}
-\nu\triangle\,\mathbf{z}^i - \alpha_1\,\mathbf{v}^{i-1}\cdot\nabla\triangle\,\mathbf{z}^i + \nabla s^i
&= \alpha_1\,\mathbf{z}^{i-1}\cdot\nabla\triangle\,\mathbf{v}^{i-1} \\
&\quad - \mathbf{v}^{i-1}\cdot\nabla\,\mathbf{z}^{i-1} - \mathbf{z}^{i-1}\cdot\nabla\,\mathbf{v}^{i-2} \\
&\quad + \mathbf{N}'(\mathbf{v}^{i-1}, \mathbf{z}^{i-1}) + \mathbf{N}'(\mathbf{z}^{i-1}, \mathbf{v}^{i-2}) \\
&=: \mathbf{f}^{i-1} \qquad \text{in } \Omega, \qquad (3.7)\\
\nabla\cdot\mathbf{z}^i &= 0 \qquad \text{in } \Omega, \qquad (3.8)\\
\mathbf{z}^i\cdot\mathbf{n} &= 0 \qquad \text{on } \Gamma, \qquad (3.9)\\
-k\,\mathbf{z}^i + \nu(1-k)[\mathbf{A}_1(\mathbf{z}^i)\,\mathbf{n}]_\tau & \\
+ \alpha_1(1-k)[(\mathbf{v}^{i-1}\cdot\nabla\,\mathbf{A}_1(\mathbf{z}^i))\,\mathbf{n}]_\tau &= -\alpha_1(1-k)\big([(\mathbf{z}^{i-1}\cdot\nabla\,\mathbf{A}_1(\mathbf{v}^{i-1}))\,\mathbf{n}]_\tau \\
&\quad + \mathbf{N}'_\Gamma(\mathbf{v}^{i-1}, \mathbf{z}^{i-1}) + \mathbf{N}'_\Gamma(\mathbf{z}^{i-1}, \mathbf{v}^{i-2})\big) \\
&=: (1-k)\,\mathbf{b}_1^{i-1} \qquad \text{on } \Gamma. \qquad (3.10)
\end{aligned}$$

By construction, $\mathbf{v}^{i-1} \in C^{m+2+\lambda}(\overline{\Omega})$, $\mathbf{v}^{i-1}\cdot\mathbf{n} = 0$ on Γ and $\|\,\mathbf{v}^{i-1}\|_\Omega^{(m+2+\lambda)} \leq \eta \leq C_5$. Furthermore, $\mathbf{f}^{i-1} \in C^{m-1+\lambda}(\overline{\Omega})$, $\mathbf{b}_1^{i-1} \in C^{m+\lambda}(\Gamma)$ and $\mathbf{b}_1^{i-1}\cdot\mathbf{n} = 0$ on Γ. Thus, we can apply Theorem 2 to problem (3.7)–(3.10). From estimates (2.14), (3.2)–(3.4) we get

$$\begin{aligned}
&\|\,\mathbf{z}^i\|_\Omega^{(m+1+\lambda)} + \|s^i\|_\Omega^{(m+\lambda)} \\
&\leq C_6\Big(\|\,\mathbf{f}^{i-1}\|_\Omega^{(m-1+\lambda)} + \|\,\mathbf{b}_1^{i-1}\|_\Gamma^{(m+\lambda)}\Big) \\
&\leq C_6\Big(C_{12}(1+|\alpha_1|+|\alpha_1+\alpha_2|)\Big[\|\,\mathbf{v}^{i-1}\|_\Omega^{(m+1+\lambda)} + \|\,\mathbf{v}^{i-2}\|_\Omega^{(m+1+\lambda)}\Big] \\
&\qquad + C_{13}|\alpha_1|\cdot\|\,\mathbf{v}^{i-1}\|_\Omega^{(m+2+\lambda)}\Big)\|\,\mathbf{z}^{i-1}\|_\Omega^{(m+1+\lambda)} \\
&\leq \Big[2C_6C_{12}(1+|\alpha_1|+|\alpha_1+\alpha_2|) + C_6C_{13}|\alpha_1|\Big]\eta\|\,\mathbf{z}^{i-1}\|_\Omega^{(m+1+\lambda)} \\
&\leq \varepsilon\|\,\mathbf{z}^{i-1}\|_\Omega^{(m+1+\lambda)}. \qquad (3.11)
\end{aligned}$$

The uniqueness of the solution follows from inequality (3.11). Indeed, the difference $(\mathbf{z}, s)$ of two solutions $(\mathbf{v}, p), (\mathbf{v}', p')$ satisfies equations

(3.7)–(3.10) with $\mathbf{v}^{i-1}, \mathbf{v}^{i-2}, \mathbf{z}^i, \mathbf{z}^{i-1}, s^i$ replaced by $\mathbf{v}, \mathbf{v}', \mathbf{z}, \mathbf{z}, s$, so that inequality (3.11) with $\mathbf{z}^i, \mathbf{z}^{i-1}, s^i$ replaced by $\mathbf{z}, \mathbf{z}, s$ holds. Since $\varepsilon \in (0,1)$, (3.11) yields $\mathbf{z} \equiv 0, s \equiv 0$. $\square$

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