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ON A STEADY THREE-DIMENSIONAL NONCOMPACT FREE BOUNDARY VALUE PROBLEM FOR THE NAVIER–STOKES EQUATIONS

ABSTRACT. A steady three-dimensional flow of a viscous incompressible fluid with a noncompact free boundary above a fixed unbounded bottom is studied. It is assumed that the motion of the fluid is generated by sources and sinks situated in a bounded part of the bottom and having zero total flux. The existence for small data of the unique solution to this problem is proved and the asymptotics of the solution is constructed.

**This paper is dedicated to
Prof. V. A. Solonnikov's 70th birthday**

1. INTRODUCTION

In the paper we study a steady flow of a viscous incompressible fluid with a noncompact free boundary above a fixed unbounded bottom $S_0 = \{x = (x', x_3) \in \mathbb{R}^3\colon x_3 = 0,\ x' \in \mathbb{R}^2\}$. It is assumed that the motion of the fluid is generated by sources and sinks situated in a bounded part of the bottom S_0 and having zero total flux. Suppose that a free boundary Γ of the fluid admits the parameterization $\Gamma = \{x \in \mathbb{R}^3\colon x_3 = \varphi(x'),\ x' \in \mathbb{R}^2\}$. We look for a free boundary that tends to a fixed height h at infinity. Without loss of generality we may take $h = 1$ (otherwise making the change of variables $x/h \to x$ and rescalling the problem we reduce h to the unit). Thus, we have to find a vector of velocity $\mathbf{v}(x)$, a pressure $p(x)$ and a function $\varphi(x')$ satisfying in the domain

$$\Omega = \{x \in \mathbb{R}^3\colon x' \in \mathbb{R}^2, 0 < x_3 < \varphi(x')\} \tag{1.1}$$

the Navier–Stokes system of equations

$$-\nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = -g\mathbf{e_3}, \quad \operatorname{div}\mathbf{v} = 0 \tag{1.2}$$

and the boundary conditions

$$\mathbf{v}|_{S_0} = \varepsilon\boldsymbol{\alpha}, \quad \mathbf{v}\cdot\mathbf{n}|_\Gamma = 0, \quad \boldsymbol{\tau}_1\cdot T(\mathbf{v})\mathbf{n}|_\Gamma = 0, \quad \boldsymbol{\tau}_2\cdot T(\mathbf{v})\mathbf{n}|_\Gamma = 0, \tag{1.3}$$

$$\frac{\sigma}{2}\nabla\cdot\frac{\nabla\varphi}{(1+|\nabla\varphi|^2)^{1/2}}=(-p+\mathbf{n}\cdot T(\mathbf{v})\mathbf{n})|_\Gamma, \tag{1.4}$$

$$\lim_{|x'|\to\infty}\varphi(x)=1, \tag{1.5}$$

where $\nabla=(\partial/\partial x_1,\partial/\partial x_2,\partial/\partial x_3)$, $\mathbf{e_3}=(0,0,1)$, g is the acceleration of the gravity, $\mathbf{n}$ is a unit vector of the outward normal to the free boundary Γ and $\boldsymbol{\tau}_1,\boldsymbol{\tau}_2$ are unit vectors tangent to Γ, $T(\mathbf{v})$ is the deformation tensor with the elements $T_{ij}(\mathbf{v})=\nu(\partial v_i/\partial x_j+\partial v_j/\partial x_i)$, $i,j=1,2,3$, ν and σ are coefficients of the viscosity and the surface tension respectively, $\varepsilon>0$ is a small parameter, supp $\boldsymbol{\alpha}\subset\{x'\in S_0\colon |x'|<1\}$ and

$$\int\limits_{\mathbb{R}^2}\alpha_3(x')\,dx'=0. \tag{1.6}$$

Note that the left-hand side of (1.4) is equal to one half of the mean curvature of the free boundary Γ and condition (1.6) expresses the fact that the total flux of sources and sinks is equal to zero.

If $\varepsilon=0$, problem (1.1)–(1.6) has an exact solution $\mathbf{v}^0(x)=0, p^0(x)=g(1-x_3),\varphi^0(x')=1$. In the paper we prove the solvability of problem (1.1)–(1.6) for small ε and we look for the solution $(\mathbf{v},p,\varphi)$ in the neighborhood of the exact solution $(\mathbf{v}^0,p^0,\varphi^0)$.

The mathematical problems for the steady flows of a viscous incompressible fluid with a free boundary have been a subject of many papers. The flows with compact free boundaries have been considered in [3, 4, 15–16, 19–22, 32, 33], the flows with moving contact points and lines have been studied in [24–28, 30] and the flows with noncompact free boundaries in [5, 10–13, 17, 23, 29, 31, 32]. In all mentioned papers the same general scheme proposed in [15] has been used. This scheme consists in the following. First, the starting free boundary problem is divided into two problems: the Navier–Stokes system for $(\mathbf{v},p)$ in a fixed domain and the problem for the function φ which describes the free boundary Γ. Using the results on the solvability of these two auxiliary problems, the starting problem is reduced to an operator equation in a certain Banach space with a contraction (for small data) operator.

In the case of problem (1.1)–(1.6) we map the unknown domain Ω given by (1.1) to the three-dimensional layer $\Pi=\{\eta=(y,z):y=(y_1,y_2)\in\mathbb{R}^2,0<z<1\}$ and make the change of the unknown functions $(\mathbf{v},p,\varphi)=(\varepsilon\mathbf{u},p^0+\varepsilon q,1+\varepsilon\Psi)$ (see Sec. 3). After this transformation the problem (1.1)–(1.6) takes the form

$$-\nu\triangle_\eta\mathbf{u}+\nabla_\eta q=\varepsilon\mathbf{M}(\mathbf{u},q,\Psi), \tag{1.7}$$

$$\operatorname{div}_\eta \mathbf{u} = 0, \tag{1.8}$$

$$\mathbf{u}|_{z=0} = \boldsymbol{\alpha} \qquad u_3|_{z=1} = 0, \tag{1.9}$$

$$\partial_3 u_1|_{z=1} = \varepsilon B_1(\mathbf{u}, q, \Psi), \quad \partial_3 u_2|_{z=1} = \varepsilon B_2(\mathbf{u}, q, \Psi); \tag{1.10}$$

$$\sigma \triangle \Psi - 2\, g \Psi = 2(2\nu \partial_3 u_3 - q)|_{z=1} + \varepsilon R(\mathbf{u}, q, \Psi), \tag{1.11}$$

$$\lim_{\sqrt{y_1^2+y_2^2}\to\infty} \Psi(y_1, y_2) = 0 \tag{1.12}$$

where $\partial_3 = \partial/\partial z$, $\mathbf{M} = (M_1, M_2, M_3)$, B_1, B_2, R are nonlinear functions of $(\mathbf{u}, q, \Psi)$. Next, assuming that the functions $\widehat{\mathbf{u}}, \widehat{q}, \widehat{\Psi}$ in the right-hand sides of (1.7)–(1.12) are given, we solve the Stokes problem (1.7)–(1.10) and find the solution $(\mathbf{u}, q)$. Then solving (1.11), (1.12) with the right-hand side $R(\mathbf{u}, q, \widehat{\Psi})$ we find the function Ψ. Thus, formally we have an operator $\mathfrak{A}\colon (\widehat{\mathbf{u}}, \widehat{q}, \widehat{\Psi}) \to (\mathbf{u}, q, \Psi)$. If we find a space $\mathfrak{E}$ in which the operator $\mathfrak{A}$ is well defined, then it could be proved for small ε that $\mathfrak{A}$ is a contraction and, therefore, the problem (1.7)–(1.12) has a unique solution. For problems with noncompact free boundaries the main difficulty is to find such a space $\mathfrak{E}$. Mainly this is related to the estimates of the pressure q. Since the function q is linearly included into the right-hand side of (1.11), q has to be a "fast" decaying at infinity function in order that (1.11) had a vanishing solution. To overcome this difficulty, we use anisotropic weighted spaces with detached asymptotics, i.e., elements $(\mathbf{u}, q, \Psi)$ of these spaces are represented as sums of the main asymptotic terms of the solutions $(\mathbf{u}, q)$ and Ψ to problems (1.7)–(1.10) and (1.11)–(1.12), respectively, and better decaying terms $(\widetilde{\mathbf{u}}, \widetilde{q})$ and $\widetilde{\Psi}$ belonging to certain weighted spaces. This gives the possibility to estimate the nonlinear terms in the right-hand sides of (1.7)–(1.10) and to derive the appropriate decay of the pressure q. The Stokes and Navier–Stokes problems in such anisotropic weighted spaces were studied in [6–8, 14].

Note that the assumption that the total flux of sources and sinks is equal to zero (condition (1.6)) is essential for our considerations. The case with the nonzero total flux is an open problem.

The paper is organized as follows. In Sec. 2, we introduce the anisotropic weighted spaces. In Sec. 3, we perform the linearization of problem (1.1)–(1.6) and reduce it to problem (1.7)–(1.12). In Secs. 4 and 5, we study linear problems (1.7)–(1.10) and (1.11)–(1.12) assuming that the right-hand sides are given. In Sec. 6, we estimate the nonlinear terms in (1.7)–(1.12). Finally, in Sec. 7, we reduce problem (1.7)–(1.12) to an operator equation and prove the main result of the paper: the existence for small ε of the unique solution to problem (1.7)–(1.12).

2. Function spaces

Let G be an arbitrary domain in $\mathbb{R}^n$, $n \geq 2$. Denote by $C^\infty(G)$ the set of all infinitely differentiable functions in G and let $C_0^\infty(G)$ be a subset of functions from $C^\infty(G)$ with compact supports in G. Let $W^{l,2}(G)$ be the Sobolev space and $W^{l-1/2,2}(G)$ be the space of traces on ∂G of functions from $W^{l,2}(G)$ (l is an integer). As usual, $W^{0,2}(G) = L^2(G)$. The norm of the element u in the function space V is denoted by $||u; V||$. Henceforth the spaces of scalar and vector-valued functions are not distinguished in notations.

Let

$$\Pi = \{\eta = (y, z) : y = (y_1, y_2) \in \mathbb{R}^2, 0 < z < 1\} \tag{2.1}$$

be an infinite layer in $\mathbb{R}^3$. We denote by $\mathcal{V}^l_{\beta,\,\kappa}(\Pi)$ ($\beta \in \mathbb{R}$ and l, κ are integers with $l \geq 0, 0 \leq \kappa \leq l$) the space of functions obtained as the closure of $C_0^\infty(\Pi)$ in the norm

$$\| v; \mathcal{V}^l_{\beta,\,\kappa}(\Pi) \| = \\ = \Big(\sum_{\alpha+|\gamma|\leq l} \int_\Omega (1+r^2)^{\beta+|\gamma|-(|\gamma|-\kappa)_+} |\partial_z^\alpha \partial_y^\gamma v(y,z)|^2 dydz \Big)^{1/2} \tag{2.2}$$

where $r = |y|$, $\alpha \geq 0$, $\gamma = (\gamma_1, \gamma_2)$, $\gamma_i \geq 0$, $|\gamma| = \gamma_1 + \gamma_2, \partial_z^\alpha = \frac{\partial^\alpha}{\partial z^\alpha}$, $\partial_y^\gamma = \frac{\partial^{|\gamma|}}{\partial_{y_1}^{\gamma_1} \partial_{y_2}^{\gamma_2}}$ and $(t)_+ = \frac{(t+|t|)}{2}$ is the positive part of a number $t \in \mathbb{R}$.

The space $\mathcal{V}^l_{\beta,\,\kappa}(\Pi)$ is a weighted space with anisotropic and step-function distribution of the weight. The direction z is singled out in the layer Π both by the geometry of set (2.1) and by the structure of norm (2.2): the differentiation ∂_z^α does not affect the exponent of the weight function $(1 + r^2)$, while the differentiation ∂_y^γ up to the order $|\gamma| \leq \kappa$ increases this exponent by $|\gamma|$. For $|\gamma| > \kappa$ the exponent of the weight function does not change any more (since $|\gamma| - (|\gamma| - \kappa)_+ = \kappa$ for $|\gamma| > \kappa$), i.e., a "step" arises in the distribution of the weight. Note that there are no step for $\kappa = l$. However, for $\kappa = 0$ we have $|\gamma| - (|\gamma|)_+ = 0$ for all $|\gamma|$, and hence, the weight factor is the same for all derivatives. Finally, we note that $\mathcal{V}^l_{0,0}(\Pi) = W^{l,2}(\Pi)$ and define $L^2_\beta(\Pi) = \mathcal{V}^0_{\beta,0}(\Pi)$.

For $l \geq 1$ it is possible to define the space $\mathcal{V}^{l-1/2}_{\beta,\,\kappa}(\partial\Pi)$ of traces on $\partial\Pi$ of functions v from $\mathcal{V}^l_{\beta,\,\kappa}(\Pi)$. The norm in the space $\mathcal{V}^{l-1/2}_{\beta,\,\kappa}(\partial\Pi)$ is given by the formula

$$\| w; \mathcal{V}^{l-1/2}_{\beta,\kappa}(\partial\Pi) \| = \inf \left\{ \| v; \mathcal{V}^l_{\beta,\,\kappa}(\Pi) \| \ : v = w \ on \ \partial\Pi \right\}. \tag{2.3}$$

The spaces $\mathcal{V}^l_{\beta,\,\kappa}(\Pi)$ were introduced in the paper [7] to reflect the anisotropic properties of the asymptotics of solutions to the Stokes problem in the layer Π.

The boundary $\partial\Pi$ of the layer Π consists of two planes $S_0 = \{(y,z) : y \in \mathbb{R}^2,\, z = 0\}$ and $\Gamma = \{(y,z) : y \in \mathbb{R}^2,\, z = 1\}$. In order to specify the norm (2.3) let us define weighted function spaces in $\mathbb{R}^2$. Denote by $\mathcal{V}^l_{\beta,\kappa}(\mathbb{R}^2)$ the space of functions obtained as the closure of $C_0^\infty(\mathbb{R}^2)$ in the norm

$$\|\, w;\, \mathcal{V}^l_{\beta,\,\kappa}(\mathbb{R}^2)\,\| = \\ = \Big(\sum_{0\le|\gamma|\le l}\int_{\mathbb{R}^2}(1+r^2)^{\beta+|\gamma|-(|\gamma|-\kappa)_+}|\partial_y^\gamma w(y)|^2 dy\Big)^{1/2} \qquad (2.4)$$

and by $\mathcal{V}^{l-1/2}_{\beta,\kappa}(\mathbb{R}^2)$ the closure of $C_0^\infty(\mathbb{R}^2)$ in the norm

$$\|\, w;\, \mathcal{V}^{l-1/2}_{\beta,\,\kappa}(\mathbb{R}^2)\,\|^2 = \|\, w;\, \mathcal{V}^{l-1}_{\beta,\,\kappa}(\mathbb{R}^2)\,\|^2 + \qquad (2.5)$$

$$+\sum_{|\gamma|=l-1}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|\partial_y^\gamma[(1+|y|^2)^{(\beta+\kappa)/2}w(y)]-\partial_{\widetilde{y}}^\gamma[(1+|\widetilde{y}|^2)^{(\beta+\kappa)/2}w(\widetilde{y})]|^2}{|y-\widetilde{y}|^3}dyd\widetilde{y}.$$

It is known (see [7]) that the norm $\|\, w;\, \mathcal{V}^{l-1/2}_{\beta,\kappa}(\partial\Pi)\,\|$ defined by (2.3) is equivalent to the sum

$$\|w;\, \mathcal{V}^{l-1/2}_{\beta,\kappa}(S_0)\| + \|w;\, \mathcal{V}^{l-1/2}_{\beta,\kappa}(\Gamma)\|$$

where the norms over planes S_0 and Γ are given by formula (2.5).

Let us formulate two lemmas concerning the properties of spaces $\mathcal{V}^l_{\beta,\,\kappa}(\Pi)$. We use these results in Sec. 5 estimating the nonlinear terms.

Lemma 1. *Let* $u \in \mathcal{V}^{l+1}_{\beta_1+2,\,l-1}(\Pi)$, $v \in \mathcal{V}^l_{\beta_2+1,\,l}(\Pi)$, $l \ge 1$. *Then* $z = u\cdot v \in \mathcal{V}^l_{\beta_1+\beta_2+3,\,l}(\Pi)$ *and there holds the inequality*

$$\|\, z;\, \mathcal{V}^l_{\beta_1+\beta_2+3,\,l}(\Pi)\,\| \le c\|\, u;\, \mathcal{V}^{l+1}_{\beta_1+2,\,l-1}(\Pi)\,\|\|\, v;\, \mathcal{V}^l_{\beta_2+1,\,l}(\Pi)\,\|. \qquad (2.6)$$

Lemma 2. *Let* $u \in \mathcal{V}^{l+1}_{\beta_1+1,\,l}(\Pi)$, $v \in \mathcal{V}^l_{\beta_2+2,\,l-1}(\Pi)$, $l \ge 1$. *Then* $z = u\cdot v \in \mathcal{V}^l_{\beta_1+\beta_2+3,\,l}(\Pi)$ *and there holds the inequality*

$$\|\, z;\, \mathcal{V}^l_{\beta_1+\beta_2+3,\,l}(\Pi)\,\| \le c\|\, u;\, \mathcal{V}^{l+1}_{\beta_1+1,\,l}(\Pi)\,\|\|\, v;\, \mathcal{V}^l_{\beta_2+2,\,l-1}(\Pi)\,\|. \qquad (2.7)$$

Lemmas 1 and 2 are proved in [14] (see Lemmas 3.2 and 3.3 in [14]).

3. Transformation of the domain Ω and linearization of the free boundary value problem

In the layer Π we define the function

$$w(y_1, y_2, z) = z \int\limits_{|\tau|\le 1} K(\tau)\Psi(y + \tau(z-1))d\tau \tag{3.1}$$

where $|\tau| = \sqrt{\tau_1^2+\tau_2^2}$, $\tau = (\tau_1, \tau_2)$, $y = (y_1, y_2)$, $\varphi(y) = 1+\varepsilon\Psi(y)$, $\varepsilon > 0$, and $K(\tau)$ is the smooth kernel, satisfying the conditions: supp $K \subset \{\,\tau = (\tau_1, \tau_2)\colon |\tau| \le 1\}$, $\int\limits_{|\tau|\le 1} K(\tau)d\tau = 1$.

It is easy to see that the function w satisfies the boundary conditions

$$w(y,z)|_{z=0} = 0, \qquad w(y,z)|_{z=1} = \Psi(y_1, y_2). \tag{3.2}$$

Lemma 3. *Assume that $\Psi \in \mathcal{V}^{l+3/2}_{\beta,\,l-1}(\mathbb{R}^2)$. Then $w \in \mathcal{V}^{l+2}_{\beta,l-1}(\Pi)$, $\partial_z^2 w \in \mathcal{V}^{l}_{\beta+1,\,l-1}(\Pi)$ and there holds the estimate*

$$\|\, w;\, \mathcal{V}^{l+2}_{\beta,\,l-1}(\Pi)\,\| + \|\partial_z^2 w;\, \mathcal{V}^{l}_{\beta+1,\,l-1}(\Pi)\| \le c\|\,\Psi;\, \mathcal{V}^{\,l+3/2}_{\beta,\,l-1}(\mathbb{R}^2)\,\|. \tag{3.3}$$

Estimate (3.3) could be proved by standard arguments (e.g., [1]) taking into account the definition of the norm in the space $\mathcal{V}^{l+2}_{\beta,l-1}(\Pi)$.

Let us define the coordinate transformation $x = \mathrm{X}(\eta)$, $\eta = (y_1, y_2, z)$ by the formulas

$$x_1 = y_1, \quad x_2 = y_2, \quad x_3 = z + \varepsilon w(y_1, y_2, z). \tag{3.4}$$

This transformation maps the layer Π onto the domain Ω given by (1.1). For sufficiently small ε the Jacobian of the transformation (3.4) is separated from zero and, therefore, there exists the inverse transformation $\eta = \mathrm{X}^{-1}(x)$. Moreover, from Lemma 3 it follows that

$$\begin{aligned}\|\,\mathrm{X}(\eta)-\eta;\mathcal{V}^{l+2}_{\beta,l-1}(\Pi)\,\| + \|\,a_{ij}(\eta)-\delta_i^j;\,\mathcal{V}^{l+1}_{\beta,l-1}(\Pi)\,\| \le\\ \le c\varepsilon\|\,\Psi;\,\mathcal{V}^{l+3/2}_{\beta,l-1}(\mathbb{R}^2)\,\|,\end{aligned} \tag{3.5}$$

where $a_{ij}(\eta) = \partial\eta_j/\partial x_i$ and δ_i^j is the Kroneker symbol.

We also introduce the vector function $\mathbf{u}$ with the components

$$u_1(\eta) = \mathrm{V}_1(\mathrm{X}(\eta))\Big(1+\varepsilon\frac{\partial w(\eta)}{\partial\eta_3}\Big), \quad u_2(\eta) = \mathrm{V}_2(\mathrm{X}(\eta))\Big(1+\varepsilon\frac{\partial w(\eta)}{\partial\eta_3}\Big),$$

$$u_3(\eta) = \mathrm{V}_3(\mathrm{X}(\eta)) - \mathrm{V}_1(\mathrm{X}(\eta)) \cdot \varepsilon \frac{\partial w(\eta)}{\partial \eta_1} - \mathrm{V}_2(\mathrm{X}(\eta)) \cdot \varepsilon \frac{\partial w(\eta)}{\partial \eta_2}. \tag{3.6}$$

We have

$$\mathrm{V}_1(x) = \left(\frac{u_1(\eta)}{1 + \varepsilon \partial w / \partial \eta_3} \right)\Big|_{\eta = \mathrm{X}^{-1}(x)},$$

$$\mathrm{V}_2(x) = \left(\frac{u_2(\eta)}{1 + \varepsilon \partial w / \partial \eta_3} \right)\Big|_{\eta = \mathrm{X}^{-1}(x)}, \tag{3.7}$$

$$\mathrm{V}_3(x) = \left[\left(u_1(\eta) \cdot \varepsilon \frac{\partial w}{\partial \eta_1} + u_2(\eta) \cdot \varepsilon \frac{\partial w}{\partial \eta_2} \right) \frac{1}{1 + \varepsilon \partial w / \partial \eta_3} + u_3(\eta) \right]\Big|_{\eta = \mathrm{X}^{-1}(x)},$$

and, if

$$\mathrm{div}_x \mathbf{V} = 0 \quad \text{and} \quad \mathbf{V} \cdot \mathbf{n}_x|_\Gamma = 0,$$

then

$$\mathrm{div}_\eta \mathbf{u} = 0 \quad \text{and} \quad u_3(\eta)|_{z=1} = 0.$$

Let us substitute into the system (1.1)–(1.5) the functions

$$\mathbf{v}(x) = \varepsilon \mathbf{V}(x),$$

$$p(x) = p^0(x) + \varepsilon q(x), \quad p^0(x) = g(1 - x_3),$$

$$\varphi(x) = 1 + \varepsilon \Psi(x).$$

After the change of variables $\eta = \mathrm{X}^{-1}(x)$ the problem $(1.1)--(1.5)$ goes over to the following equivalent boundary value problem in the layer Π:

$$-\nu \triangle_\eta \mathbf{u} + \nabla_\eta q = \varepsilon \mathbf{M}(\mathbf{u}, q, \Psi), \tag{3.8}$$

$$\mathrm{div}_\eta \mathbf{u} = 0, \tag{3.9}$$

$$\mathbf{u}|_{z=0} = \boldsymbol{\alpha} \qquad u_3|_{z=1} = 0, \tag{3.10}$$

$$\partial_3 u_1|_{z=1} = \varepsilon B_1(\mathbf{u}, q, \Psi), \ \ \partial_3 u_2|_{z=1} = \varepsilon B_2(\mathbf{u}, q, \Psi); \tag{3.11}$$

$$\sigma \triangle \Psi - 2\, g \Psi = 2(2\nu \partial_3 u_3 - q)|_{z=1} + \varepsilon \widetilde{R}(\mathbf{u}, q, \Psi), \tag{3.12}$$

$$\lim_{\sqrt{y_1^2 + y_2^2} \to \infty} \Psi(y_1, y_2) = 0 \tag{3.13}$$

where $\mathbf{M} = (M_1, M_2, M_3)$,

$$\begin{aligned} M_1 = -\nu \triangle_\eta \left(\frac{\partial_3 w}{1 + \varepsilon \partial_3 w} u_1 \right) + \frac{\nu}{\varepsilon} (\widehat{\nabla}_\eta^2 - \nabla_\eta^2) \widehat{\mathrm{V}}_1 - \\ -(\widehat{\mathrm{V}} \cdot \widehat{\nabla}_\eta) \widehat{\mathrm{V}}_1 + \frac{1}{\varepsilon} (\partial_1 - \widehat{\partial}_1) q, \end{aligned} \tag{3.14}$$

$$M_2 = -\nu\Delta_\eta\left(\frac{\partial_3 w}{1+\varepsilon\partial_3 w}u_2\right)+\frac{\nu}{\varepsilon}(\,\widehat{\nabla}_\eta^2-\nabla_\eta^2)\widehat{\mathrm{V}}_2-$$
$$-(\widehat{\mathrm{V}}\cdot\widehat{\nabla}_\eta)\widehat{\mathrm{V}}_2+\frac{1}{\varepsilon}(\partial_2-\widehat{\partial}_2)q, \tag{3.15}$$

$$M_3 = \nu\Delta_\eta\left(\frac{\partial_1 w}{1+\varepsilon\partial_3 w}u_1\right)+\nu\Delta_\eta\left(\frac{\partial_2 w}{1+\varepsilon\partial_3 w}u_2\right)+$$
$$+\frac{\nu}{\varepsilon}(\,\widehat{\nabla}_\eta^2-\nabla_\eta^2)\widehat{\mathrm{V}}_3-(\widehat{\mathrm{V}}\cdot\widehat{\nabla}_\eta)\widehat{\mathrm{V}}_3+\frac{1}{\varepsilon}(\partial_3-\widehat{\partial}_3)q, \tag{3.16}$$

$\partial_i = \partial/\partial_{\eta_i}$, $\widehat{\nabla} = (\widehat{\partial}_1,\widehat{\partial}_2,\widehat{\partial}_3) = A\nabla_\eta$, $A = (a_{ij})$, $a_{ij} = \frac{\partial\eta_j}{\partial x_i}$, $\widehat{\mathrm{V}}(\eta) = \mathrm{V}(x)\big|_{x=\mathrm{X}(\eta)}$,

$$B_1 = \frac{1}{\varepsilon}\Big[(1-a_{33})\partial_3\widehat{\mathrm{V}}_1 - a_{13}\partial_3\widehat{\mathrm{V}}_3\Big]\Big|_{z=1} + \frac{\varepsilon}{\nu}\Big[\partial_1\Psi\widehat{T}_{11}(\widehat{\mathrm{V}})+ \tag{3.17}$$
$$+\partial_2\Psi\widehat{T}_{12}(\widehat{\mathrm{V}})-\partial_1\Psi\widehat{T}_{33}(\widehat{\mathrm{V}})+(\partial_1\Psi)^2\varepsilon\widehat{T}_{13}(\widehat{\mathrm{V}})+\partial_1\Psi\partial_2\Psi\varepsilon\widehat{T}_{23}(\widehat{\mathrm{V}})\Big]\Big|_{z=1}+$$
$$+\left[\partial_3\left(\frac{\partial_3 w}{1+\varepsilon\partial_3 w}u_1\right)-\partial_1\left(\frac{\partial_1 w}{1+\varepsilon\partial_3 w}u_1\right)-\partial_1\left(\frac{\partial_2 w}{1+\varepsilon\partial_3 w}u_2\right)\right]\Big|_{z=1},$$

$$B_2 = \frac{1}{\varepsilon}\Big[(1-a_{33})\partial_3\widehat{\mathrm{V}}_2 - a_{23}\partial_3\widehat{\mathrm{V}}_3\Big]\Big|_{z=1} + \frac{\varepsilon}{\nu}[\partial_2\Psi\widehat{T}_{22}(\widehat{\mathrm{V}})+ \tag{3.18}$$
$$+\partial_1\Psi\widehat{T}_{12}(\widehat{\mathrm{V}})-\partial_2\Psi\widehat{T}_{33}(\widehat{\mathrm{V}})+\partial_1\Psi\partial_2\Psi\varepsilon\widehat{T}_{13}(\widehat{\mathrm{V}})+(\partial_2\Psi)^2\varepsilon\widehat{T}_{23}(\widehat{\mathrm{V}})\Big]\Big|_{z=1}+$$
$$+\left[\partial_3\left(\frac{\partial_3 w}{1+\varepsilon\partial_3 w}u_2\right)-\partial_2\left(\frac{\partial_1 w}{1+\varepsilon\partial_3 w}u_1\right)-\partial_2\left(\frac{\partial_2 w}{1+\varepsilon\partial_3 w}u_2\right)\right]\Big|_{z=1},$$

$$\widehat{T}_{ij}(\widehat{\mathrm{V}}) = \nu\sum_{m=1}^{3}\Big(a_{jm}\partial_m\widehat{\mathrm{V}}_i + a_{im}\partial_m\widehat{\mathrm{V}}_j\Big),$$

$$\widetilde{R} = \frac{2}{\varepsilon}\Bigg[\varepsilon^2((\partial_1\Psi)^2\widehat{T}_{11}(\widehat{\mathrm{V}})+(\partial_2\Psi)^2\widehat{T}_{22}(\widehat{\mathrm{V}}))-2\varepsilon^2\partial_1\Psi\partial_2\Psi\widehat{T}_{12}(\widehat{\mathrm{V}})-$$
$$-2\partial_1\Psi\varepsilon\widehat{T}_{13}(\widehat{\mathrm{V}})-2\partial_2\Psi\varepsilon\widehat{T}_{23}(\widehat{\mathrm{V}})-2\nu\varepsilon^2(\partial_1\Psi)^2\partial_3\widehat{\mathrm{V}}_3-2\nu\varepsilon^2(\partial_2\Psi)^2\partial_3\widehat{\mathrm{V}}_3+$$
$$+2\nu(a_{33}-1)\partial_3\widehat{\mathrm{V}}_3\Bigg]\Bigg|_{z=1}\frac{1}{1+\varepsilon^2(\partial_1\Psi)^2+\varepsilon^2(\partial_2\Psi)^2}+$$

$$+4\nu\partial_3\left(\frac{\partial_1 w}{1+\varepsilon\partial_3 w}u_1\right)\bigg|_{z=1}+4\nu\partial_3\left(\frac{\partial_2 w}{1+\varepsilon\partial_3 w}u_2\right)\bigg|_{z=1}+$$

$$+\frac{1}{2}\sigma\varepsilon\partial_1\left(\partial_1\Psi|\nabla'\Psi|^2\int\limits_0^1\frac{ds}{(1+s\varepsilon^2|\nabla'\Psi|^2)^{3/2}}\right)+$$

$$+\frac{1}{2}\sigma\varepsilon\partial_2\left(\partial_2\Psi|\nabla'\Psi|^2\int\limits_0^1\frac{ds}{(1+s\varepsilon^2|\nabla'\Psi|^2)^{3/2}}\right), \tag{3.19}$$

$\nabla' = (\partial_1,\ \partial_2)$.

It is easy to see from (3.8)–(3.13) that the original problem (1.1)–(1.5) could be divided into two problems: the Stokes problem (3.8)–(3.11) and the problem (3.12)–(3.13) for the function Ψ describing the free boundary Γ.

4. Solvability of the Stokes problem

Let us consider in the layer Π the Stokes problem

$$-\nu\triangle\mathbf{u}+\nabla q=\mathbf{M}, \qquad \operatorname{div}\,\mathbf{u}=0, \tag{4.1}$$

$$\mathbf{u}|_{z=0}=\boldsymbol{\alpha} \qquad u_3|_{z=1}=0, \tag{4.2}$$

$$\partial_3 u_1|_{z=1}=B_1 \qquad \partial_3 u_2|_{z=1}=B_2. \tag{4.3}$$

First we study the Stokes system with homogenous boundary conditions, i.e, we assume that the functions $\boldsymbol{\alpha}$, B_1, B_2 in (4.2), (4.3) are equal to zero. Denote this system by (4.1), $(4.2)_0$, $(4.3)_0$.

It is obvious that in the layer Π the Korn's inequality is valid. Therefore, the following theorem is true (see [18]).

Theorem 1. *Let* $\mathbf{M}\in L^2(\Pi)$. *Then there exists a unique weak solution* $\mathbf{u}\in W^{1,2}(\Pi)$ *of the problem* (4.1), $(4.2)_0$, $(4.3)_0$ *satisfying the equation* $\operatorname{div}\mathbf{u}=0$, *the boundary conditions* $(4.2)_0$ *and the integral identity*

$$-\frac{\nu}{2}\sum_{i,j=1}^{3}\int\limits_{\Pi}T_{ij}(\mathbf{u})T_{ij}(\boldsymbol{\xi})\,d\eta=\int\limits_{\Pi}\mathbf{M}\cdot\boldsymbol{\xi}\,d\eta \tag{4.4}$$

for all $\boldsymbol{\xi}\in W^{1,2}(\Pi)$ *such that* $\operatorname{div}\boldsymbol{\xi}=0$, $\boldsymbol{\xi}|_{z=0}=0$, $\xi_3|_{z=1}=0$. *There holds the estimate*

$$\|\nabla\mathbf{u};\ L^2(\Pi)\|\le c\,\|\mathbf{M};\ L^2(\Pi)\|. \tag{4.5}$$

Theorem 2. *Let* $\mathbf{M} \in L^2_{\beta+2}(\Pi) \subset L^2(\Pi)$, $\beta \in (-2,-1)$, *and let* $\mathbf{u}$ *be the corresponding weak solution of the problem* (4.1), $(4.2)_0$, $(4.3)_0$. *There exists a positive number* $\mu_* \le \beta+2$ *such that* $\nabla\mathbf{u} \in L^2_\mu(\Pi)$ *for all* $\mu \in (0,\mu_*]$ *and there holds the estimate*

$$\int\limits_{\Pi} (1+r^2)^\mu |\nabla\mathbf{u}|^2 d\eta \le c \int\limits_{\Pi} (1+r^2)^{\beta+2} |\mathbf{M}|^2 \, d\eta. \tag{4.6}$$

Proof. Let us define the functions

$$R_\mu(\eta,k) = \begin{cases} 5^\mu, & r \le 2, \\ (1+r^2)^\mu, & 2 \le r \le 2^k, \\ (1+2^{2k})^\mu, & r \ge 2^k \end{cases}$$

and

$$\boldsymbol{\xi}_0(\eta) = \begin{cases} R_\mu(\eta,k)\mathbf{u}(\eta), & \eta \in \omega_0, \\ R_\mu(\eta,k)\mathbf{u}(\eta) + \mathbf{W}_j(\eta), & \eta \in \omega_j, \ j=1,2,\dots,k-1, \\ R_\mu(\eta,k)\mathbf{u}(\eta), & \eta \in \Pi\backslash\Pi_k, \end{cases}$$

where $\omega_0 = \{x \in \Pi : |y| = r < 2\}$, $\omega_j = \{x \in \Pi : 2^j < r < 2^{j+1}\}$, $j = 1,2,\dots$, $\Pi_l = \{x \in \Pi : r < 2^j\}$, and $\mathbf{W}_j$ are the solutions of the problems

$$\begin{cases} \operatorname{div} \mathbf{W}_j = -\operatorname{div}\,(R_\mu(\eta,k)\mathbf{u}) = -\nabla R_\mu \cdot \mathbf{u}, & x \in \omega_j, \\ \mathbf{W}_j = 0, & x \in \partial\omega_j. \end{cases} \tag{4.7}$$

Since the flux of $\mathbf{u}$ is equal to zero, i.e., $\int_{\Sigma_R} \mathbf{u}\cdot\mathbf{n}\, dS = 0 \ \forall R > 2$, where $\Sigma_R = \{(y,z) \in \Pi \colon R = r\}$, we get

$$\int\limits_{\omega_j} \operatorname{div}\,(R_\mu(\eta,k)\mathbf{u})\, d\eta =$$

$$= (1+2^{2j+2})^\mu \int\limits_{\Sigma_{2^{j+1}}} \mathbf{u}\cdot\mathbf{n}\, dS - (1+2^{2j})^\mu \int\limits_{\Sigma_{2^j}} \mathbf{u}\cdot\mathbf{n}\, dS = 0.$$

Therefore (see [9]), there exist solutions $\mathbf{W}_j$ of (4.7) satisfying the estimates

$$\|\nabla\mathbf{W}_j;\ L^2(\omega_j)\| \le c\, 2^j \|\nabla R_\mu \cdot \mathbf{u};\ L^2(\omega_j)\| \tag{4.8}$$

with constant c independent of j. Since $(1+r^2)\sim 2^{2j}$ for $x\in\omega_j$ and

$$R_\mu(x,k)\le c(1+r^2),\ |\nabla R_\mu(x,k)|\le \\ \le c\mu(1+r^2)^{\mu-1/2},\ \forall x\in\Pi,\ k\ge 1, \quad (4.9)$$

this solution satisfies the inequalities

$$\|(1+r^2)^{-\mu/2}\nabla\mathbf{W}_j;\ L^2(\omega_j)\|\le c\,2^{-j\mu}\,\|\nabla\mathbf{W}_j;\ L^2(\omega_j)\|\le \\ \le c\,2^{-j\mu+j}\|\nabla R_\mu\cdot\mathbf{u};\ L^2(\omega_j)\|\le c\mu\,2^{-j\mu+j}\|(1+r^2)^{\mu-1/2}\mathbf{u};\ L^2(\omega_j)\|\le \\ \le c\mu\|(1+r^2)^{\mu/2}\mathbf{u};\ L^2(\omega_j)\|\le c\mu\|(1+r^2)^{\mu/2}\nabla\mathbf{u};\ L^2(\omega_j)\|. \quad (4.10).$$

Substituting $\boldsymbol{\xi}_0$ into the identity (4.4) we obtain

$$\frac{\nu}{2}\sum_{i,j=1}^{3}\int\limits_\Pi R_\mu T_{ij}(\mathbf{u})^2\,d\eta=\frac{\nu}{2}\sum_{i,j=1}^{3}\int\limits_\Pi T_{ij}(\mathbf{u})(u_j\partial_i R_\mu+u_i\partial_j R_\mu)\,d\eta+ \\ +\frac{\nu}{2}\sum_{l=1}^{k-1}\int\limits_{\omega_l}\sum_{i,j=1}^{3}T_{ij}(\mathbf{u})\cdot T_{ij}(\mathbf{W}_l)\,d\eta-\int\limits_\Pi \mathbf{M}\cdot R_\mu\mathbf{u}\,d\eta- \\ -\sum_{l=1}^{k-1}\int\limits_{\omega_l}\mathbf{M}\cdot\mathbf{W}_j\,d\eta. \quad (4.11)$$

Estimating the right-hand side of (4.11) we get

$$\left|\sum_{i,j=1}^{3}\int\limits_\Pi T_{ij}(\mathbf{u})(u_j\partial_i R_\mu+u_i\partial_j R_\mu)\,d\eta\right|\le c\mu\int\limits_\Pi R_\mu|\nabla\mathbf{u}|^2 d\eta, \quad (4.12)$$

$$\left|\sum_{l=1}^{k-1}\int\limits_{\omega_l}\sum_{i,j=1}^{3}T_{ij}(\mathbf{u})\cdot T_{ij}(\mathbf{W}_l)\,d\eta\right|\le c\mu\int\limits_\Pi R_\mu|\nabla\mathbf{u}|^2 d\eta, \quad (4.13)$$

$$\left|\int\limits_\Pi \mathbf{M}\cdot R_\mu\mathbf{u}\,d\eta\right|\le c\delta\int\limits_\Pi R_\mu|\nabla\mathbf{u}|^2\,d\eta+c_\delta\int\limits_\Pi(1+r^2)^{\beta+2}|\mathbf{M}|^2\,d\eta, \quad (4.14)$$

$$\left|\sum_{j=1}^{k-1}\int\limits_{w_j}\mathbf{M}\cdot\mathbf{W}_j\,d\eta\right|\le c\mu\int\limits_\Pi R_\mu|\nabla\mathbf{u}|^2\,d\eta+c\int\limits_\Pi(1+r^2)^{\beta+2}|\mathbf{M}|^2\,d\eta. \quad (4.15)$$

In (4.12)–(4.15) we have used (4.9)–(4.10), the Poincare inequality with respect to z and the Young unequally. From (4.11)–(4.15) it follows that

$$\sum_{i,j=1}^{3}\int\limits_{\Pi} R_\mu\, T_{ij}(\mathbf{u})^2\, d\eta \le c_1(\delta+\mu)\int\limits_{\Pi} R_\mu |\nabla\mathbf{u}|^2\, d\eta +$$

$$+c_\delta \int\limits_{\Pi}(1+r^2)^{\beta+2}|\mathbf{M}|^2\, d\eta. \tag{4.16}$$

In Π_k there holds the weighted Korn inequality

$$\int\limits_{\Pi_k} R_\mu |\nabla\mathbf{u}|^2\, d\eta \le c\sum_{i,j=1}^{3}\int\limits_{\Pi_k} R_\mu T_{ij}(\mathbf{u})^2\, d\eta.$$

Therefore, (4.16) implies

$$\int\limits_{\Pi} R_\mu |\nabla\mathbf{u}|^2\, d\eta \le c_2(\delta+\mu)\int\limits_{\Pi} R_\mu |\nabla\mathbf{u}|^2\, d\eta + c_\delta\int\limits_{\Pi}(1+r^2)^{\beta+2}|\mathbf{M}|^2\, d\eta.$$

If $\delta < \frac{1}{4c_2}, \mu_* \le \min(\beta+2,\ \frac{1}{2c_2})$, from the last inequality we derive that

$$\int\limits_{\Pi_k}(1+r^2)^{\mu}|\nabla\mathbf{u}|^2 d\eta \le c\int\limits_{\Pi} R_\mu |\nabla\mathbf{u}|^2 d\eta \le c\int\limits_{\Pi}(1+r^2)^{\beta+2}|\mathbf{M}|^2\, d\eta. \tag{4.17}$$

Passing in (4.17) to a limit as $k\to\infty$ we get

$$\int\limits_{\Pi}(1+r^2)^{\mu}|\nabla\mathbf{u}|^2 d\eta \le c\int\limits_{\Pi}(1+r^2)^{\beta+2}|\mathbf{M}|^2\, d\eta.$$

The theorem is proved.

In what follows we suppose that

$$\mu=\mu_*, \qquad \beta=\mu-2. \tag{4.18}$$

From the definition of the spaces $\mathcal{V}^l_{\beta,\,\kappa}(\Pi)$ we conclude that the weak solution $\mathbf{u}$ satisfies the inclusions

$$\mathbf{u}'\in\mathcal{V}^1_{\beta+1,\,1}(\Pi), \quad u_3\in\mathcal{V}^1_{\beta+2,\,0}(\Pi), \tag{4.19}$$

where $\mathbf{u}'=(u_1,\,u_2)$.

In order to estimate the pressure q we apply the following result proved in [6, 14].

Lemma 4. *Let* $\mathbf{M} \in L^2_{\beta+2}(\Pi)$, $-2 < \beta < -1$ *and let* $\mathbf{u}$ *be the weak solution of the problem* (4.1), $(4.2)_0$, (4.3_0) *satisfying the inclusions* (4.19). *Then there exists a function* $\widetilde{q} \in L^2_\beta(\Pi)$ *satisfying together with* $\mathbf{u}$ *the integral identity*

$$-\frac{\nu}{2}\sum_{i,j=1}^{3}\int_\Pi T_{ij}(\mathbf{u})T_{ij}(\boldsymbol{\xi})\,d\eta = \int_\Pi \widehat{q}\operatorname{div}\boldsymbol{\xi}d\eta + \int_\Pi \mathbf{M}\cdot\boldsymbol{\xi}d\eta \tag{4.20}$$

for every $\boldsymbol{\xi}$ *with* $\nabla\boldsymbol{\xi} \in L^2_{-\beta-1}(\Pi)$ *and* $\boldsymbol{\xi}|_{z=0} = 0$, $\xi_3|_{z=1} = 0$. *There holds the estimate*

$$\|\widehat{q}\,;\,L^2_\beta(\Pi)\| \le c\,\|\mathbf{M}\,;\,L^2_{\beta+2}(\Pi)\|. \tag{4.21}$$

Remark 1. Lemma 4 is also true under the weaker condition $\mathbf{M} \in L^2_{\beta+1}(\Pi)$, $-2 < \beta < -1$ (see [14]). However, inclusions (4.19) are proved only under the condition $\mathbf{M} \in L^2_{\beta+2}(\Pi)$.

Remark 2. As usual, the pressure q is defined by problem (4.1)–(4.3) up to an additive constant, i.e., $q = \widehat{q} + \mathrm{const}$. Since constants belong to the space $L^2_\beta(\Pi)$ with $-2 < \beta < -1$, we have $q \in L^2_\beta(\Pi)$.

Thus, we have proved the existence of the solution $(\mathbf{u}, q)$ of the problem (4.1), $(4.2)_0$, $(4.3)_0$ satisfying the conditions

$$\begin{aligned}&(\mathbf{u}, q) = (\mathbf{u}', u_3, q) \in \\ &\quad\in \mathcal{V}^1_{\beta+1,\,1}(\Pi) \times \mathcal{V}^1_{\beta+2,\,0}(\Pi) \times L^2_\beta(\Pi) \subset L^2_\beta(\Pi) \times L^2_\beta(\Pi) \times L^2_\beta(\Pi).\end{aligned}$$

Next, we shall investigate the regularity and decay properties of this solution. To do this let us introduce the domain $\mathcal{D}^l_\beta(\Pi)$ and the range $\mathcal{R}^l_\beta(\Pi;\partial\Pi)$ of the Stokes operator (4.1)–(4.3) in weighted function spaces. In the domain $\mathcal{D}^l_\beta(\Pi)$ of the Stokes operator we put all the pair $(\mathbf{u}, q)$ such that

$$\mathbf{u}' \in \mathcal{V}^{l+1}_{\beta+1,\,l}(\Pi), \quad u_3 \in \mathcal{V}^{l+1}_{\beta+2,\,l-1}(\Pi), \quad q \in \mathcal{V}^l_{\beta,\,l}(\Pi), \quad \partial_z q \in \mathcal{V}^{l-1}_{\beta+2,\,l-1}(\Pi).$$

The norm in $\mathcal{D}^l_\beta(\Pi)$ is given by the formula

$$\begin{aligned}\|\,(\mathbf{u}, q)\,;\mathcal{D}^l_\beta(\Pi)\,\| &= \|\,\mathbf{u}';\,\mathcal{V}^{l+1}_{\beta+1,\,l}(\Pi)\,\| + \|\,u_3;\,\mathcal{V}^{l+1}_{\beta+2,\,l-1}(\Pi)\,\| + \\ &+ \|\,q;\mathcal{V}^l_{\beta,\,l}(\Pi)\,\| + \|\,\partial_z q;\,\mathcal{V}^{l-1}_{\beta+2,\,l-1}(\Pi)\,\|.\end{aligned}$$

The range $\mathcal{R}^l_\beta(\Pi;\partial\Pi)$ consists of vector functions $(\mathbf{M}, \boldsymbol{\alpha}, B_1, B_2)$ such that $\mathbf{M}$ admits the representation

$$\mathbf{M} = \mathbf{M}_0 + \partial_z\mathbf{M}_1 + \nabla\Phi \tag{4.22}$$

with

$$\mathbf{M}_0 \in \mathcal{V}^{l-1}_{\beta+2,\,l-1}(\Pi), \quad \mathbf{M}'_1 \in \mathcal{V}^{l}_{\beta+1,\,l}(\Pi), \quad M_{13} \in \mathcal{V}^{l}_{\beta+2,\,l-1}(\Pi),$$

$$\Phi \in \mathcal{V}^{l}_{\beta,\,l}(\Pi), \quad \partial_z \Phi \in \mathcal{V}^{l-1}_{\beta+2,\,l-1}(\Pi), \tag{4.23}$$

$$\boldsymbol{\alpha}' \in \mathcal{V}^{l+1/2}_{\beta+1,\,l}(\partial\Pi), \quad \alpha_3 \in \mathcal{V}^{l+1/2}_{\beta+2,\,l-1}(\partial\Pi), \qquad B_1, B_2 \in \mathcal{V}^{l-1/2}_{\beta+1,\,l}(\partial\Pi).$$

The norm in $\mathcal{R}^l_\beta(\Pi;\partial\Pi)$ is defined by the formula

$$\|(\mathbf{M},\boldsymbol{\alpha},B_1,B_2);\mathcal{R}^l_\beta(\Pi;\partial\Pi)\| =$$

$$= \inf\Big\{ \|\mathbf{M}_0;\mathcal{V}^{l-1}_{\beta+2,\,l-1}(\Pi)\| + \|\mathbf{M}'_1;\mathcal{V}^{l}_{\beta+1,\,l}(\Pi)\| +$$

$$+ \|M_{13};\mathcal{V}^{l}_{\beta+2,\,l-1}(\Pi)\| + \|\eta;\mathcal{V}^{l}_{\beta,\,l}(\Pi)\| + \|\partial_z\eta;\mathcal{V}^{l-1}_{\beta+2,\,l-1}(\Pi)\|\Big\}$$

$$+ \|\boldsymbol{\alpha}';\mathcal{V}^{l+1/2}_{\beta+1,\,l}(\partial\Pi)\| + \|\alpha_3;\mathcal{V}^{l+1/2}_{\beta+2,\,l-1}(\partial\Pi)\| + \sum_{i=1}^{2}\|B_i;\mathcal{V}^{l-1/2}_{\beta+1,\,l}(\Pi)\|$$

where the infimum is taken over all possible representations of $\mathbf{M}$ in the form (4.22).

Theorem 3. *Let $(\mathbf{M},\mathbf{0},0,0) \in \mathcal{R}^l_{\beta+2}(\Pi;\partial\Pi)$, $l \ge 1$, $\beta \in (-2,-1)$. Then there exists a unique solution $(\mathbf{u},q)$ of problem (4.1), $(4.2)_0$, $(4.3)_0$ admitting the asymptotic representation*

$$\begin{pmatrix} \mathbf{u}(y,z) \\ q(y) \end{pmatrix} = \chi(r)\begin{pmatrix} \mathbf{U}_{-1}(y,z) \\ P_{-1}(y) \end{pmatrix} + \begin{pmatrix} \widetilde{\mathbf{u}}(y,z) \\ \widetilde{q}(y) \end{pmatrix}, \tag{4.24}$$

where χ is a smooth cut-off function, $\chi(r)=1$ for $r\ge 2$ and $\chi(r)=0$ for $r\le 1$,

$$\mathbf{U}'_{-1}(y,z) = \frac{1}{2\nu}z\Big(\frac{z}{2}-1\Big)\nabla'_y P_{-1}(y), \quad U_{-1,3}(y,z)\equiv 0,$$

$$P_{-1}(y) = c^+_{-1}r^{-1}\cos(\varphi) + c^-_{-1}r^{-1}\sin(\varphi) \tag{4.25}$$

$(\widetilde{\mathbf{u}},\widetilde{q}) \in \mathcal{D}^l_{\beta+2}(\Pi)$ and the constants $c^\pm_{-1}$ are linear bounded functionals on the data of the problem. Moreover, there holds the estimate

$$\|(\widetilde{\mathbf{u}},\widetilde{q});\mathcal{D}^l_{\beta+2}(\Pi)\| + (|c^+_{-1}|+|c^-_{-1}|) \le$$
$$\le c\,\|(\mathbf{M},\mathbf{0},0,0);\mathcal{R}^l_{\beta+2}(\Pi;\partial\Pi)\|. \tag{4.26}$$

Proof. From the condition $(\mathbf{M}, \mathbf{0}, 0, 0) \in \mathcal{R}^l_{\beta+2}(\Pi; \partial\Pi)$ it follows that $\mathbf{M} \in L^2_{\beta+2}(\Pi)$. Therefore, by Theorems 1 and 2, there exists a weak solution $(\mathbf{u}, q)$ of problem (4.1), $(4.2)_0$, $(4.3)_0$ satisfying the inclusion $(\mathbf{u}, q) \in L^2_\beta(\Pi) \times L^2_\beta(\Pi)$, by Theorem 3.1 from [7], we conclude that $(\mathbf{u}, q) \in \mathcal{D}^l_\beta(\Pi)$. Moreover, since $(\mathbf{M}, \mathbf{0}, 0, 0) \in \mathcal{R}^l_{\beta+2}(\Pi; \partial\Pi)$, by Theorem 4.1 [7], the solution $(\mathbf{u}, q)$ admits the asymptotic representation

$$\begin{pmatrix} \mathbf{u}(y,z) \\ q(y,z) \end{pmatrix} = \chi(r) \begin{pmatrix} c_0^+ \mathbf{U}_0^+(y,z) + c_0^- \mathbf{U}_0^-(y,z) \\ c_0^+ P_0^+(y) + c_0^- P_0^-(y) \end{pmatrix} +$$

$$+\chi(r) \begin{pmatrix} \mathbf{U}_{-1}(y,z) \\ P_{-1}(y) \end{pmatrix} + \begin{pmatrix} \widetilde{\mathbf{u}}(y,z) \\ \widetilde{q}(y,z) \end{pmatrix},$$

where

$$\mathbf{U}_0^{\pm\prime}(y,z) = \frac{1}{2\nu} z \Big(\frac{z}{2} - 1\Big) \nabla_y{}' P_0^\pm(y), \quad U_{03}^\pm(y,z) = 0,$$

$$P_0^+(y) = 1, \qquad P_0^-(y) = -(1/2\pi)\ln r,$$

and $(\widetilde{\mathbf{u}}, \widetilde{q}) \in \mathcal{D}^l_{\beta+2}(\Pi)$. There holds the estimate

$$\|(\widetilde{\mathbf{u}}, \widetilde{q}); \mathcal{D}^l_{\beta+2}(\Pi)\| + (|c_{-1}^+| + |c_{-1}-| + |c_0^-|) \le$$

$$\le c\Big(\|(\mathbf{M}, \mathbf{0}, 0, 0); \mathcal{R}^l_\beta(\Pi; \partial\Pi)\| +$$

$$+\|\mathbf{u}; L^2_\beta(\Pi)\| + \|q; L^2_\beta(\Pi)\| + |c_0^+|\Big). \tag{4.27}$$

Since the constant c_0^- is proportional to the flux of $\mathbf{u}$ over cylindrical surfaces $\{\eta \in \Pi\colon |y| = t,\ t > 1\}$ (see [7, 8]) we obtain $c_0^- = 0$. The constant c_0^+ could be prescribed arbitrary. Normalizing the pressure function q by the condition

$$\lim_{r\to\infty} q(y,z) = 0,$$

we get $c_0^+ = 0$. Thus, we have proved the asymptotic representation formula (4.24). Estimate (4.26) follows now from (4.27), (4.6), (4.18), and (4.21). The theorem is proved.

Let us consider now the general case of nonhomogeneous boundary conditions (4.2) and (4.3). First, we prove the following extension result.

Lemma 5. *Let* $\boldsymbol{\alpha} \in W^{l+1/2,2}(\mathbb{R}^2)$, $\operatorname{supp} \boldsymbol{\alpha} \subset \{y \in \mathbb{R}^2 : |y| < 1\}$,

$$\int\limits_{\mathbb{R}^2} \alpha_3(y)\, dy = 0, \tag{4.28}$$

$B_1,\ B_2 \in \mathcal{V}^{l-1/2}_{\beta+3,l}(\mathbb{R}^2)$. *There exists a solenoidal vector function* $\mathbf{w} = (\mathbf{w}',\ w_3) \in \mathcal{V}^{l+1}_{\beta+3,l}(\Pi) \times \mathcal{V}^{l+1}_{\beta+4,l-1}(\Pi)$ *satisfying the boundary conditions* (4.2) *and* (4.3), *and the estimate*

$$\|(\mathbf{w}',\ w_3);\ \mathcal{V}^{l+1}_{\beta+3,l}(\Pi) \times \mathcal{V}^{l+1}_{\beta+4,l-1}(\Omega)\| \le c\|(\mathbf{0}, \boldsymbol{\alpha}, B_1, B_2)\ \mathcal{R}^{\,l}_{\beta+2}(\Pi; \partial\Pi)\| \le$$

$$\le c\big(\|\boldsymbol{\alpha};\ W^{l+1/2,2}(\mathbb{R}^2)\| + \|B_1;\ \mathcal{V}^{l-1/2}_{\beta+3,l}(\mathbb{R}^2)\| + \|B_2;\ \mathcal{V}^{l-1/2}_{\beta+3,l}(\mathbb{R}^2)\|\big). \tag{4.29}$$

Proof. Since $\boldsymbol{\alpha}$ has zero flux over S_0 (see (4.28)) and $\operatorname{supp} \boldsymbol{\alpha} \subset \{y \in \mathbb{R}^2\colon |y| < 1\}$, there exists (see [2]) an extension $\mathbf{w}^{(1)}$ such that

$$\operatorname{div}\ \mathbf{w}^{(1)} = 0, \quad \mathbf{w}^{(1)}|_{z=0} = \boldsymbol{\alpha},$$

$$\operatorname{supp} \mathbf{w}^{(1)} \subset \{(y, z) \in \Pi : |y| < 2, 0 \le z < 1/2\}$$

and

$$\|\mathbf{w}^{(1)};\ W^{l+1,2}(\mathbb{R}^2)\| \le c\|\boldsymbol{\alpha};\ W^{l+1/2,2}(\mathbb{R}^2)\|.$$

Let us construct the function $\mathbf{w}^{(2)}$ such that $\operatorname{div}\ \mathbf{w}^{(2)} = 0$ and

$$\mathbf{w}^{(2)}|_{z=0} = 0,\ \ w_3^{(2)}|_{z=1} = 0,\ \ \partial_3 w_1^{(2)}|_{z=1} = B_1,\ \ \partial_3 w_2^{(2)}|_{z=1} = B_2 \tag{4.30}$$

Define a vector function $\boldsymbol{\omega}$ with the components

$$\omega_1(y, z) =$$
$$= \zeta(z)\frac{(1-z)^2}{2} \int\limits_{|\tau|<1} B_2(y_1 + \tau_1(1-z), y_2 + \tau_2(1-z))\, K(\tau_1, \tau_2)\, d\tau_1 d\tau_2,$$

$$\omega_2(y, z) =$$
$$= -\zeta(z)\frac{(1-z)^2}{2} \int\limits_{|\tau|<1} B_1(y_1 + \tau_1(1-z), y_2 + \tau_2(1-z))\, K(\tau_1, \tau_2)\, d\tau_1 d\tau_2,$$

$$\omega_3(y, z) = 0,$$

where ζ is a smooth function with $\zeta(1) = 1$, $\zeta'(1) = 0$, $\zeta''(1) = 0$, and $\zeta(z) = 0$ for $z < 1/2$ and K is a smooth kernel satisfying the conditions: $\operatorname{supp} K \subset \{\ \tau = (\tau_1, \tau_2)\colon |\tau| \le 1\}$, $\int\limits_{|\tau|\le 1} K(\tau)d\tau = 1$. By direct computations it is easy to verify that

$$\boldsymbol{\omega}|_{z=1} = 0, \quad \partial_3^2\omega_1|_{z=1} = B_2, \quad \partial_3^2\omega_1|_{z=1} = -B_1.$$

Moreover, taking into account the definition of the weighted norms, by standard arguments (e.g., [1]) we get the estimate

$$\|\boldsymbol{\omega};\ \mathcal{V}^{l+2}_{\beta+3,l}(\Pi)\| \le c\big(\|B_1;\ \mathcal{V}^{l-1/2}_{\beta+3,l}(\mathbb{R}^2)\| + \|B_2;\ \mathcal{V}^{l-1/2}_{\beta+3,l}(\mathbb{R}^2)\|\big).$$

Let us put $\mathbf{w}^{(2)} = \operatorname{curl}\boldsymbol{\omega}$, i.e., $w_1^{(2)} = -\partial_3\omega_2$, $w_2^{(2)} = \partial_3\omega_1$, $w_1^{(3)} = \partial_1\omega_2 - \partial_2\omega_1$. Then $\operatorname{div}\mathbf{w}^{(2)} = 0$, $\mathbf{w}^{(2)}$ satisfies the boundary conditions (4.30) and $\mathbf{w}^{(2)'} \in \mathcal{V}^{l+1}_{\beta+3,l}(\Pi)$, $w_3^{(2)} \in \mathcal{V}^{l+1}_{\beta+4,l-1}(\Pi)$,

$$\|\mathbf{w}^{(2)'};\ \mathcal{V}^{l+1}_{\beta+3,l}(\Pi)\| + \|w_3^{(2)};\ \mathcal{V}^{l+1}_{\beta+4,l-1}(\Pi)\| \le$$
$$\le c\big(\|B_1;\ \mathcal{V}^{l-1/2}_{\beta+3,l}(\mathbb{R}^2)\| + \|B_2;\ \mathcal{V}^{l-1/2}_{\beta+3,l}(\mathbb{R}^2)\|\big).$$

Obviously, the function $\mathbf{w} = \mathbf{w}^{(1)} + \mathbf{w}^{(2)}$ is solenoidal and satisfies the boundary conditions (4.2) and (4.3). Estimate (4.29) for $\mathbf{w}$ follows from estimates for $\mathbf{w}^{(1)}$ and $\mathbf{w}^{(2)}$ and from the fact that for functions with a compact support the norm $\|\mathbf{w}^{(1)};\ W^{l+1,2}(\Pi)\|$ is equivalent to the norm $\|(\mathbf{w}^{(1)'},\ w_3^{(1)});\ \mathcal{V}^{l+1}_{\beta+3,l}(\Pi)\times\mathcal{V}^{l+1}_{\beta+4,l-1}(\Omega)\|$ and $\|\boldsymbol{\alpha};\ W^{l+1/2,2}(\mathbb{R}^2)\|$ is equivalent to $\|(\mathbf{0},\boldsymbol{\alpha},0,0)\,\mathcal{R}^{\,l}_{\beta+2}(\Pi;\partial\Pi)\|$. The lemma is proved.

From Theorem 3 and Lemma 5 we derive the main result of this section.

Theorem 4. *Let* $(\mathbf{M},\boldsymbol{\alpha},B_1,B_2) \in \mathcal{R}^{\,l}_{\beta+2}(\Pi;\partial\Pi)$, $l \ge 1$, $\beta \in (-2,-1)$, *and* $\boldsymbol{\alpha}$ *satisfies the conditions of Lemma 5. Then there exists a unique solution* $(\mathbf{u}, q)$ *of the problem* (4.1), (4.2), (4.3) *admitting the asymptotic representation* (4.24)–(4.25) *and satisfying the estimate*

$$\|\,(\widetilde{\mathbf{u}},\widetilde{q});\ \mathcal{D}^{\,l}_{\beta+2}(\Pi)\,\| + (|c^+_{-1}| + |c^-_{-1}|) \le$$
$$\le c\,\|\,(\mathbf{M},\boldsymbol{\alpha},B_1,B_2);\ \mathcal{R}^{\,l}_{\beta+2}(\Pi;\partial\Pi)\,\|. \tag{4.31}$$

Remark 3. Condition (4.28) is essential for our considerations. Only under this condition the flux of $\mathbf{u}$ is equal to zero and, therefore, $c_0^- = 0$. Otherwise, we have (see [8])

$$c_0^- = -6\nu\int\limits_{\partial\Pi}\boldsymbol{\alpha}\cdot\mathbf{n}ds.$$

5. Solvability of problem (3.12), (3.13)

Let us consider in $\mathbb{R}^2$ the boundary value problem

$$\sigma \triangle \Psi - k\Psi = R, \tag{5.1}$$

$$\lim_{\sqrt{y_1^2+y_2^2}\to\infty} \Psi(y_1, y_2) = 0, \tag{5.2}$$

where $k = 2g$. Obviously, for every $R \in L^2(\mathbb{R}^2)$ problem (5.1), (5.2) has a unique weak solution $\Psi \in W^{1,2}(\mathbb{R}^2)$ satisfying the integral identity

$$\sigma \int_{\mathbb{R}^2} \nabla\Psi\nabla\xi dy + k\int_{\mathbb{R}^2} \Psi\xi dy = -\int_{\mathbb{R}^2} R\xi dy \quad \forall \xi \in W^{1,2}(\mathbb{R}^2).$$

Lemma 6. *Let $R \in L^2_\gamma(\mathbb{R}^2), \gamma > 0$. Then the weak solution Ψ of (5.1), (5.2) belongs to the space $\mathcal{V}^1_{\gamma,0}(\mathbb{R}^2)$ and*

$$||\Psi;\ \mathcal{V}^1_{\gamma,0}(\mathbb{R}^2)|| \le c||R;\ L^2_\gamma(\mathbb{R}^2)||. \tag{5.3}$$

Proof. Since $\gamma > 0$, it is enough to prove inequality (5.3) for the weak solution Ψ. For every positive ρ and δ we define the function

$$S_\gamma(y,\delta,\rho) = \begin{cases} (1+\delta r^2)^\gamma, & r < \rho \\ (1+\delta\rho^2)^\gamma, & r \ge \rho \end{cases}$$

Let us multiply the equation (5.1) by $S_\gamma\Psi$ and integrate by parts in $\mathbb{R}^2$:

$$\sigma\int_{\mathbb{R}^2} |\nabla\Psi|^2 S_\gamma dy + k\int_{\mathbb{R}^2} \Psi^2 S_\gamma dy =$$

$$= -2\sigma\gamma \int_{\mathbb{R}^2\cap B_\rho} \nabla\Psi\cdot(1+\delta r^2)^{\gamma-1}\Psi\delta\, r\,\nabla r dy - \int_{\mathbb{R}^2} R\Psi S_\gamma dy \tag{5.4}$$

where $B_\rho = \{y \in \mathbb{R}^2 :\ |y| < \rho\}$. Estimating the right-hand side of (5.4) we get

$$\sigma\int_{\mathbb{R}^2} |\nabla\Psi|^2 S_\gamma dy + k\int_{\mathbb{R}^2} \Psi^2 S_\gamma dy \le \frac{k}{2}\int_{\mathbb{R}^2} \Psi^2 S_\gamma dy + \frac{1}{2k}\int_{\mathbb{R}^2} R^2 S_\gamma dy +$$

$$+\sigma\gamma\sqrt{\delta}\Big(\int_{\mathbb{R}^2\cap B_\rho} |\nabla\Psi|^2(1+\delta r^2)^\gamma dy + \int_{\mathbb{R}^2\cap B_\rho} \Psi^2(1+\delta r^2)^{\gamma-1} dy\Big) \le$$

$$\le \Big(\frac{k}{2}+\sigma\gamma\sqrt{\delta}\Big)\int\limits_{\mathbb{R}^2}\Psi^2 S_\gamma\, dy+\sigma\gamma\sqrt{\delta}\int\limits_{\mathbb{R}^2}|\nabla\Psi|^2 S_\gamma\, dy+$$

$$+\frac{1}{2k}\int\limits_{\mathbb{R}^2}R^2(1+\delta r^2)^\gamma\, dy. \tag{5.5}$$

Hence, if $\delta=\min(k^2/16\sigma^2\gamma^2,\ 1/4\gamma^2)$, from (5.5) follows the inequality

$$\frac{\sigma}{2}\int\limits_{\mathbb{R}^2}|\nabla\Psi|^2 S_\gamma\, dy+\frac{k}{4}\int\limits_{\mathbb{R}^2}\Psi^2 S_\gamma\, dy\le c\int\limits_{\mathbb{R}^2}R^2(1+r^2)^\gamma\, dy.$$

Passing in the last inequality to a limit as $\rho\to\infty$ we derive (5.3). The lemma is proved.

Lemma 7. *Let $R\in\mathcal{V}^{1/2}_{\gamma,0}(\mathbb{R}^2),\gamma>0$ and let Ψ be the weak solution of the problem* (5.1), (5.2). *Then $\Psi\in\mathcal{V}^{5/2}_{\gamma,0}(\mathbb{R}^2)$ and there holds the inequality*

$$\|\,\Psi;\,\mathcal{V}^{5/2}_{\gamma,0}(\mathbb{R}^2)\,\|\le c(\,\|\,R;\,\mathcal{V}^{1/2}_{\gamma,0}(\mathbb{R}^2)\,\|+\|\,\Psi;\,L^2_\gamma(\mathbb{R}^2)\,\|\,) \tag{5.6}$$

Proof. Let us consider Ψ as a weak solution of the Poisson problem

$$\sigma\triangle\Psi=k\Psi+R=\widetilde{R},\quad y\in\mathbb{R}^2. \tag{5.7}$$

We introduce the covering of the layer Π by the infinite union of the quadrats $Q_{s,l}=\{y\in\mathbb{R}^2:\ |y_1-s|<1/2, |y_2-l|<1/2\}$, i.e., $\mathbb{R}^2=\bigcup\limits_{s,l\in\mathbb{Z}}Q_{s,l}$. Let $\widehat{Q}_{s,l}$ be the union of the quadrat $Q_{s,l}$ with its eight neighbours, i.e.,

$$\widehat{Q}_{s,l}=\bigcup_{i=l-1}^{l+1}\ \bigcup_{j=s-1}^{s+1}Q_{j,i}\supset Q_{s,l}.$$

By local regularity results for problem (5.7) there hold the estimates

$$\|\,\Psi;\,W^{5/2,2}(Q_{s,l})\,\|^2\le$$

$$\le c\Big(\|\,R;\,W^{1/2,2}(\widehat{Q}_{s,l})\,\|^2+\|\,\Psi;\,L^2(\widehat{Q}_{s,l})\,\|^2\Big), \tag{5.8}$$

where the constant c is independent of s and l. Multiplying (5.8) by $(s^2+l^2)^\gamma$, taking into account that $|y|=r\sim(s^2+l^2)^{1/2}$ in $\widehat{Q}_{s,l}$, and summing the obtained inequalities over $s,\,l\in\mathbb{Z}$, we derive the estimate

$$\|\,(1+r^2)^{\frac{\gamma}{2}}\Psi;\,W^{5/2,2}(\mathbb{R}^2)\,\|^2\le$$

$$\leq c(\|\,(1+r^2)^{\frac{\gamma}{2}}R;\,W^{1/2,\,2}(\mathbb{R}^2)\,\|^2 + \|\,(1+r^2)^{\frac{\gamma}{2}}\Psi;\,L^2(\mathbb{R}^2)\,\|^2)$$

which is equivalent to (5.6). The lemma is proved.

Theorem 5. *Let $R \in \mathcal{V}^{l-1/2}_{\beta+2,l-1}(\mathbb{R}^2), \beta \in (-2,-1)$. Then the problem (5.1), (5.2) has a unique solution $\Psi \in \mathcal{V}^{l+3/2}_{\beta+2,l-1}(\mathbb{R}^2)$ and there holds the estimate*

$$\|\,\Psi;\,\mathcal{V}^{l+3/2}_{\beta+2,l-1}(\mathbb{R}^2)\,\| \leq c\,\|\,R;\,\mathcal{V}^{l-1/2}_{\beta+2,l-1}(\mathbb{R}^2)\,\|. \tag{5.9}$$

Moreover, if the right-hand side R admits a representation

$$R(y) = \chi(r)P_{-1}(y) + R_1(y), \tag{5.10}$$

where χ is a smooth cut-off function, $\chi(r) = 1$ for $r \geq 2$ and $\chi(r) = 0$ for $r \leq 1$, $P_{-1}(y) = c^+_{-1}r^{-1}\cos(\varphi) + c^-_{-1}r^{-1}\sin(\varphi)$, $R_1 \in \mathcal{V}^{l-1/2}_{\beta+2,l-1}(\mathbb{R}^2)$, $\beta \in (-2,-1)$, then Ψ could be represented as a sum

$$\Psi(y) = -\frac{1}{k}\chi(r)P_{-1}(y) + \widetilde{\Psi}(y) \tag{5.11}$$

with $\widetilde{\Psi}$ satisfying the estimate

$$\|\,\widetilde{\Psi};\,\mathcal{V}^{l+3/2}_{\beta+2,l-1}(\mathbb{R}^2)\,\| \leq c\left(\|\,R_1;\,\mathcal{V}^{l-1/2}_{\beta+2,l-1}(\mathbb{R}^2)\,\| + |c^+_{-1}| + |c^-_{-1}|\right). \tag{5.12}$$

Proof. Since $R \in \mathcal{V}^{l-1/2}_{\beta+2,l-1}(\mathbb{R}^2) \subset L^2(\mathbb{R}^2)$ for $\beta \in (-2,-1)$ it is enough to prove estimates (5.9) and (5.11) for the weak solution of problem (5.1), (5.2). It is well known that the week solution is locally smooth, i.e., $\Psi \in W^{l+3/2,2}_{\text{loc}}(\mathbb{R}^2)$. Therefore, we may differentiate equation (5.1) with respect to y up to the order $|\alpha| \leq l-1$:

$$\sigma\triangle D^\alpha\Psi - kD^\alpha\Psi = D^\alpha R, \qquad |\alpha| \leq l-1.$$

Denote $D^\alpha\Psi = \Psi_\alpha$, $D^\alpha R = R_\alpha$. By the definition of the norm in the space $\mathcal{V}^{l-1/2}_{\beta+2,l-1}(\mathbb{R}^2)$, we have $R_\alpha \in L^2_{\beta+2+|\alpha|}(\mathbb{R}^2)$, $|\alpha| \leq l-1$. Therefore, applying to Ψ_α Lemma 6 at $\gamma = \beta+2, \dots, \gamma = \beta+2+|\alpha|, \dots, \gamma = \beta+2+(l-1)$, we obtain $\Psi_\alpha \in L^2_{\beta+2+|\alpha|}(\mathbb{R}^2)$, $|\alpha| \leq l-1$, $\nabla\Psi_\alpha \in L^2_{\beta+2+(l-1)}(\mathbb{R}^2)$, $|\alpha| = l-1$, and there holds the estimate

$$\sum_{|\alpha|\leq l-1}\int_{\mathbb{R}^2} |D^\alpha\Psi|^2(1+r^2)^{\beta+2+|\alpha|}\,dy +$$

$$+\sum_{|\alpha|=l-1}\int_{\mathbb{R}^2}|\nabla D^\alpha\Psi|^2(1+r^2)^{\beta+2+(l-1)}\,dy\le$$

$$\le c\sum_{|\alpha|\le l-1}\int_{\mathbb{R}^2}|D^\alpha R|^2(1+r^2)^{\beta+2+|\alpha|}\,dy. \tag{5.13}$$

Further, since $R_\alpha\in\mathcal{V}^{1/2}_{\beta+2+(l-1),0}(\mathbb{R}^2)$, $|\alpha|=l-1$, by Lemma 7 at $\gamma=\beta+2+(l-1)$, we conclude that $\Psi_\alpha\in\mathcal{V}^{5/2}_{\beta+2+(l-1),0}(\mathbb{R}^2)$, $|\alpha|=l-1$, and

$$\|\Psi_\alpha;\ \mathcal{V}^{5/2}_{\beta+2+(l-1),0}(\mathbb{R}^2)\|\le c\|R_\alpha;\ \mathcal{V}^{1/2}_{\beta+2+(l-1),0}(\mathbb{R}^2)\|,\quad |\alpha|=l-1. \tag{5.14}$$

Thus, $\Psi\in\mathcal{V}^{l+3/2}_{\beta+2,l-1}(\mathbb{R}^2)$ and estimate (5.9) follows from (5.13) and (5.14).

Let us assume now that the right-hand side R admits the representation (5.10). We look for the solution Ψ in form (5.11). Substituting sum (5.11) into Eq. (5.1), we get

$$\sigma\triangle\widetilde{\Psi}-k\widetilde{\Psi}=R_1-\sigma\triangle(\chi(r)P_{-1}(y)).$$

Since $P_{-1}(y)$ is a harmonic function in $\mathbb{R}^2$, the function $\triangle(\chi(r)P_{-1}(y))$ has a compact support and it is easy to verify that $\triangle(\chi(r)P_{-1}(y))\in\mathcal{V}^{l-1/2}_{\beta+2,l-1}(\mathbb{R}^2)$ and

$$\|\triangle(\chi(r)P_{-1}(y));\ \mathcal{V}^{l-1/2}_{\beta+2,l-1}(\mathbb{R}^2)\,\|\le c(|c^+_{-1}|+|c^-_{-1}|). \tag{5.15}$$

Therefore, $\widetilde{\Psi}\in\mathcal{V}^{l+3/2}_{\beta+2,l-1}(\mathbb{R}^2)$ and from (5.15) and (5.9), follows estimate (5.12). The theorem is proved.

6. Estimates of the nonlinear terms

Let us estimate the right-hand sides $\mathbf{M}$, B_1, B_2, and $\widetilde{R}$ in (3.8)–(3.13). Assume that the functions $\mathbf{u}$, q, and Ψ admit the representations (4.24) and (5.11), i.e.,

$$\mathbf{u}(\eta)=\chi(r)\mathbf{U}_{-1}(y,z)+\widetilde{\mathbf{u}}(\eta),\quad q(\eta)=\chi(r)P_{-1}(y)+\widetilde{q}(y,z),$$

$$\Psi(y)=g\chi(r)P_{-1}(y)+\widetilde{\Psi}(y), \tag{6.1}$$

where $\mathbf{U}_{-1}$ and P_{-1} are defined by (4.25) with some given constants c^+_{-1} and c^-_{-1}, and $(\widetilde{\mathbf{u}},\widetilde{q})\in\mathcal{D}^l_{\beta+2}(\Pi)$, $\widetilde{\Psi}\in\mathcal{V}^{l+3/2}_{\beta+2,l-1}(\mathbb{R}^2)$, $\beta\in(-2,-1)$. Let

$$\|\widetilde{\mathbf{u}},\widetilde{q}),\ \mathcal{D}^l_{\beta+2}(\Pi)\|+\|\widetilde{\Psi};\ \mathcal{V}^{l+3/2}_{\beta+2,l-1}(\mathbb{R}^2)\|+(|c^+_{-1}|+|c^-_{-1}|)\le A. \tag{6.2}$$

Substituting the expression of Ψ into formula (3.1) for the function w we obtain

$$w(y,z) = g\,z \int\limits_{|\tau|\le 1} K(\tau)\chi(y+\tau(z-1))P_{-1}(y+\tau(z-1))d\tau +$$

$$+z \int\limits_{|\tau|\le 1} K(\tau)\widetilde{\Psi}(y+\tau(z-1))d\tau \equiv I_1(y,z) + \widetilde{w}(y,z). \qquad (6.3)$$

By direct computation we get

$$|\partial_z^\alpha \partial_y^\gamma I_1(y,z)| \le c\,\frac{(|c_{-1}^+| + |c_{-1}^-|)}{(1+r^2)^{(1+|\gamma|)/2}}. \qquad (6.4)$$

Moreover, the function $\widetilde{w}$ satisfies the estimate (3.3) with $\widetilde{\Psi}$ instead of Ψ and $\beta+2$ instead of β.

Let us start with the estimate of M_1 defined by (3.14). Since

$$\begin{aligned}
&a_{11} = 1, \qquad a_{12} = 0, \qquad a_{13} = \frac{\partial z}{\partial x_1} = -\varepsilon\frac{\partial_1 w}{1+\varepsilon\partial_3 w},\\
&a_{21} = 0, \qquad a_{22} = 1, \qquad a_{23} = \frac{\partial z}{\partial x_2} = -\varepsilon\frac{\partial_2 w}{1+\varepsilon\partial_3 w},\\
&a_{31} = 0, \qquad a_{32} = 0, \qquad a_{33} = \frac{\partial z}{\partial x_3} = \frac{1}{1+\varepsilon\partial_3 w},
\end{aligned}$$

from (3.7) it follows that

$$(\widehat{\mathrm{V}}\cdot\widehat{\nabla}_\eta)\widehat{\mathrm{V}}_1 = \widehat{\mathrm{V}}_1\frac{\partial\widehat{\mathrm{V}}_1}{\partial y_1} + \widehat{\mathrm{V}}_2\frac{\partial\widehat{\mathrm{V}}_2}{\partial y_2} + \frac{u_3}{1+\varepsilon\partial_3 w}\frac{\partial\widehat{\mathrm{V}}_1}{\partial z} \qquad (6.5)$$

Note that in virtue of Lemmas 1, 2, and 3, the conditions $\widetilde{u}_i \in \mathcal{V}^{l+1}_{\beta+3,\,l}(\Pi)$, $i = 1,2$, $\widetilde{u}_3 \in \mathcal{V}^{l+1}_{\beta+4,\,l-1}(\Pi)$ imply the inclusions

$$\frac{\widetilde{u}_1}{1+\varepsilon\partial_3 w} \in \mathcal{V}^{l+1}_{\beta+3,\,l}(\Pi),\ i = 1,2,\ \frac{\widetilde{u}_3}{1+\varepsilon\partial_3 w} \in \mathcal{V}^{l+1}_{\beta+4,\,l-1}(\Pi).$$

Here, we have used the fact that by construction $\partial_z^2 w$ has better decay properties than w, i.e., $\partial_z^2 w \in \mathcal{V}^{l}_{\beta+3,\,l-1}\Pi$ (see Lemma 3 with β changed to $\beta+2$). Moreover, by the definition of the norms, it follows that $\frac{1}{r^\alpha}v \in \mathcal{V}^l_{\mu+\alpha,\,\kappa}(\Pi)$ for any function $v \in \mathcal{V}^l_{\mu,\,\kappa}(\Pi)$. Therefore, using the representation formulas (6.1) and (6.3) for $\mathbf{u}$ and w we derive $(\widehat{\mathrm{V}}\cdot\widehat{\nabla}_\eta)\widehat{\mathrm{V}}_1 \in \mathcal{V}^{l-1}_{2\beta+7,\,l-1}(\Pi) \subset \mathcal{V}^{l-1}_{\beta+4,\,l-1}(\Pi)$,

$$\|(\widehat{\mathrm{V}}\cdot\widehat{\nabla}_\eta)\widehat{\mathrm{V}}_1;\, \mathcal{V}^{l-1}_{\beta+4,\,l-1}(\Pi)\| \le c(A)\Big[|\mathbf{c}_{-1}|^2+$$

$$+|\mathbf{c}_{-1}|\Big(\sum_{i=1}^{2}\|\widetilde{u}_i;\, \mathcal{V}^{l+1}_{\beta+3,\,l}(\Pi)\| + \|\widetilde{u}_3;\, \mathcal{V}^{l+1}_{\beta+4,\,l-1}(\Pi)\|\Big)+$$

$$+\sum_{i=1}^{2}\|\widetilde{u}_i;\, \mathcal{V}^{l+1}_{\beta+3,\,l}(\Pi)\|^2 + \|\widetilde{u}_3;\, \mathcal{V}^{l+1}_{\beta+4,\,l-1}(\Pi)\|^2\Big], \tag{6.6}$$

where $|\mathbf{c}_{-1}| = (|c^+_{-1}|^2 + |c^-_{-1}|^2)^{1/2}$ and the constant $c(A)$ polynomially depends on A from (6.2). Note that deriving estimate (6.6) we have used the explicit form of separated asymptotic part in representations (6.1) and (6.3) for $\mathbf{u}$, q, Ψ, and w. Further,

$$(\widehat{\nabla}^2_\eta - \nabla^2_\eta)\widehat{\mathrm{V}}_1 = \left(\frac{\partial a_{13}}{\partial y_1}\right)\frac{\partial\widehat{\mathrm{V}}_1}{\partial z} + 2a_{13}\frac{\partial^2\widehat{\mathrm{V}}_1}{\partial y_1\partial z} + a_{13}\left(\frac{\partial a_{13}}{\partial z}\right)\frac{\partial\widehat{\mathrm{V}}_1}{\partial z} + a_{13}^2\frac{\partial^2\widehat{\mathrm{V}}_1}{\partial z^2}+$$

$$+\left(\frac{\partial a_{23}}{\partial y_2}\right)\frac{\partial\widehat{\mathrm{V}}_1}{\partial z} + 2a_{23}\frac{\partial^2\widehat{\mathrm{V}}_1}{\partial y_2\partial z} + a_{23}\left(\frac{\partial a_{23}}{\partial z}\right)\frac{\partial\widehat{\mathrm{V}}_1}{\partial z} + a_{23}^2\frac{\partial^2\widehat{\mathrm{V}}_1}{\partial z^2}+$$

$$+a_{33}\frac{\partial}{\partial z}\left(a_{33}\frac{\partial\widehat{\mathrm{V}}_1}{\partial z}\right) - \frac{\partial^2\widehat{\mathrm{V}}_1}{\partial z^2} \equiv \sum_{i=1}^{8} J_i + a_{33}\frac{\partial}{\partial z}\left(a_{33}\frac{\partial\widehat{\mathrm{V}}_1}{\partial z}\right) - \frac{\partial^2\widehat{\mathrm{V}}_1}{\partial z^2} =$$

$$= \sum_{i=1}^{8} J_i - (a_{33}-1)\partial_3(a_{33}-1)\partial_3\mathrm{V}_1 - (\partial_3(a_{33}-1))\partial_3\widehat{\mathrm{V}}_1+$$

$$+\partial_3((a_{33}-1)^2\partial_3\widehat{\mathrm{V}}_1) + 2\partial_3((a_{33}-1)\partial_3\widehat{\mathrm{V}}_1) \equiv$$

$$\equiv \sum_{i=1}^{10} J_i + \partial_3((a_{33}-1)^2\partial_3\widehat{\mathrm{V}}_1) + 2\partial_3((a_{33}-1)\partial_3\widehat{\mathrm{V}}_1). \tag{6.7}$$

Analogously, as above by Lemmas 1, 2, and 3, we obtain $\sum_{i=1}^{10} J_i \in \mathcal{V}^{l-1}_{2\beta+6,\,l-1}(\Pi) \subset \mathcal{V}^{l-1}_{\beta+4,\,l-1}(\Pi)$ and

$$\|\sum_{i=1}^{10} J_i;\, \mathcal{V}^{l-1}_{\beta+4,\,l-1}(\Pi)\| \le$$

$$\leq \varepsilon c(A)\Big[|\mathbf{c}_{-1}| + \|\widetilde{w};\ \mathcal{V}^{l+2}_{\beta+2,\,l-1}(\Pi)\|\Big]\Big[|\mathbf{c}_{-1}| + \|\widetilde{u}_1;\ \mathcal{V}^{l+1}_{\beta+3,\,l}(\Pi)\|\Big] \leq$$

$$\leq \varepsilon c(A)\Big[|\mathbf{c}_{-1}| + \|\widetilde{\Psi};\ \mathcal{V}^{l+3/2}_{\beta+2,\,l-1}(\mathbb{R}^2)\|\Big]\Big[|\mathbf{c}_{-1}| + \|\widetilde{u}_1;\ \mathcal{V}^{l+1}_{\beta+3,\,l}(\Pi)\|\Big]. \quad (6.8)$$

For the last two terms in (6.7) we get

$$(a_{33}-1)^2\partial_3\widehat{\nabla}_1 \in \mathcal{V}^l_{2\beta+5,\,l}(\Pi) \subset \mathcal{V}^l_{\beta+3,\,l}(\Pi),$$

$$(a_{33}-1)\partial_3\widehat{\nabla}_1 \in \mathcal{V}^l_{2\beta+5,\,l}(\Pi) \subset \mathcal{V}^l_{\beta+3,\,l}(\Pi)$$

and, since $a_{33}-1 = -\varepsilon\partial_3 w/(1+\varepsilon\partial_3 w)$, there holds the estimate

$$\|(a_{33}-1)^2\partial_3\widehat{\nabla}_1 + (a_{33}-1)\partial_3\widehat{\nabla}_1;\ \mathcal{V}^l_{\beta+3,\,l}(\Pi)\| \leq$$

$$\leq \varepsilon c(A)\Big[|\mathbf{c}_{-1}| + \|\widetilde{\Psi};\ \mathcal{V}^{l+3/2}_{\beta+2,\,l-1}(\mathbb{R}^2)\|\Big]\Big[|\mathbf{c}_{-1}| + \|\widetilde{u}_1;\ \mathcal{V}^{l+1}_{\beta+3,\,l}(\Pi)\|\Big]. \quad (6.9)$$

Further, since $\frac{\partial_3 w}{1+\varepsilon\partial_3 w}u_1 \in \mathcal{V}^{l+1}_{2\beta+5,\,l}(\Pi) \subset \mathcal{V}^{l+1}_{\beta+3,\,l}(\Pi)$, by Lemmas 1, 2, and 3, we obtain

$$\Delta'_\eta\left(\frac{\partial_3 w}{1+\varepsilon\partial_3 w}u_1\right) \in \mathcal{V}^{l-1}_{\beta+5,\,l-2}(\Pi) \subset \mathcal{V}^{l-1}_{\beta+4,\,l-1}(\Pi),$$

$$\frac{\partial}{\partial z}\left(\frac{\partial_3 w}{1+\varepsilon\partial_3 w}u_1\right) \in \mathcal{V}^l_{\beta+3,\,l}(\Pi),$$

$$\left\|\Delta'_\eta\left(\frac{\partial_3 w}{1+\varepsilon\partial_3 w}u_1\right);\ \mathcal{V}^{l-1}_{\beta+4,\,l-1}(\Pi)\right\| \leq$$

$$\leq c(A)\Big[|\mathbf{c}_{-1}| + \|\widetilde{\Psi};\ \mathcal{V}^{l+3/2}_{\beta+2,\,l-1}(\mathbb{R}^2)\|\Big]\Big[|\mathbf{c}_{-1}| + \|\widetilde{u}_1;\ \mathcal{V}^{l+1}_{\beta+3,\,l}(\Pi)\|\Big], \quad (6.10)$$

where $\Delta'_\eta = \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}$,

$$\left\|\frac{\partial}{\partial z}\left(\frac{\partial_3 w}{1+\varepsilon\partial_3 w}u_1\right);\ \mathcal{V}^l_{\beta+3,\,l}(\Pi)\right\| \leq$$

$$\leq c(A)\Big[|\mathbf{c}_{-1}| + \|\widetilde{\Psi};\ \mathcal{V}^{l+3/2}_{\beta+2,\,l-1}(\mathbb{R}^2)\|\Big]\Big[|\mathbf{c}_{-1}| + \|\widetilde{u}_1;\ \mathcal{V}^{l+1}_{\beta+3,\,l}(\Pi)\|\Big] \quad (6.11)$$

Finally, $(\partial_1 - \widehat{\nabla}_1)q = -a_{13}\partial_3 q \in \mathcal{V}^{l-1}_{2\beta+7,\,l-1}(\Pi) \subset \mathcal{V}^{l-1}_{\beta+4,\,l-1}(\Pi)$ and

$$\|(\partial_1 - \widehat{\nabla}_1)q;\ \mathcal{V}^{l-1}_{\beta+4,\,l-1}(\Pi)\| \leq$$

$$\leq \varepsilon c(A)\Big[|\mathbf{c}_{-1}| + \|\widetilde{\Psi};\ \mathcal{V}^{l+3/2}_{\beta+2,\,l-1}(\mathbb{R}^2)\| + \|q;\ \mathcal{V}^l_{\beta+2,\,l}(\Pi)\|\Big]^2. \quad (6.12)$$

Thus, denoting

$$M_{11} = \frac{1}{\varepsilon}(a_{33}-1)^2\partial_3\widehat{\mathrm{V}}_1 + \frac{1}{\varepsilon}2(a_{33}-1)\partial_3\widehat{\mathrm{V}}_1 + \frac{\partial}{\partial z}\left(\frac{\partial_3 w}{1+\varepsilon\partial_3 w}u_1\right)$$

we see that M_1 is represented in the form

$$M_1 = M_{10} + \partial_z M_{11}$$

where $M_{10} \in \mathcal{V}^{l-1}_{\beta+4,\,l-1}(\Pi)$ and $M_{11} \in \mathcal{V}^{l}_{\beta+3,\,l}(\Pi)$ and from (6.6)–(6.12) it follows that

$$\|M_{10};\ \mathcal{V}^{l-1}_{\beta+4,\,l-1}(\Pi)\| + \|M_{11}; \mathcal{V}^{l}_{\beta+3,\,l}(\Pi)\| \le$$

$$\le c(A)\Big(|\mathbf{c}_{-1}| + \|\widetilde{\mathbf{\Psi}};\ \mathcal{V}^{l+3/2}_{\beta+2,\,l-1}(\mathbb{R}^2)\| + \|(\widetilde{\mathbf{u}},\widetilde{q});\ \mathcal{D}^{l}_{\beta+2}(\Pi)\|\Big)^2.$$

Estimates for M_1 and M_2 are similar and we derive that $\mathbf{M}$ is represented in the form

$$\mathbf{M} = \mathbf{M}_0 + \partial_z\mathbf{M}_1,$$

where $(\mathbf{M}_0, \mathbf{M}_1) \in \mathcal{V}^{l-1}_{\beta+4,\,l-1}(\Pi) \times \mathcal{V}^{l}_{\beta+3,\,l}(\Pi)$. Thus, $(\mathbf{M}, \mathbf{0}, 0, 0) \in \mathcal{R}^{\,l}_{\beta+2}(\Pi;\partial\Pi)$. Moreover, there holds the estimate

$$\|(\mathbf{M}, \mathbf{0}, 0, 0);\ \mathcal{R}^{\,l}_{\beta+2}(\Pi;\partial\Pi)\| \le$$

$$\le c(A)\Big(|\mathbf{c}_{-1}| + \|\widetilde{\mathbf{\Psi}};\ \mathcal{V}^{l+3/2}_{\beta+2,\,l-1}(\mathbb{R}^2)\| + \|(\widetilde{\mathbf{u}},\widetilde{q});\ \mathcal{D}^{l}_{\beta+2}(\Pi)\|\Big)^2. \qquad (6.13)$$

Arguing analogously and using the fact that B_1, B_2, and $\widetilde{R}$ include the products of functions that are the traces of functions from the corresponding weighted spaces and the equivalence of norms (2.3) and (2.5) we conclude that $(\mathbf{0}, \mathbf{0}, B_1, B_2) \in \mathcal{R}^{\,l}_{\beta+2}(\Pi;\partial\Pi)$, $\widetilde{R} \in \mathcal{V}^{\,l-1/2}_{\beta+2,\,l-1}(\partial\Pi)$, and

$$\|(\mathbf{0}, \mathbf{0}, B_1, B_2);\ \mathcal{R}^{\,l}_{\beta+2}(\Pi;\partial\Pi)\| + \|\widetilde{R};\ \mathcal{V}^{\,l-1/2}_{\beta+2,\,l-1}(\mathbb{R}^2)\|$$

$$\le c(A)\Big(|\mathbf{c}_{-1}| + \|\widetilde{\mathbf{\Psi}};\ \mathcal{V}^{l+3/2}_{\beta+2,\,l-1}(\mathbb{R}^2)\| + \|(\widetilde{\mathbf{u}},\widetilde{q});\ \mathcal{D}^{l}_{\beta+2}(\Pi)\|\Big)^2. \qquad (6.14)$$

Thus, we have proved

Lemma 8. *Let functions* $\mathbf{u}$, q, *and* Ψ *admit representations* (6.1) *with* $(\widetilde{\mathbf{u}}, q) \in \mathcal{D}^l_{\beta+2}(\Pi)$, $\widetilde{\Psi} \in \mathcal{V}^{l+3/2}_{\beta+2,l-1}(\mathbb{R}^2)$, *and* $\beta \in (-2,-1)$ *and let there holds condition* (6.2). *Then the functions* $\mathbf{M}$, B_1, B_2, *and* $\widetilde{R}$ *defined by* (3.14)–(3.19) *satisfy the estimate*

$$\|(\mathbf{M}, \mathbf{0}, B_1, B_2);\, \mathcal{R}^l_{\beta+2}(\Pi;\partial\Pi)\| + \|\widetilde{R};\, \mathcal{V}^{l-1/2}_{\beta+2,\,l-1}(\mathbb{R}^2)\|$$

$$\leq c(A)\Big(|\mathbf{c}_{-1}| + \|\widetilde{\Psi};\, \mathcal{V}^{l+3/2}_{\beta+2,\,l-1}(\mathbb{R}^2)\| + \|(\widetilde{\mathbf{u}}, \widetilde{q});\, \mathcal{D}^l_{\beta+2}(\Pi)\|\Big)^2 \qquad (6.15)$$

where the constant $c(A)$ *depends on* A *polynomially and does not depend on* ε.

Let $(\mathbf{u}^*, q^*, \Psi^*)$ and $(\mathbf{u}^{**}, q^{**}, \Psi^{**})$ be two triples of functions admitting representations (6.1) with constants $c^{\pm *}_{-1}$ and $c^{\pm **}_{-1}$, respectively, and satisfying condition (6.2). Denote

$$\mathcal{M} = \mathbf{M}(\mathbf{u}^*, q^*, \Psi^*) - \mathbf{M}(\mathbf{u}^{**}, q^{**}, \Psi^{**}),$$

$$\mathcal{B}_i = B_i(\mathbf{u}^*, q^*, \Psi^*) - B_i(\mathbf{u}^{**}, q^{**}, \Psi^{**}), \quad i = 1, 2,$$

$$\widetilde{\mathcal{R}} = \widetilde{R}(\mathbf{u}^*, q^*, \Psi^*) - \widetilde{R}(\mathbf{u}^{**}, q^{**}, \Psi^{**}).$$

By the same arguments as before we get

Lemma 9. *Let functions* $(\mathbf{u}^*, q^*, \Psi^*)$ *and* $(\mathbf{u}^{**}, q^{**}, \Psi^{**})$ *satisfy the conditions of Lemma* 8. *Then there holds the estimate*

$$\|(\mathcal{M}, \mathbf{0}, \mathcal{B}_1, \mathcal{B}_2);\, \mathcal{R}^l_{\beta+2}(\Pi;\partial\Pi)\| + \|\widetilde{\mathcal{R}};\, \mathcal{V}^{l-1/2}_{\beta+2,\,l-1}(\mathbb{R}^2)\| \leq$$

$$\leq c(A)\Big(|\mathbf{c}^*_{-1} - \mathbf{c}^{**}_{-1}| + \|\widetilde{\Psi}^* - \widetilde{\Psi}^{**};\, \mathcal{V}^{l+3/2}_{\beta+2,\,l-1}(\mathbb{R}^2)\| +$$

$$+ \|(\widetilde{\mathbf{u}}^* - \widetilde{\mathbf{u}}^{**}, \widetilde{q}^* - \widetilde{q}^{**});\, \mathcal{D}^l_{\beta+2}(\Pi)\|\Big)^2 \qquad (6.16)$$

where the constant $c(A)$ *depends on* A *polynomially and does not depend on* ε.

7. Solvability of the free boundary value problem

In this section, we prove the main result of the paper: the solvability for small ε of the free boundary value problem (1.1)–(1.6) or what is the same the solvability of the equivalent problem (3.8)–(3.13).

Let us denote by $\mathfrak{E}^l_\beta(\Pi)$ the space of triples $(\mathbf{u}, q, \Psi)$ satisfying representations (6.1) with some constants $c^{\pm}_{-1}$. The space $\mathfrak{E}^l_{\beta+2}(\Pi)$ could be identified with the direct product $\mathcal{D}^l_{\beta+2}(\Pi) \times \mathcal{V}^{l+3/2}_{\beta+2,\, l-1}(\mathbb{R}^2) \times \mathbb{R} \times \mathbb{R}$. The norm in $\mathfrak{E}^l_{\beta+2}(\Pi)$ is given by the formula

$$\|(\mathbf{u}, q, \Psi);\ \mathfrak{E}^l_{\beta+2}(\Pi)\| =$$
$$= \Big(\|(\widetilde{\mathbf{u}}, \widetilde{q};\ \mathcal{D}^l_{\beta+2}(\Pi)\|^2 + \|\widetilde{\Psi};\ \mathcal{V}^{l+3/2}_{\beta+2,\, l-1}(\mathbb{R}^2\|^2 + |\mathbf{c}_{-1}|^2\Big)^{1/2}.$$

Theorem 6. *Let $\boldsymbol{\alpha} \in W^{l+1/2,2}(\mathbb{R}^2)$, $l \geq 1$,supp $\boldsymbol{\alpha} \subset \{y \in \mathbb{R}^2 :\ |y| < 1\}$,*

$$\int\limits_{\mathbb{R}^2} \alpha_3(y)\, dy = 0. \tag{7.1}$$

Then for sufficiently small ε problem (3.8)–(3.13) has a unique solution $(\mathbf{u}, q, \Psi) \in \mathfrak{E}^l_{\beta+2}(\Pi)$. There holds the estimate

$$\|(\mathbf{u}, q, \Psi);\ \mathfrak{E}^l_{\beta+2}(\Pi)\| \leq c\|\boldsymbol{\alpha};\ W^{l+1/2,2}(\mathbb{R}^2)\|. \tag{7.2}$$

Proof. Let us consider problem (3.8)–(3.11) with the right-hand sides $\varepsilon\mathbf{M}(\mathbf{u}^*, q^*, \Psi^*)$, $\boldsymbol{\alpha}$, $\varepsilon B_1(\mathbf{u}^*, q^*, \Psi^*)$, and $\varepsilon B_2(\mathbf{u}^*, q^*, \Psi^*)$, where the functions $(\mathbf{u}^*, q^*, \Psi^*)$ are given and belong to the ball

$$\mathfrak{B}_A = \{(\mathbf{u}^*, q^*, \Psi^*) \in \mathfrak{E}^l_{\beta+2}(\Pi) : \|(\mathbf{u}^*, q^*, \Psi^*); \mathfrak{E}^l_{\beta+2}(\Pi)\| \leq A\} \subset \mathfrak{E}^l_{\beta+2}(\Pi).$$

By Theorem 4, problem (3.8)–(3.11) has a unique solution $(\mathbf{u}, q)$ satisfying the asymptotic representation (4.24), (4.25). Moreover, because of (4.31), (6.15), we have the estimate

$$\|(\widetilde{\mathbf{u}}, \widetilde{q}); \mathcal{D}^l_{\beta+2}(\Pi)\| + |\mathbf{c}_{-1}| \leq c\,\|(\varepsilon\mathbf{M}, \boldsymbol{\alpha}, \varepsilon B_1, \varepsilon B_2); \mathcal{R}^l_{\beta+2}(\Pi; \partial\Pi)\| \leq$$
$$\leq \varepsilon c(A)\Big(|\mathbf{c}^*_{-1}| + \|\widetilde{\Psi}^*;\ \mathcal{V}^{l+3/2}_{\beta+2,\, l-1}(\mathbb{R}^2)\| + \|(\widetilde{\mathbf{u}}^*, \widetilde{q}^*); \mathcal{D}^l_{\beta+2}(\Pi)\|\Big)^2 +$$
$$+ c_1\|\boldsymbol{\alpha};\ W^{l+1/2,2}(\mathbb{R}^2)\|. \tag{7.3}$$

Consider now problem (3.12), (3.13) with the right–hand side $R = 2(2\nu\partial_3 u_3 - q)|_{z=1} + \varepsilon\widetilde{R}(\mathbf{u}, q, \Psi^*)$. Then R admits the representation (5.10) with $R_1 = 2(2\nu\partial_3 u_3 - \widetilde{q}) + \varepsilon\widetilde{R}(\mathbf{u}, q, \Psi^*) \in \mathcal{V}^{l-1/2}_{\beta+2,l-1}(\mathbb{R}^2)$ and by Theorem 5, there exists a unique solution Ψ of (3.12), (3.13) satisfying the representation $\Psi(y) = g\chi(r)P_{-1}(y) + \widetilde{\Psi}(y)$ and the estimate

$$\|\,\widetilde{\Psi};\ \mathcal{V}^{l+3/2}_{\beta+2,l-1}(\mathbb{R}^2)\,\| \le$$

$$\le c\Big(\varepsilon\|\,\widetilde{R};\ \mathcal{V}^{l-1/2}_{\beta+2,l-1}(\mathbb{R}^2)\,\| + \|\,(\widetilde{\mathbf{u}}, \widetilde{q});\ \mathcal{D}^l_{\beta+2}(\Pi)\,\| + |\mathbf{c}_{-1}|\,\Big). \tag{7.4}$$

Estimating in the right-hand side of (7.4) the term $\varepsilon\|\,\widetilde{R};\ \mathcal{V}^{l-1/2}_{\beta+2,l-1}(\mathbb{R}^2)\|$ by (6.15), from (7.3) and (7.4) we derive

$$\|\,(\widetilde{\mathbf{u}}, \widetilde{q});\ \mathcal{D}^l_{\beta+2}(\Pi)\,\| + |\mathbf{c}_{-1}| + \|\,\widetilde{\Psi};\ \mathcal{V}^{l+3/2}_{\beta+2,l-1}(\mathbb{R}^2)\| \le$$

$$\le \varepsilon c^{(1)}(A)\Big(|\mathbf{c}^*_{-1}| + \|\widetilde{\Psi}^*;\ \mathcal{V}^{l+3/2}_{\beta+2,\,l-1}(\mathbb{R}^2)\| + \|(\widetilde{\mathbf{u}}^*, \widetilde{q}^*);\ \mathcal{D}^l_{\beta+2}(\Pi)\|\Big)^2 +$$

$$+ c_2\|\boldsymbol{\alpha};\ W^{l+1/2,2}(\mathbb{R}^2)\|. \tag{7.5}$$

Thus, we have defined an operator

$$\mathfrak{A} : (\mathbf{u}^*, q^*, \Psi^*) \in \mathfrak{E}^l_{\beta+2}(\Pi) \longmapsto (\mathbf{u}, q, \Psi) \in \mathfrak{E}^l_{\beta+2}(\Pi).$$

Moreover, taking

$$A = 2c_2\|\boldsymbol{\alpha};\ W^{l+1/2,2}(\mathbb{R}^2)\|, \quad \varepsilon < \frac{1}{2c^{(1)}(A)A},$$

we get that $\mathfrak{A}$ maps the ball $\mathfrak{B}_A$ to $\mathfrak{B}_A$.

Obviously, the fixed point $(\mathbf{u}, q, \Psi) = \mathfrak{A}(\mathbf{u}, q, \Psi)$ of $\mathfrak{A}$ is the solution of problem (3.8)–(3.13). Let as prove for sufficiently small ε that $\mathfrak{A}$ is a contraction operator in $\mathfrak{B}_A$. Let $(\mathbf{u}^*, q^*, \Psi^*)$ and $(\mathbf{u}^{**}, q^{**}, \Psi^{**}) \in \mathfrak{B}_A$. The difference $(\mathbf{w}, s, \Theta) = \mathfrak{A}(\mathbf{u}^*, q^*, \Psi^*) - \mathfrak{A}(\mathbf{u}^{**}, q^{**}, \Psi^{**})$ is the solution of the boundary value problem

$$-\nu\triangle_\eta\mathbf{w} + \nabla_\eta s = \varepsilon(\mathbf{M}(\mathbf{u}^*, q^*, \Psi^*) - \mathbf{M}(\mathbf{u}^{**}, q^{**}, \Psi^{**})), \quad \mathrm{div}_\eta\mathbf{w} = 0,$$

$$\mathbf{w}|_{z=0} = \mathbf{0}, \qquad w_3|_{z=1} = 0,$$

$$\partial_3 w_1|_{z=1} = \varepsilon(B_1(\mathbf{u}^*, q^*, \Psi^*) - B_1(\mathbf{u}^{**}, q^{**}, \Psi^{**})),$$

$$\partial_3 w_2|_{z=1} = \varepsilon(B_2(\mathbf{u}^*, q^*, \Psi^*) - B_2(\mathbf{u}^{**}, q^{**}, \Psi^{**})),$$

$$\sigma\triangle\Theta - 2\,g\Theta = 2(2\nu\partial_3 w_3 - s)|_{z=1} + \varepsilon(\widetilde{R}(\mathbf{u}^*, q^*, \Psi^*) - \widetilde{R}(\mathbf{u}^{**}, q^{**}, \Psi^{**})),$$

$$\lim_{\sqrt{y_1^2+y_2^2}\to\infty} \Theta(y_1, y_2) = 0.$$

Therefore, arguing in the same way as above and using instead of (6.15) estimate (6.16), we obtain

$$\|\mathfrak{A}(\mathbf{u}^*, q^*, \Psi^*) - \mathfrak{A}(\mathbf{u}^{**}, q^{**}, \Psi^{**});\ \mathfrak{E}^l_{\beta+2}(\Pi)\| \le$$

$$\le \varepsilon c^{(2)}(A)\|(\mathbf{u}^*, q^*, \Psi^*) - \mathfrak{A}(\mathbf{u}^{**}, q^{**}, \Psi^{**});\ \mathfrak{E}^l_{\beta+2}(\Pi)\|.$$

Hence, if $\varepsilon < c^{(2)}(A)^{-1}$, the operator $\mathfrak{A}$ is a contraction in the ball $\mathfrak{B}_A$ and by Banach contraction principle $\mathfrak{A}$ has a unique fixed point in this ball. The theorem is proved.

Remark 4. Note that the operator $\mathfrak{A}$ could not be defined without separating the asymptotic part of the functions $\mathbf{u}$, q, and Ψ. The whole $(\mathbf{u}, q, \Psi)$ belongs only to the space $\mathcal{D}^l_{\beta+1}(\Pi) \times \mathcal{V}^{l+3/2}_{\beta+1,l-1}(\mathbb{R}^2)$ with $\beta \in (-2, -1)$ and this is not enough to show that the right-hand side $(\mathbf{M}, \mathbf{0}, B_1, B_2, R)$ of Eqs. (3.8)–(3.13) hits into the appropriate space $\mathcal{R}^l_{\beta+1}(\Pi, \partial\Pi) \times \mathcal{V}^{l-1/2}_{\beta+1,l-1}(\mathbb{R}^2)$.

Moreover, choosing the weighted spaces with detached asymptotics we not only prove the existence of the unique solution, but simultaneously also get the asymptotic representation of it.

Remark 5. Without condition (7.1), the asymptotic part of the pressure q contains the logarithmically growing term (see the proof of Theorem 3 and the definition of $P_0^-(y)$) and, therefore, Eq. (3.12) has no solutions satisfying condition (3.13). The case of sources and sinks with nonzero total flux remans an open problem.

References

1. D. K. Faddeev, B. Z. Vulich, V. A. Solonnikov, N. N. Uralceva, *Selected topics in analysis and higher algebra*, Ch. 3. Sobolev spaces, Leningrad (1981), pp. 130–190.
2. L. V. Kapitanskii, K. Pileckas, *Certain problems of vector analysis*, — Zap. Nauchn. Semin. LOMI **138** (1984), 65–85.
3. O. A. Ladyzhenskaya, V. G. Osmolovskii, *On the free surface of a fluid layer over a solid sphere*, — Vestnik Leningrad. Univ. Math. **13** (1976), 25–30.
4. V. G. Osmolovskii, *The free surface of a drop in a symmetric force field*, — Zap. Nauchn. Semin. LOMI **52** (1975), 160–174.
5. S. A. Nazarov, K. Pileckas, *On noncompact free boundary problems for the plane stationary Navier–Stokes equations*, — J. für reine und angewandte Math. **438** (1993), 103–141.
6. S. A. Nazarov, K. Pileckas, *On the solvability of the Stokes and Navier–Stokes problem in the domains that are layer-like at infinity*, — J. Math. Fluid Mech. **1**, No. 1 (1999), 78–116.

7. S. A. Nazarov, K. Pileckas, *The asymptotic properties of the solutions to the Stokes problem in domains that are layer-like at infinity*,— J. Math. Fluid Mech. **1**, No. 2 (1999), 131–167.
8 S. A. Nazarov, K. Pileckas, *On the Fredholm property of the Stokes operator in a layer-like domain*, — ZAA **20**, No. 1 (2001), 155–182.
9. K. Pileckas, *Existence of solutions for the Navier–Stokes equations having an infinite dissipation of energy in a class of domains with noncompact boundaries*, — Zap. Nauchn. Semin. LOMI **110** (1981), 180–202.
10. K. Pileckas, *Solvability of a problem of plane motion of a viscous incompressible fluid with noncompact free boundary*, — Diff. Eqs. Appl. **30**, (1981), 57–96; Zap. Nauchn. Semin. LOMI **110** (1981), 174–179.
11. K. Pileckas, *On the problem of motion of heavy viscous incompressible fluid with noncompact free boundary*, — Litovskii Mat. Sb. **28**, No. 2 (1988), 315–333.
12. K. Pileckas, *On a plane motion of a viscous incompressible capillary liquid with a noncompact free boundary*, — Arch. Mech. **41**, Nos. 2–3 (1989), 329–342.
13. K. Pileckas, *The example of nonuniqeness of the solution to a noncompact free boundary problem for the stationary Navier–Stokes system*, — Diff. Eqs. Appl. **42** (1988), 59–64.
14. K. Pileckas, *Asymptotics of solutions of the stationary Navier–Stokes system of equations in a domain of layer type* — Mat. Sb. **193**, No. 12 (2002), 69–104.
15. V. V. Puchnachov, *The plane stationary problem with a free boundary for the Navier–Stokes equations*, — Z. Prikl. Meh. i Tehn. Fiz. **3** (1972), 91–102.
16. D. Sattinger, *On the free surface of a viscous fluid*, — Proc. R. Soc. London A **349** (1976), 183–204.
17. J. Socolowsky, *Solvability of a two-dimensional problem for the motion of two viscous incompressible liquids with non-compact free boundaries*, — Zap. Nauchn. Semin. LOMI **163** (1987), 146–153.
18. V. A. Solonnikov, V. E. Shchadilov, *On a boundary value problem for a stationary system of Navier–Stokes equations*, — Trudy Steklov Mat. Inst. **125** (1973), 196–210.
19. V. A. Solonnikov, *Solvability of the problem of plane motion of heavy viscous incompressible capillary liquid partially filling a container*, — Izv. Akad. Nauk. SSSR, Ser. Mat. **43** (1979), 203–236.
20. V. A. Solonnikov, *On a free boundary problem for the system of Navier–Stokes equations*, — Trudy S. L. Sobolev Sem. **2** (1978), 127–140.
21. V. A. Solonnikov, *Solvability of a three-dimensional boundary value problem with a free surface for the stationary Navier–Stokes system*,— Zap. Nauchn. Semin. LOMI, **84** (1979), 252–285.
22. V. A. Solonnikov, *On the Stokes equations in domains with nonsmooth boundaries and on a viscous incompressible flow with a free surface*,— Nonlin. Part. Diff. Eqs. Appl., Collège de France Seminar **3** (1980/81), 340–423.
23. V. A. Solonnikov, *Solvability of the problem on the effluence of a viscous incompressible fluid into an open basin*, — Trudy Steklov Mat. Inst. **179** (1988), 174–202.
24. V. A. Solonnikov, *On the solvability of some two-dimensional quasistationary free boundary problems for the Navier–Stokes equations with moving contact point*, — Zap. Nauchn. Semin. POMI **206** (1993), 119–126.

25. V. A. Solonnikov, *On the problems with free boundaries and moving contact points for two-dimensional stationary Navier–Stokes equations*, — Zap. Nauchn. Semin. POMI **213** (1994), 179–205.
26. V. A. Solonnikov, *Some free boundary problems for the Navier–Stokes equations with moving contact points and lines*, — in: Partial differential equations: models in physics and biology, G. Lumer, S. Nicaise, B.-W. Schulze (Eds) **82** (1994), pp. 329–350.
27. V. A. Solonnikov, *On some free boundary problems for the Navier–Stokes equations with moving contact points and lines*, Math. Annal. **302** (1995), 743–772.
28. V. A. Solonnikov, *Free boundary problems for the Navier–Stokes equations with moving contact points*, — in: Free boundary problems: theory and applications, Pitman Research Notes in Math. Series **323** (1995), 213–214.
29. V. A. Solonnikov, *Solvability of the free boundary problem governing a viscous incompressible flow through an aperture*, — Algebra Analiz **9**, No. 2 (1997), 169–191.
30. V. A. Solonnikov, *Solvability of the two-dimensional free boundary problem for the Navier–Stokes equations for the limiting values of the contact angle*, — Quaderni di matematica **2** (1998), 163–210.
31. V. A. Solonnikov, *Solvability of two stationary free boundary problems for the Navier–Stokes equations*, — Boll. UMI (8) **1-B** (1998), 223–262.
32. V. A. Solonnikov, *Stationary free boundary problems for the Navier–Stokes equations, Advanced topics in theoretical fluid mechanics*, — Pitman Research Notes Math. Series **392** (1998), 147–212.
33. V. A. Solonnikov, *On the problem of a steady fall of a drop in a liquid medium*, — J. Math. Fluid Mech. **1** (1999), 326–355.

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