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ON THE NAVIER–STOKES EQUATIONS WITH THE ENERGY-DEPENDENT NONLOCAL VISCOSITIES

ABSTRACT. We discuss the mathematical derivation of incompressible viscous flows where the viscosity depends on the total dissipation energy. In the two-dimensional periodic case, we consider first the case of temperature dependent viscosities with very large thermal conductivity in the heat convective equation, in which we obtain as an asymptotic limit the Navier–Stokes system coupled with and ordinary differential equation involving the dissipation energy. Letting further the latent heat vanish, we derive the Navier–Stokes equations with a nonlocal viscosity depending on the total dissipation of energy.

Dedicated to V. A. Solonnikov on occasion of his 70th birthday

1. INTRODUCTION

In this paper, we obtain a nonlocal model (see Ladyzhenskaya [4]) as a limit of the coupled system of motion and energy equations to Navier–Stokes flows. The plane motion of a homogeneous viscous incompressible fluid in a periodic cell can be described by the system

$$\left.\begin{array}{c} \partial_t v + \operatorname{div}(v \otimes v) - \operatorname{div}(\mu(\theta)D(v)) = f - \nabla p \\ \operatorname{div} v = 0 \end{array}\right\} \quad \text{in} \quad Q_T, \tag{1.1}$$

$$\delta(\partial_t \theta + \operatorname{div}(v\theta)) - \sigma\Delta\theta + \theta = \mu(\theta)|D(v)|^2 \quad \text{in} \quad Q_T, \tag{1.2}$$

$$v\big|_{t=0} = v_0, \tag{1.3}$$

$$\theta\big|_{t=0} = \theta_0. \tag{1.4}$$

Here $Q_T \equiv \mathbb{T} \times (0,T)$, $\mathbb{T}$ is a two-dimensional unit torus, $v : Q_T \to \mathbb{R}^2$ is the velocity field, $p : Q_T \to \mathbb{R}$ is the pressure, the strain velocity

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tensor $D(v) = \frac{1}{2}(\nabla v + (\nabla v)^T)$, μ the viscosity, f the external force, θ the temperature, and δ and σ some positive constants. We assume there exist positive constants μ_0 and μ_1 such that

$$0 < \mu_0 \leqslant \mu(s) \leqslant \mu_1. \tag{1.5}$$

Passing to the limit on σ ($\sigma \to +\infty$) and sucessively on δ ($\delta \to 0^+$), the local system (1.1)–(1.2) becomes the nonlocal problem

$$\left.\begin{aligned} \partial_t v + \operatorname{div}(v \otimes v) - \nu(\|D(v)\|^2_{2,\mathbb{T}})\Delta v = f - \nabla p \\ \operatorname{div} v = 0 \end{aligned}\right\} \quad \text{in} \quad Q_T, \tag{1.6}$$

where ν is a new viscosity related with the initial one μ by the functional relation $\nu = \mu \circ \alpha$, where α is the inverse function of $\beta(\theta) = \theta/\mu(\theta)$.

This problem was already studied in the steady-state case, for Navier–Stokes and Bingham fluids in [1]. This kind of problems can be regarded as a particular asymptotic limit case of certain nonisothermal flows with very high thermal conductivity in the fluid ($\sigma \to \infty$). In the evolutionary problem, the second limit $\delta \to 0$ corresponds essentially to a very small specific heat.

The paper is structured as follows. The functional space framework is introduced and the main results are stated in the next Section. Sections 3, 4, and 5 are concerned with their proofs. Namely, in Sec. 3 the existence of a weak solution to the coupled system of partial differential equations (1.1)–(1.2) with periodical and initial conditions (1.3) and (1.4) is proved by a fixed point argument. In Sec. 4, we present the passage to the limit when σ tends to infinity. So it is obtained a shadow system constituted by a partial and an ordinary differential equation (see [2], for further references). In Sec. 5, the passage to the limit as δ tends to zero in this shadow system yields the unique partial differential equation (1.6) in which there exists a relation between the temperature and the energy dissipation. Our proof of this second limit requires additionally that μ is a monotone nonincreasing function. The removal of this restriction as well as the corresponding three dimensional problem are interesting open questions.

2. Notations and Main Results

Let $\mathbb{T}$ be a two-dimensional unit torus (i.e., meas $\mathbb{T} = 1$), $Q_T = \mathbb{T} \times (0, T)$. For k, $m \in \mathbb{N}$ and q, r, $s \geq 1$, let us define

- $L^q(\mathbb{T})$, $W^k_q(\mathbb{T})$ are the usual Lebegue and Sobolev spaces, $\|\cdot\|_{q,\mathbb{T}}$ is the norm of $L^q(\mathbb{T})$, and $(\cdot,\cdot)$ is the scalar product in $L^2(\mathbb{T})$,

- $C^l(\mathbb{T};\mathbb{R}^m)$, $l \leqslant \infty$ is the space of all C^l-smooth functions $v : \mathbb{R}^2 \to \mathbb{R}^m$, $l = 0, 1, 2, \dots$ often we omit the image space $\mathbb{R}^m$ for simplicity,
- $C^l(\bar{Q}_T) = \{v \in C^l(\bar{Q}_T) : \forall t \in [0,T]\ v(\cdot,t) \in C^l(\mathbb{T})\}$, $l = 0, 1, 2, \dots$,
- $J^\infty(\mathbb{T}) = \{u \in C^\infty(\mathbb{T};\mathbb{R}^2), \operatorname{div} u = 0\}$,
- $J^k_q(\mathbb{T}) =$ Closure of $J^\infty(\mathbb{T})$ in $W^k_q(\mathbb{T})$, $k = 0, 1, 2, \dots$,
- $W^{-1}_q(\mathbb{T}) = (W^1_{q'}(\mathbb{T}))^*$, $J^{-1}_q(\mathbb{T}) = (J^1_{q'}(\mathbb{T}))^*$, $\frac{1}{q} + \frac{1}{q'} = 1$, and we keep the same notation $(\cdot,\cdot)$ for the duality between these spaces,
- $L^{r,s}(Q_T) \equiv L^s(0,T;L^r(\mathbb{T}))$, $\|f\|_{L^{r,s}(Q_T)} \equiv \left(\int_0^T \|f(\cdot,t)\|^s_{r,\mathbb{T}}\, dt\right)^{1/s}$,
- $W^{1,0}_q(Q_T) \equiv L^q(0,T;W^1_q(\mathbb{T})) = \{u \in L^q(Q_T) : \nabla u \in L^q(Q_T)\}$,
  $\|u\|_{W^{1,0}_q(Q_T)} \equiv \|u\|_{q,Q_T} + \|\nabla u\|_{q,Q_T}$,
- $W^{2,1}_q(Q_T) = \{u \in L^q(Q_T) : \nabla u, \nabla^2 u, \partial_t u \in L^q(Q_T)\}$,
  $\|u\|_{W^{2,1}_q(Q_T)} \equiv \|u\|_{W^{1,0}_q(Q_T)} + \|\nabla^2 u\|_{q,Q_T} + \|\partial_t u\|_{q,Q_T}$,
  $W^{2,1}_q(Q_T) \equiv L^q(0,T;W^2_q(\mathbb{T})) \cap W^{2,1}_q(Q_T)$.

The main results of this paper are the following theorems.

Theorem 2.1. *Under the assumption* (15), *for any given* $\delta > 0$, $\sigma > 0$ *and* f, v_0, θ_0 *such that*

$$f \in L^2(0,T;W^{-1}_2(\mathbb{T})), \quad v_0 \in J_2(\mathbb{T}), \quad \theta_0 \in L^1(\mathbb{T}) \tag{2.1}$$

there exists at least one solution (v,p,θ) *of system* (1.1)–(1.4) *having the following properties:*

$$v \in C([0,T];L^2(\mathbb{T})) \cap L^2(0,T;J^1_2(\mathbb{T})), \quad \partial_t v \in L^2(0,T;W^{-1}_2(\mathbb{T})),$$

$$p \in L^2(Q_T), \quad \int\limits_{\mathbb{T}} p(x,t)\, dx = 0,$$

$$\theta \in L^\infty(0,T;L^1(\mathbb{T})) \cap W^{1,0}_q(Q_T), \quad \partial_t\theta \in L^1(0,T;W^{-1}_q(\mathbb{T})),$$

$$1 \leqslant q < \frac{4}{3}$$

satisfying the relations

$$\begin{gathered}(\partial_t v, w) + (\mu(\theta)D(v), D(w)) - (v\otimes v, \nabla w) = (f,w) + (p, \operatorname{div} w),\\ \forall w \in W^1_2(\mathbb{T}), \quad \text{a.e.} \quad t \in (0,T),\end{gathered} \tag{2.2}$$

$$v|_{t=0} = v_0, \tag{2.3}$$

$$\begin{gathered}\int_{Q_T} (-\delta\theta\partial_t\eta + \sigma\nabla\theta\cdot\nabla\eta - \delta\theta v\cdot\nabla\eta + \theta\eta)\;dx\;dt = \\ = \int_{Q_T} \mu(\theta)|D(v)|^2\eta\;dx\;dt + \delta\int_{\mathbb{T}} \theta_0(x)\eta(x,0)\;dx, \\ \forall\eta\in C^1(\bar{Q}_T):\;\; \eta|_{t=T} = 0.\end{gathered} \tag{2.4}$$

and satisfying the following estimates

$$\|v\|_{L^\infty(0,T;L^2(\mathbb{T}))} + \|v\|_{W_2^{1,0}(Q_T)} \leqslant C\Big(\|f\|_{2,Q_T} + \|v_0\|_{2,\mathbb{T}}\Big), \tag{2.5}$$

$$\|p\|_{2,Q_T} \leqslant C\Big(\|D(v)\|_{2,Q_T} + \|v\|^2_{4,Q_T} + \|f\|_{L^2(0,T;W_2^{-1}(\mathbb{T}))}\Big), \tag{2.6}$$

$$\begin{gathered}\|\theta\|_{L^\infty(0,T;L^1(\mathbb{T}))} + \sqrt{\sigma}\|\nabla\theta\|_{q,Q_T} + \|\theta\|_{q,Q_T} \leqslant \\ \leqslant \mathcal{F}\left(\|D(v)\|^2_{2,Q_T}, \|\theta_0\|_{1,\mathbb{T}}\right),\end{gathered} \tag{2.7}$$

where constants C and the majorant $\mathcal{F}$ (which is a nondecreasing continuous function of its arguments) depend only on μ_0, μ_1, and δ.

Theorem 2.2. *Let $\delta > 0$ be fixed and $(v_\sigma, p_\sigma, \theta_\sigma)$ be a solution to (1.1)–(1.4) corresponding to each $\sigma > 0$ and assume*

$$\sigma \to +\infty.$$

Then there exist

$$v \in C([0,T];L^2(\mathbb{T})) \cap L^2(0,T;J_2^1(\mathbb{T})), \quad \partial_t v \in L^2(0,T;W_2^{-1}(\mathbb{T})),$$

$$p \in L^2(Q_T), \quad \int_{\mathbb{T}} p(x,t)\;dx = 0, \quad \text{and} \quad \xi \in W_1^1(0,T),$$

such that

$$v_\sigma \rightharpoonup v \quad in \quad W_2^{1,0}(Q_T), \quad v_\sigma \to v \quad in \quad L^3(Q_T),$$

$$\partial_t v_\sigma \rightharpoonup \partial_t v \quad in \quad L^2(0,T;W_2^{-1}(\mathbb{T})),$$

$$p_\sigma \rightharpoonup p \quad in \quad L^2(Q_T),$$

$$\theta_\sigma \to \xi \quad in \quad W_q^{1,0}(Q_T), \quad \nabla\theta_\sigma \to 0 \quad in \quad L^q(Q_T),$$

and (v, p, ξ) satisfy the equations

$$
\begin{aligned}
(\partial_t v, w) - (v \otimes v, \nabla w) + \frac{1}{2}\mu(\xi)(\nabla v, \nabla w) &= (f, w) + (p, \operatorname{div} w), \\
\forall w \in W_2^1(\mathbb{T}), \quad \text{for a.e.} \quad t &\in (0, T), \\
\operatorname{div} v = 0 \quad &\text{in} \quad Q_T, \\
v|_{t=0} &= v_0,
\end{aligned}
\tag{2.8}
$$

$$
\delta \frac{d\xi}{dt} + \xi = \mu(\xi)\|D(v)\|^2_{2,\mathbb{T}}, \quad \text{in} \quad (0, T), \tag{2.9}
$$

$$
\xi|_{t=0} = \int_{\mathbb{T}} \theta_0 \; dx. \tag{2.10}
$$

Theorem 2.3. *For each $\delta > 0$ let $(v_\delta, p_\delta, \xi_\delta)$ be a solution of (2.8)–(2.10). Suppose additionally*

$$
f \in L^2(0, T; W_2^1(\mathbb{T})) \quad \text{and} \quad v_0 \in J_2^1(\mathbb{T}),
$$

$$
\mu \quad \text{is monotone nonincreasing.}
$$

Then for any $\delta > 0$, $t_0 > 0$

$$
\xi_\delta \in W_2^1(t_0, T), \qquad \|\xi_\delta\|_{W_2^1(t_0,T)} \leqslant C,
$$

where C does not depend on δ. Moreover, if

$$
\delta \to 0
$$

then there exists (v, p) such that for some subsequence

$$
v_\delta \to v \quad \text{in} \quad W_2^{1,0}(Q_T), \quad p_\delta \to p \quad \text{in} \quad L^2(Q_T),
$$

and (v, p) satisfy the equation

$$
\left.
\begin{aligned}
\partial_t v + \operatorname{div}(v \otimes v) - \nu(\|D(v)\|^2_{2,\mathbb{T}})\Delta v &= f - \nabla p \\
\operatorname{div} v &= 0
\end{aligned}
\right\} \quad \text{in} \quad Q_T, \tag{2.11}
$$

where

$$
\nu(s) = \mu \circ \alpha(s),
$$

where α is the inverse function of $\beta(\theta) = \theta/\mu(\theta)$.

3. Proof of Theorem 2.1

The proof is based on the Schauder fixed point theorem. Let $\zeta \in L^q(Q_T)$ and (v_ζ, p_ζ) be a solution of the system

$$\begin{gathered}(\partial_t v_\zeta, w) - (v_\zeta \otimes v_\zeta, \nabla w) + (\mu(\zeta)D(v_\zeta), D(w)) = (f, w) + (p_\zeta, \operatorname{div} w),\\ \forall w \in W_2^1(\mathbb{T}), \quad \text{a.e.} \quad t \in (0,T),\\ \operatorname{div} v_\zeta = 0, \qquad (3.1)\\ v_\zeta(0) = v_0.\end{gathered}$$

Indeed, the existence follows from classical theory of fluid mechanics (see [3], for instance) as stated in the following proposition (the existence of pressure in $L^2(Q_T)$ is a consequence of the periodical boundary conditions).

Proposition 3.1. *There exists a unique solution v_ζ of (3.1) such that*

$$v_\zeta \in C([0,T]; L^2(\mathbb{T})) \cap L^2(0,T; J_2^1(\mathbb{T})), \quad \partial_t v_\zeta \in L^2(0,T; W_2^{-1}(\mathbb{T})),$$

$$p_\zeta \in L^2(Q_T), \qquad \int\limits_{\mathbb{T}} p_\zeta(x,t)\, dx = 0,$$

and the following estimate holds

$$\|v_\zeta\|_{L^\infty(0,T;L^2(\mathbb{T}))} + \|v_\zeta\|_{W_2^{1,0}(Q_T)} + \|p_\zeta\|_{2,Q_T} + \|\partial_t v_\zeta\|_{L^2(0,T;W_2^{-1}(\mathbb{T}))} \leqslant$$
$$\leqslant C\Big(\|f\|_{2,Q_T} + \|v_0\|_{2,\mathbb{T}}\Big).$$

Let θ_ζ be a solution of the problem

$$\int\limits_{Q_T} (-\delta\theta_\zeta \partial_t \eta + \sigma\nabla\theta_\zeta \cdot \nabla\eta - \delta\theta_\zeta v_\zeta \cdot \nabla\eta + \theta_\zeta \eta)\, dx\, dt =$$
$$= \int\limits_{Q_T} \mu(\zeta)|D(v_\zeta)|^2 \eta\, dx\, dt + \delta \int\limits_{\mathbb{T}} \theta_0(x)\eta(x,0)\, dx, \qquad (3.2)$$
$$\forall \eta \in C^1(\bar{Q}_T): \ \eta|_{t=T} = 0.$$

Proposition 3.2. *There is a unique solution θ_ζ of (3.2) such that*

$$\theta_\zeta \in L^\infty(0,T;L^1(\mathbb{T})) \cap W_q^{1,0}(Q_T), \quad \partial_t\theta_\zeta \in L^1(0,T;W_q^{-1}(\mathbb{T})),$$

and for every fixed $1 \leqslant q < \frac{4}{3}$, $\delta > 0$ there is a majorant $\mathcal{F}$ (which is a nondecreasing continuous function of its arguments) such that the estimate

$$\|\theta_\zeta\|_{L^\infty(0,T;L^1(\mathbb{T}))} + \sqrt{\sigma}\|\nabla\theta_\zeta\|_{q,Q_T} + \|\theta_\zeta\|_{\frac{3q}{2},Q_T} \leqslant \mathcal{F}\left(\|D(v_\zeta)\|^2_{2,Q_T}, \|\theta_0\|_{1,\mathbb{T}}\right)$$

holds. The function $\mathcal{F}$ depends only on q, δ, μ_1 and does not depend on σ.

Proof of Proposition 3.2. Denote $g_\zeta = \mu(\zeta)|D(v_\zeta)|^2$ and consider $g_m \in C^\infty(Q_T)$, $\theta_{0m} \in C^\infty(\mathbb{T})$, and $v_m \in C^\infty(\bar{Q}_T)$, $\operatorname{div} v_m = 0$ such that

$$g_m \to g_\zeta \quad \text{in} \quad L^1(Q_T), \qquad \theta_{0m} \to \theta_0 \quad \text{in} \quad L^1(\mathbb{T}),$$
$$v_m \to v_\zeta \quad \text{in} \quad W_2^{1,0}(Q_T).$$

Let θ_m solve the problem

$$\begin{gathered} \delta(\partial_t\theta_m + v_m\cdot\nabla\theta_m) - \sigma\Delta\theta_m + \theta_m = g_m, \quad \text{in} \quad Q_T, \\ \theta_m|_{t=0} = \theta_{0m}. \end{gathered} \tag{3.3}$$

The existence is consequence of classical theory for parabolic equations (see [5], for instance).

In order to pass to the limit on m ($m \to \infty$) let us establish some estimates in the following lemmas.

Lemma 3.1.

$$\|\theta_m\|_{L^\infty(0,T;L^1(\mathbb{T}))} \leqslant \frac{1}{\delta}\|g_\zeta\|_{1,Q_T} + \|\theta_0\|_{1,\mathbb{T}}.$$

Proof of Lemma 3.1. Define ϕ_ε and Φ_ε by

$$\phi_\varepsilon(s) := \begin{cases} -1, & s < -\varepsilon, \\ s/\varepsilon, & -\varepsilon \leqslant s \leqslant \varepsilon, \\ 1, & s > \varepsilon \end{cases} \qquad \Phi_\varepsilon(s) := \int_0^s \phi_\varepsilon(\tau)\,d\tau.$$

Multiplying (3.3) by the test function $\eta = \phi_\varepsilon(\theta_m) \in W_2^1(\mathbb{T})$ and integrating over $\mathbb{T}$ we obtain

$$\delta\frac{d}{dt}\int_{\mathbb{T}} \Phi_\varepsilon(\theta_m)\,dx + \sigma\int_{\mathbb{T}\cap[|\theta_m|<1]} |\nabla\theta_m|^2\,dx + \delta\int_{\mathbb{T}} v_m\cdot\nabla\theta_m\phi_\varepsilon(\theta_m)\,dx +$$

$$+\int_{\mathbb{T}} \theta_m \phi_\varepsilon(\theta_m)\ dx = \int_{\mathbb{T}} g_m \phi_\varepsilon(\theta_m)\ dx.$$

Note that

$$\int_{\mathbb{T}} v_m \cdot \nabla\theta_m \phi_\varepsilon(\theta_m)\ dx = \int_{\mathbb{T}} v_m \cdot \nabla\Phi_\varepsilon(\theta_m)\ dx = 0.$$

Hence

$$\delta \frac{d}{dt} \int_{\mathbb{T}} \Phi_\varepsilon(\theta_m)\ dx \leqslant \int_{\mathbb{T}} |g_m|\ dx$$

and we get the inequality

$$\|\Phi_\varepsilon(\theta_m)\|_{L^\infty(0,T;L^1(\mathbb{T}))} \leqslant \frac{1}{\delta}\|g_m\|_{1,Q_T} + \|\Phi_\varepsilon(\theta_{0m})\|_{1,\mathbb{T}}.$$

Passing to the limit as $\varepsilon \to 0$ we may obtain the conclusion of Lemma 3.1.

Lemma 3.2. *Let $s \in (0,1)$ then there exists a constant $C = C(s,\delta)$ such that*

$$\sigma \int_{Q_T} \frac{|\nabla\theta_m|^2}{(1+|\theta_m|)^{s+1}}\ dx\ dt \leqslant C\Big(\|\theta_m\|_{L^\infty(0,T;L^1(\mathbb{T}))} + \|g_\zeta\|_{1,Q_T} + \|\theta_0\|_{L^1(\mathbb{T})}\Big).$$

Proof of Lemma 3.2. Define

$$\psi(\tau) := \int_0^\tau \frac{dr}{(1+|r|)^{s+1}} = \operatorname{sign}\tau \int_0^{|\tau|} \frac{dr}{(1+r)^{s+1}}, \quad \Psi(\tau) := \int_0^\tau \psi(r)\ dr.$$

Note that

$$|\psi(\tau)| \leqslant C, \quad |\Psi(\tau)| \leqslant C(1+|\tau|).$$

Multiplying (3.3) by the test function $\eta = \psi(\theta_m)$ and integrating by parts, we obtain

$$\delta \frac{d}{dt} \int_{\mathbb{T}} \Psi(\theta_m)\ dx + \sigma \int_{\mathbb{T}} \psi'(\theta_m)|\nabla\theta_m|^2\ dx + \delta \int_{\mathbb{T}} v_m \cdot \nabla\theta_m \psi(\theta_m)\ dx +$$

$$+\int_{\mathbb{T}} \theta_m \psi(\theta_m)\ dx = \int_{\mathbb{T}} g_m \psi(\theta_m)\ dx.$$

Note that

$$\int_{\mathbb{T}} v_m \cdot \nabla\theta_m \psi(\theta_m)\ dx = \int_{\mathbb{T}} v_m \cdot \nabla\Psi(\theta_m)\ dx = 0.$$

Integrating the relation

$$\sigma \int_{\mathbb{T}} \frac{|\nabla\theta_m|^2}{(1+|\theta_m|)^{s+1}} dx = \int_{\mathbb{T}} g_m \psi(\theta_m)\ dx - \delta\frac{d}{dt}\int_{\mathbb{T}} \Psi(\theta_m)\ dx - \int_{\mathbb{T}} \theta_m\psi(\theta_m)\ dx$$

over $(0,T)$ and using Lemma 3.1, we may obtain the conclusion of Lemma 3.2.

Lemma 3.3. *For any $1 \leqslant q < \frac{4}{3}$ there exists a constant $C = C(q,\delta)$, independently of m and $\sigma > 1$, such that*

$$\|\sqrt{\sigma}\ \nabla\theta_m\|_{q,Q_T} \leqslant C.$$

Proof of Lemma 3.3. For any $\theta \in W^1_q(\mathbb{T})$ the interpolation inequality

$$\|\theta\|^{\frac{3q}{2}}_{\frac{3q}{2},\mathbb{T}} \leqslant C\|\theta\|^{\frac{q}{2}}_{1,\mathbb{T}}(\|\nabla\theta\|^q_{q,\mathbb{T}} + \|\theta\|^q_{1,\mathbb{T}}) \tag{3.4}$$

holds. Since $1 \leqslant q < \frac{4}{3}$ there exists $s \in (0,1)$ such that $(s+1)\frac{q}{2-q} = \frac{3q}{2}$. By Hölder inequality, we get

$$\sigma^{\frac{q}{2}} \int_{Q_T} |\nabla\theta_m|^q\ dxdt \leqslant$$

$$\leqslant \left(\sigma\int_{Q_T} \frac{|\nabla\theta_m|^2}{(1+|\theta_m|)^{s+1}}\ dxdt\right)^{\frac{q}{2}} \left(\int_{Q_T} (1+|\theta_m|)^{(s+1)\frac{q}{2-q}}\ dxdt\right)^{\frac{2-q}{2}} =$$

$$= \left(\sigma\int_{Q_T} \frac{|\nabla\theta_m|^2}{(1+|\theta_m|)^{s+1}}\ dx\ dt\right)^{\frac{q}{2}} \left(\int_{Q_T} (1+|\theta_m|)^{\frac{3q}{2}}\ dx\ dt\right)^{\frac{2-q}{2}}.$$

Using (3.4), we obtain

$$\sigma^{\frac{q}{2}}\|\nabla\theta_m\|^q_{q,Q_T} \leqslant$$

$$\leqslant C\left(\sigma\int_{Q_T} \frac{|\nabla\theta_m|^2}{(1+|\theta_m|)^{s+1}}\ dx\ dt\right)^{\frac{q}{2}} \left(1 + \|\theta_m\|^{\frac{q}{2}}_{L^\infty(0,T;L^1(\mathbb{T}))}\right)^{\frac{2-q}{2}} \times$$

$$\times \left(\|\nabla\theta_m\|^q_{q,Q_T} + \|\theta_m\|^q_{L^\infty(0,T;L^1(\mathbb{T}))}\right)^{\frac{2-q}{2}}.$$

Using the Young inequality $ab \leqslant \varepsilon a^{\frac{2}{2-q}} + C_\varepsilon b^{\frac{2}{q}}$, we obtain

$$(\sigma^{\frac{q}{2}} - 1)\|\nabla\theta_m\|^q_{q,Q_T} \leqslant$$

$$\leqslant C\left(1 + \|\theta_m\|^{\frac{2-q}{2}}_{L^\infty(0,T;L^1(\mathbb{T}))}\right)\sigma \int\limits_{Q_T} \frac{|\nabla\theta_m|^2}{(1+|\theta_m|)^{s+1}}\, dx\ dt +$$

$$+ C\|\theta_m\|^q_{L^\infty(0,T;L^1(\mathbb{T}))}.$$

Hence recalling the estimates of Lemmas 3.1 and 3.2 the conclusion of Lemma 3.3 follows.

Lemma 3.4.

$$\|\theta_m\|_{\frac{3q}{2},Q_T} \leqslant C\|\theta_m\|^{\frac{1}{3}}_{L^\infty(0,T;L^1(\mathbb{T}))}\left(\|\nabla\theta_m\|^{\frac{2}{3}}_{q,Q_T} + \|\theta_m\|^{\frac{2}{3}}_{L^\infty(0,T;L^1(\mathbb{T}))}\right).$$

Proof of Lemma 3.4. This estimate results directly from the interpolation inequality (3.4).

Proof of Proposition 3.2 (continuation). From Lemma 3.3, there exists a subsequence $\{\theta_m\}$ such that

$$\theta_m \rightharpoonup \theta_\zeta \quad \text{in} \quad W^{1,0}_q(Q_T).$$

Passing to the limit in the identity

$$\int\limits_{Q_T} (-\delta\theta_m\partial_t\eta + \sigma\nabla\theta_m\cdot\nabla\eta - \delta\theta_m v_m\cdot\nabla\eta + \theta_m\eta)\ dx\ dt =$$

$$= \int\limits_{Q_T} g_m\eta\ dx\ dt + \delta\int\limits_{\mathbb{T}} \theta_{0m}(x)\eta(x,0)\ dx,$$

$$\forall \eta \in C^1(\bar{Q}_T): \quad \eta|_{t=T} = 0,$$

we obtain (3.2). The uniqueness of a solution of this problem follows from the estimate obtained in Lemma 3.1 for θ_ζ.

In order to prove that $\partial_t\theta_\zeta \in L^1(0,T;W^{-1}_q(\mathbb{T}))$ we note that from (3.2) and the definition of the distribution derivative $\partial_t\theta_\zeta$ the identity

$$\delta\langle\partial_t\theta_\zeta, w\rangle = \int\limits_{\mathbb{T}} (g_\zeta w - \sigma\nabla\theta_\zeta\cdot\nabla w + \delta\theta_\zeta v_\zeta\cdot\nabla w - \theta_\zeta w)\ dx$$

$$\forall w \in W^1_{q'}(\mathbb{T})$$

follows. Hence, it yields

$$\delta\|\partial_t\theta_\zeta\|_{W_q^{-1}(\mathbb{T})} = \delta \sup_{\|\eta\|_{W^1_{q'}(\mathbb{T})}\leqslant 1} |\langle \partial_t\theta_\zeta, \eta\rangle| =$$

$$= \sup_{\|\eta\|_{W^1_{q'}(\mathbb{T})}\leqslant 1} \Big| \int_{\mathbb{T}} (g_\zeta w - \sigma\nabla\theta_\zeta\cdot\nabla w + \delta\theta_\zeta v_\zeta\cdot\nabla w - \theta_\zeta w)\, dx\Big|.$$

Note that for $1 \leqslant q < \frac{4}{3}$ and $n = 2$, we have the estimates

$$\|w\|_{\infty,\mathbb{T}} \leqslant C\|w\|_{W^1_{q'}(\mathbb{T})},$$

$$\|\theta_\zeta v_\zeta\|_{q,\mathbb{T}} \leqslant \|v_\zeta\|_{2,\mathbb{T}}\|\theta_\zeta\|_{\frac{2q}{2-q},\mathbb{T}} \leqslant C\|v_\zeta\|_{2,\mathbb{T}}\|\theta_\zeta\|_{W^1_q(\mathbb{T})}.$$

Thus, we get

$$\|\partial_t\theta_\zeta\|_{W_q^{-1}(\mathbb{T})} \leqslant$$
$$\leqslant C(\delta)\Big(\|g_\zeta\|_{1,\mathbb{T}} + \sigma\|\nabla\theta_\zeta\|_{q,\mathbb{T}} + \|v_\zeta\|_{2,\mathbb{T}}\|\theta_\zeta\|_{W^1_q(\mathbb{T})} + \|\theta_\zeta\|_{1,\mathbb{T}}\Big)$$

and consequently the estimate

$$\|\partial_t\theta_\zeta\|_{L^1(0,T;W_q^{-1}(\mathbb{T}))} \leqslant C\Big(\|D(v_\zeta)\|^2_{2,Q_T} + \sigma\|\nabla\theta_\zeta\|_{L^1(0,T;L^q(\mathbb{T}))} +$$
$$+\|v_\zeta\|_{L^\infty(0,T;L^2(\mathbb{T}))}\|\theta_\zeta\|_{L^1(0,T;W^1_q(\mathbb{T}))} + \|\theta_\zeta\|_{1,Q_T}\Big) \tag{3.5}$$

holds. Proposition 3.2 is proved.

Proof of Theorem 2.1. Consider $\zeta \in L^q(Q_T)$ and let v_ζ, θ_ζ be the solutions of problems (3.1), (3.2), respectively. Define the mapping

$$\mathcal{T} : L^q(Q_T) \to L^q(Q_T), \quad \mathcal{T}(\zeta) = \theta_\zeta \in W^{1,0}_q(Q_T).$$

In order to prove that this operator is compact, we recall from Proposition 3.2 that

$$\begin{aligned} &\theta_\zeta && \text{is bounded in} \quad L^q(0,T;W^1_q(\mathbb{T})),\\ &\partial_t\theta_\zeta && \text{is bounded in} \quad L^1(0;T;W^{-1}_q(\mathbb{T})) \end{aligned} \tag{3.6}$$

and we note that we have also the imbeddings

$$W^1_q(\mathbb{T}) \hookrightarrow\hookrightarrow L^q(\mathbb{T}) \hookrightarrow W^{-1}_q(\mathbb{T}).$$

By Corollary 4, §8 of [7] we get

$$\theta_\zeta \quad \text{is precompact in} \quad L^q(0,T;L^q(\mathbb{T})), \tag{3.7}$$

and hence $\mathcal{T}$ is compact.

In order to prove that this operator is continuous, let us take $\zeta_m \to \zeta$ in $L^q(Q_T)$. For each $m \in \mathbb{N}$, let us consider $v_m = v_{\zeta_m}$ the solution of system (3.1) corresponding to ζ_m. We claim that $\theta_m \equiv \theta_{\zeta_m} \to \theta \equiv \theta_\zeta$ in $L^q(Q_T)$. Note that by Proposition 3.1 we have the estimate

$$\|v_m\|_{W_2^{1,0}(Q_T)} + \|p_m\|_{2,Q_T} + \|\partial_t v_m\|_{L^2(0,T;W_2^{-1}(\mathbb{T}))} \leqslant C,$$

where C does not depend on m. Hence we can extract a subsequence $\{v_m\}$ such that

$$\begin{gathered} v_m \rightharpoonup v \quad \text{in} \quad W_2^{1,0}(Q_T), \\ v_m \quad \text{is bounded in} \quad L^4(Q_T), \\ v_m \to v \quad \text{in} \quad L^3(Q_T), \\ p_m \rightharpoonup p \quad \text{in} \quad L^2(Q_T). \end{gathered} \tag{3.8}$$

Obviously (v,p) is the unique solution of the problem (3.1) corresponding to ζ, i.e., $v = v_\zeta$, $p = p_\zeta$. The lack of strong convergence of $D(v_m)$ in $L^2(Q_T)$ does not allow us to pass to the limit in the right-hand side of the energy transfer equation (3.2) directly. Instead of it we use the special representation of the term $\mu(\zeta_m)|D(v_m)|^2$ which is analogue to the "local energy identity" in the Navier–Stokes theory. Let $\eta \in C^1(\bar{Q}_T)$, $\eta|_{t=0} = 0$ be an arbitrary scalar function and take in (3.1) $w = v_m\eta \in \overset{\circ}{W}{}_2^1(\mathbb{T})$ for a.e. $t \in (0,T)$. We obtain the identity

$$\int_{\mathbb{T}} \mu(\zeta_m)|D(v_m)|^2\eta \, dx = (f, v_m\eta) + (p_m, v_m \cdot \nabla\eta) - (\partial_t v_m, v_m\eta) +$$
$$+ (v_m \otimes v_m, \eta\nabla v_m + v_m \otimes \nabla\eta) - (\mu(\zeta_m)D(v_m), v_m \otimes \nabla\eta).$$

Note that

$$(v_m \otimes v_m, \eta\nabla v_m) = -\frac{1}{2}(|v_m|^2 v_m, \nabla\eta)$$

and also

$$2\int_0^T (\partial_t v_m, v_m\eta) \, dt =$$
$$= (|v_m(T)|^2, \eta(T)) - (|v_m(0)|^2, \eta(0)) - \int_{Q_T} |v_m|^2 \partial_t\eta dx dt.$$

(For smooth v_m the last identity is obvious and for $v_m \in W_2^{1,0}(Q_T)$ such that $\partial_t v_m \in L^2(0,T;W_2^{-1}(\mathbb{T}))$ it can be justified by approximations since the imbedding of $\mathcal{W} \equiv \{v \in W_2^{1,0}(Q_T) \colon \partial_t v \in L^2(0,T;W_2^{-1}(\mathbb{T}))\}$ into $C([0,T];L^2(\mathbb{T}))$ is continuous.) Hence we arrive at the identity

$$\int\limits_{Q_T} \mu(\zeta_m)|D(v_m)|^2\eta \; dxdt = \int\limits_{Q_T} f\cdot v_m\eta \; dxdt +$$
$$+\int\limits_{Q_T} p_m v_m\cdot\nabla\eta \; dxdt + \frac{1}{2}(|v_0|^2,\eta(0)) + \frac{1}{2}\int\limits_{Q_T}|v_m|^2\partial_t\eta dxdt + \tag{3.9}$$
$$+\frac{1}{2}\int\limits_{Q_T}|v_m|^2 v_m\cdot\nabla\eta \; dxdt - \int\limits_{Q_T}\mu(\zeta_m)D(v_m) : v_m\otimes\nabla\eta \; dxdt.$$

The convergences (3.8) allow us to pass to the limit in the right hand side of (3.9). Using for (v,ζ) the identity which is similar to (3.9) we obtain

$$\lim_{m\to\infty}\int\limits_{Q_T}\mu(\zeta_m)|D(v_m)|^2\eta \; dxdt = \int\limits_{Q_T}\mu(\zeta)|D(v)|^2\eta \; dxdt,$$
$$\forall \eta \in C(\bar{Q}_T): \; \eta|_{t=T} = 0. \tag{3.10}$$

Moreover, from (3.6), (3.7) applied to θ_m the convergences

$$\theta_m \rightharpoonup \theta \quad \text{in} \quad W_q^{1,0}(Q_T),$$
$$\theta_m \to \theta \quad \text{in} \quad L^q(Q_T) \tag{3.11}$$

follow. Passing to the limit as $m \to \infty$ in the identity

$$\int\limits_{Q_T}(-\delta\theta_m\partial_t\eta + \sigma\nabla\theta_m\cdot\nabla\eta - \delta\theta_m v_m\cdot\nabla\eta + \theta_m\eta) \; dx \; dt =$$
$$= \int\limits_{Q_T}\mu(\zeta_m)|D(v_m)|^2\eta \; dx \; dt + \delta\int\limits_{\mathbb{T}}\theta_0(x)\eta(x,0) \; dx, \tag{3.12}$$
$$\forall \eta \in C^1(\bar{Q}_T): \; \eta|_{t=T} = 0,$$

taking into account (3.10) and the uniqueness of solution we obtain that $\theta = \theta_\zeta = \mathcal{T}(\zeta)$. From (3.11) we conclude that $\mathcal{T}$ is continuous in $L^q(Q_T)$.

Finally, if we define a ball of the space $L^q(Q_T)$ such that, its radius being sufficiently large, $\mathcal{T}$ maps that ball into itself. Hence by Schauder fixed point theorem there exists $\zeta \in L^q(Q_T)$ such that $\mathcal{T}(\zeta) = \zeta$. Obviously $(v,\theta) = (v_\zeta,\theta_\zeta)$ is the required solution of the problem (1.1)–(1.4) having all necessary properties. Theorem 2.1 is proved.

4. Proof of Theorem 2.2

Let $\sigma \to \infty$. Estimates (2.5), (2.7) allow us to extract a subsequence such that

$$\begin{gathered} v_\sigma \rightharpoonup v \quad \text{in} \quad W_2^{1,0}(Q_T), \quad \partial_t v_\sigma \rightharpoonup \partial_t v \quad \text{in} \quad L^2(0,T;W_2^{-1}(\mathbb{T})), \\ v_\sigma \quad \text{is bounded in} \quad L^4(Q_T), \\ v_\sigma \to v \quad \text{in} \quad L^3(Q_T), \\ p_\sigma \rightharpoonup p \quad \text{in} \quad L^2(Q_T), \\ \theta_\sigma \rightharpoonup \theta \quad \text{in} \quad W_q^{1,0}(Q_T), \\ \nabla\theta_\sigma \to 0 \quad \text{in} \quad L^q(Q_T). \end{gathered} \tag{4.1}$$

Hence, $\nabla\theta = 0$ and there exists $\xi \in L^q(0,T)$ such that $\theta = \xi$. Denote by

$$\tilde\theta_\sigma(t) \equiv \int\limits_{\mathbb{T}} \theta_\sigma(x,t)\, dx.$$

Then

$$\tilde\theta_\sigma \rightharpoonup \xi \quad \text{in} \quad L^q(0,T).$$

Let $\varphi \in C^1([0,T])$, $\varphi(T) = 0$ be an arbitrary function and take in (2.4) $\eta(x,t) = \varphi(t)$, $\forall x \in \mathbb{T}$, $\forall t \in (0,T)$. Hence we obtain the relation

$$\begin{gathered} \int\limits_0^T (-\delta\tilde\theta_\sigma\varphi' + \tilde\theta_\sigma\varphi)\, dt = \int\limits_0^T \varphi(t) \int\limits_{\mathbb{T}} \mu(\theta_\sigma)|D(v_\sigma)|^2\, dxdt + \\ +\delta\varphi(0)\int\limits_{\mathbb{T}} \theta_0(x)\, dx, \quad \forall\varphi \in C^1([0,T]):\ \varphi(T) = 0, \end{gathered} \tag{4.2}$$

from which the inclusion $\frac{d\tilde\theta_\sigma}{dt} \in L^1(0,T)$ and the identities

$$\delta\frac{d\tilde\theta_\sigma}{dt} + \tilde\theta_\sigma = \int\limits_{\mathbb{T}} \mu(\theta_\sigma)|D(v_\sigma)|^2\, dx, \quad \tilde\theta_\sigma(0) = \int\limits_{\mathbb{T}} \theta_0(x)\, dx$$

follow. Thus, for a fixed $\delta > 0$

$$\tilde\theta_\sigma \quad \text{is bounded in} \quad W_1^1(0,T)$$

and we can extract a subsequence $\{\tilde\theta_\sigma\}$ such that

$$\tilde\theta_\sigma \to \xi \quad \text{in} \quad L^q(0,T)$$

and using Poincaré inequality, it yields

$$\int\limits_{Q_T} |\theta_\sigma - \xi|^q \, dxdt \leqslant \int\limits_{Q_T} |\theta_\sigma - \tilde\theta_\sigma|^q \, dxdt + \int\limits_0^T |\tilde\theta_\sigma - \xi|^q \, dt \leqslant$$

$$\leqslant C \int\limits_{Q_T} |\nabla\theta_\sigma|^q \, dxdt + \int\limits_0^T |\tilde\theta_\sigma - \xi|^q \, dt.$$

Hence

$$\theta_\sigma \to \xi \quad \text{in} \quad L^q(Q_T).$$

This convergence allows to pass to the limit as $\sigma \to \infty$ in the equation

$$\int\limits_0^T (\partial_t v_\sigma, w) \, dt + \int\limits_{Q_T} \mu(\theta_\sigma) D(v_\sigma) : D(w) \, dxdt - \int\limits_{Q_T} v_\sigma \otimes v_\sigma : \nabla w \, dxdt =$$

$$= \int\limits_{Q_T} (f \cdot w + p_\sigma \mathrm{div}\ w) \, dxdt, \quad \forall w \in C^1(\bar Q_T),$$

and hence (2.8) follows.

To pass to the limit in (4.2), we shall use the special representation of type (3.9) for the right-hand side of (4.2) again. For any scalar $\varphi \in C^1([0,T])$, $\varphi(T) = 0$ we have

$$\int\limits_{Q_T} \mu(\theta_\sigma) |D(v_\sigma)|^2 \varphi \, dxdt =$$

$$= \int\limits_{Q_T} f \cdot v_\sigma \varphi \, dxdt + \frac{1}{2} \|v_0\|^2_{2,\mathbb{T}} \varphi(0) + \frac{1}{2} \int\limits_0^T \|v_\sigma\|^2_{2,\mathbb{T}} \varphi' dt. \quad (4.3)$$

Passing to the limit in this identity and taking into account the similar relation for (v,ξ):

$$\int_0^T \mu(\xi)\|D(v)\|^2_{2,\mathbb{T}}\varphi\ dt =$$

$$= \int_{Q_T} f\cdot v\varphi\ dxdt + \frac{1}{2}\|v_0\|^2_{2,\mathbb{T}}\varphi(0) + \frac{1}{2}\int_0^T \|v\|^2_{2,\mathbb{T}}\varphi' dt. \quad (4.4)$$

we obtain the identity

$$\lim_{\sigma\to\infty}\int_{Q_T}\mu(\theta_\sigma)|D(v_\sigma)|^2\varphi\ dxdt = \int_0^T \mu(\xi)\|D(v)\|^2_{2,\mathbb{T}}\varphi\ dt,$$

$$\forall\varphi\in C^1([0,T]),\quad \varphi(T)=0.$$

Passing to the limit as $\sigma\to\infty$ in (4.2) we get the identity from which (2.9), (2.10) follows. Theorem 2.2 is proved.

5. Proof of Theorem 2.3

Let us begin to state some additional regularity.

Proposition 5.1. *Assume that $\delta>0$ and let $(v_\delta,p_\delta,\xi_\delta)$ be a solution of* (2.8)–(2.10).

(i) *Let $f\in L^2(Q_T)$, $v_0\in J^1_2(\mathbb{T})$. Then*

$$\nabla v_\delta\in L^\infty(0,T;L^2(\mathbb{T}))\cap W^{1,0}_2(Q_T),\quad \partial_t v_\delta\in L^2(Q_T),$$

$$\nabla p_\delta\in L^2(Q_T),$$

and the following estimate holds:

$$\|\nabla v_\delta\|_{L^\infty(0,T;L^2(\mathbb{T}))}+\|\nabla^2 v_\delta\|_{2,Q_T}+\|\partial_t v_\delta\|_{2,Q_T}+\|\nabla p_\delta\|_{2,Q_T}\leqslant C_1(f,v_0).$$

(ii) *Assume additionally $f\in L^2(0,T;W^1_2(\mathbb{T}))$. Then for any smooth cut-off function $\chi\in C^\infty([0,T])$ vanishing at $t=0$*

$$\chi\nabla^2 v_\delta\in L^\infty(0,T;L^2(\mathbb{T}))\cap W^{1,0}_2(Q_T),\quad \chi\partial_t\nabla v_\delta\in L^2(Q_T),$$

$$\chi\nabla^2 p_\delta\in L^2(Q_T),$$

and the following estimate holds:

$$\|\chi\nabla^2 v_\delta\|_{L^\infty(0,T;L^2(\mathbb{T}))}+\|\chi\nabla^3 v_\delta\|_{2,Q_T}+\|\chi\partial_t\nabla v_\delta\|_{2,Q_T}+$$

$$+\|\chi \nabla^2 p_\delta\|_{2,Q_T} \leqslant C_2(f, v_0),$$

for some positive constants C_1 and C_2.

Proposition 5.1 is well-known in the case of the $2D$-Navier-Stokes system under periodical boundary conditions and its proof is standard. For the reader convenience we derive here required estimates assuming (v,p) are smooth enough. These estimates can be easily justified, for instance, with the help of finite differences.

Sketch of the proof of Proposition 5.1.

Part (i): Within this part we skip the index δ in notations: $v \equiv v_\delta$ etc. Taking in (NS1) $w = \Delta v \in J_2^1(\mathbb{T})$ after integration by parts we obtain for a.e. $t \in (0,T)$

$$\frac{d}{dt}\|\nabla v\|^2_{2,\mathbb{T}} + \mu(\xi)\|\nabla^2 v\|^2_{2,\mathbb{T}} = 2((v\otimes v)_{,k}, \nabla v_{,k}) - 2(f, \Delta v),$$

and consequently

$$\frac{d}{dt}\|\nabla v\|^2_{2,\mathbb{T}} + \mu_0\|\nabla^2 v\|^2_{2,\mathbb{T}} \leq C\|\nabla(v\otimes v)\|^2_{2,\mathbb{T}} + C\|f\|^2_{2,\mathbb{T}}.$$

By Ladyzhenskaya inequality (see 3)

$$\|\nabla(v\otimes v)\|^2_{2,\mathbb{T}} \leqslant C\|v\|^2_{4,\mathbb{T}}\|\nabla v\|^2_{4,\mathbb{T}} \leqslant C\|v\|^2_{4,\mathbb{T}}\|\nabla v\|_{2,\mathbb{T}}\|\nabla v\|_{W^1_2(\mathbb{T})} \leqslant$$
$$\leqslant \varepsilon\|\nabla^2 v\|^2_{2,\mathbb{T}} + C_\varepsilon(1+\|v\|^4_{4,\mathbb{T}})\|\nabla v\|^2_{2,\mathbb{T}} \tag{5.1}$$

and hence it follows

$$\frac{d}{dt}\|\nabla v\|^2_{2,\mathbb{T}} + \frac{\mu_0}{2}\|\nabla^2 v\|^2_{2,\mathbb{T}} \leq C(1+\|v\|^4_{4,\mathbb{T}})\|\nabla v\|^2_{2,\mathbb{T}} + C\|f\|^2_{2,\mathbb{T}}.$$

Using the Gronwall lemma we obtain the estimate

$$\|\nabla v\|_{L^\infty(0,T;L^2(\mathbb{T}))} + \|\nabla^2 v\|_{2,Q_T} \leqslant C(f, \mu_0, v_0).$$

Taking in (2.8) the test function $w = \nabla\varphi$ where $\varphi \in W^2_2(\mathbb{T})$ is an arbitrary scalar function we derive the identity

$$(p, \Delta\varphi) = (g, \nabla^2\varphi) - (f, \nabla\varphi), \quad \forall \varphi \in W^2_2(\mathbb{T}),$$

where $g \equiv \frac{1}{2}\mu(\xi)D(v) - v\otimes v$. As $g \in W^1_2(\mathbb{T};\mathbb{M}^{2\times 2})$ for a.e. $t \in (0,T)$ we can find $q \in W^1_2(\mathbb{T})$ such that $\int_{\mathbb{T}} q\,dx = 0$ and

$$(\nabla q, \nabla\varphi) = (\operatorname{div} g + f, \nabla\varphi), \quad \forall \varphi \in W^1_2(\mathbb{T}), \tag{5.2}$$

$$\|q\|_{W_2^1(\mathbb{T})} \leqslant C(\|f\|_{2,\mathbb{T}} + \|\mathrm{div}\ g\|_{2,\mathbb{T}}).$$

Existence and uniqueness of q follows from the fact that the right hand side of (davlenie) is a bounded linear functional over $W_2^1(\mathbb{T})$. It is easy to see that

$$(p-q, \Delta\varphi) = 0 \quad \forall \varphi \in W_2^2(\mathbb{T}),$$

and choosing $\varphi \in W_2^2(\mathbb{T})$, $\int_{\mathbb{T}} \varphi\ dx = 0$ satisfying the condition $\Delta\varphi = p - q$ we obtain $p = q$ a.e. in $\mathbb{T}$. Hence $p \in W_2^1(\mathbb{T})$ and the estimate

$$\|p\|_{W_2^1(\mathbb{T})} \leqslant C(\|v \otimes v\|_{W_2^1(\mathbb{T})} + \|f\|_{2,\mathbb{T}}), \qquad \text{a.e. } t \in (0,T)$$

holds. Taking into account (5.1), we obtain for pressure the required estimate.

From the equation $\partial_t v = f - \nabla p + \frac{1}{2}\mu(\xi)\Delta v - \mathrm{div}(v \otimes v)$ a.e. in Q_T we get an estimate for $\partial_t v$. Part (i) is proved.

Part (ii). Similar to Part (i). Proposition 5.1 is proved.

Proof of Theorem 2.3.

Let $(v_\delta, p_\delta, \xi_\delta)$ be a solution of the system (2.8)–(2.3) and denote $g_\delta(t) = \|D(v_\delta(t))\|^2_{2,\mathbb{T}}$. Multiplying the relation

$$\delta \frac{d\xi_\delta}{dt} + \xi_\delta = \mu(\xi_\delta) g_\delta \tag{5.3}$$

by ξ_δ and integrating over $(0,T)$, we obtain

$$\delta \sup_{t\in[0,T]} |\xi_\delta(t)|^2 + \int_0^T |\xi_\delta(t)|^2\ dt \leqslant \delta|\xi_0|^2 + C\int_0^T |g_\delta(t)|^2\ dt.$$

As g_δ is bounded in $L^\infty(0,T)$ (see Part (i) of Proposition 5.1) we conclude

$$\xi_\delta \quad \text{is bounded in} \quad L^2(0,T).$$

Multiplying (5.3) by $\chi\frac{d\xi_\delta}{dt}$, where $\chi \in C^\infty([0,T])$ is a nonnegative cut-off function vanishing at $t = 0$, we obtain the identity

$$\delta\chi\left|\frac{d\xi_\delta}{dt}\right|^2 + \frac{1}{2}\frac{d}{dt}(\chi(t)|\xi_\delta(t)|^2) = \frac{1}{2}\frac{d\chi}{dt}|\xi_\delta(t)|^2 + \chi g_\delta \frac{d}{dt}M(\xi_\delta), \tag{5.4}$$

where $M(t) \equiv \int_0^t \mu(s)\ ds$. Integrating (5.4) over $(0,t)$ and taking into account relations

$$\int_0^t \chi g_\delta \frac{d}{d\tau} M(\xi_\delta)\ d\tau =$$

$$= \chi(t) g_\delta(t) M(\xi_\delta(t)) - \int_0^t \frac{d\chi}{dt} g_\delta M(\xi_\delta)\, d\tau - \int_0^t M(\xi_\delta)\chi \frac{dg_\delta}{dt}\, d\tau,$$

$|M(\xi_\delta)| \leqslant \mu_1 |\xi_\delta|$, $ab \leqslant \varepsilon a^2 + C_\varepsilon b^2$, and $|\frac{d\chi}{dt}| \leqslant L$ we deduce the estimate

$$\delta \int_0^T \chi \Big|\frac{d\xi_\delta}{dt}\Big|^2\, dt + \sup_{t\in[0,T]} (\chi(t)|\xi_\delta(t)|^2) \leqslant$$

$$\leqslant C \left\{ \sup_{t\in[0,T]} (\chi(t)|g_\delta(t)|^2) + (L+1) \int_0^T (|g_\delta|^2 + |\xi_\delta|^2)\, dt + \int_0^T \chi \Big|\frac{dg_\delta}{dt}\Big|^2\, dt \right\}.$$

We have

$$\int_0^T \chi \Big|\frac{dg_\delta}{dt}\Big|^2\, dt = \int_0^T \chi \Big|\frac{d}{dt} \|D(v_\delta)\|^2_{2,\mathbb{T}}\Big|^2\, dt =$$

$$= 4 \int_0^T \chi \Big| \int_{\mathbb{T}} D(v_\delta) : \partial_t D(v_\delta)\, dx \Big|^2\, dt.$$

Since

$$\Big| \int_{\mathbb{T}} D(v_\delta) : \partial_t D(v_\delta)\, dx \Big|^2 \leqslant \|\nabla v_\delta\|^2_{2,\mathbb{T}} \|\partial_t \nabla v_\delta\|^2_{2,\mathbb{T}}$$

and from Proposition 5.1, we finally obtain

$$\int_0^T \chi \Big|\frac{dg_\delta}{dt}\Big|^2\, dt \leqslant C \sup_{t\in[0,T]} \|\nabla v_\delta(t)\|^2_{2,\mathbb{T}} \int_0^T \chi \|\partial_t \nabla v_\delta\|^2_{2,\mathbb{T}}\, dt.$$

Hence

$$\sqrt{\chi}\, \frac{dg_\delta}{dt} \quad \text{is bounded in} \quad L^2(0,T), \tag{5.5}$$

and consequently

$$\sqrt{\delta\chi}\, \frac{d\xi_\delta}{dt} \quad \text{is bounded in} \quad L^2(0,T). \tag{5.6}$$

Assume that μ is of class C^1, $\mu'(s) \leqslant 0$.

Differentiating (5.3) with respect to t, multiplying by $\chi^2 \frac{d\xi_\delta}{dt}$ and integrating over t we get the identity

$$\frac{\delta}{2}\frac{d}{dt}\Big(\chi^2\Big|\frac{d\xi_\delta}{dt}\Big|^2\Big) + \chi^2\Big|\frac{d\xi_\delta}{dt}\Big|^2 =$$

$$= \chi^2\mu'(\xi_\delta)\Big|\frac{d\xi_\delta}{dt}\Big|^2 g_\delta + \mu(\xi_\delta)\chi^2\frac{d\xi_\delta}{dt}\frac{dg_\delta}{dt} + \delta\chi\frac{d\chi}{dt}\Big|\frac{d\xi_\delta}{dt}\Big|^2.$$

As $\mu' \leqslant 0$ and $g_\delta \geqslant 0$, we get the inequality

$$\frac{\delta}{2}\frac{d}{dt}\Big(\chi^2\Big|\frac{d\xi_\delta}{dt}\Big|^2\Big) + \chi^2\Big|\frac{d\xi_\delta}{dt}\Big|^2 \leqslant \mu(\xi_\delta)\chi^2\frac{d\xi_\delta}{dt}\frac{dg_\delta}{dt} + \delta\chi\frac{d\chi}{dt}\Big|\frac{d\xi_\delta}{dt}\Big|^2. \qquad (5.7)$$

Integrating this identity over $(0,T)$, using $ab \leqslant \varepsilon a^2 + C_\varepsilon b^2$ and $|\frac{d\chi}{dt}| \leqslant L$ we find the estimate

$$\delta \sup_{t\in(0,T)} \chi^2\Big|\frac{d\xi_\delta}{dt}\Big|^2 + \int_0^T \chi^2\Big|\frac{d\xi_\delta}{dt}\Big|^2\, dt \leqslant$$

$$\leqslant C\int_0^T \chi^2\Big|\frac{dg_\delta}{dt}\Big|^2\, dt + L\delta\int_0^T \chi\Big|\frac{d\xi_\delta}{dt}\Big|^2\, dt.$$

Taking into account (5.5) and (5.6), we get

$$\chi\frac{d\xi_\delta}{dt} \quad \text{is bounded in} \quad L^2(0,T).$$

Hence the estimate

$$\|\chi\xi_\delta\|_{W_2^1(0,T)} \leqslant C \qquad (5.8)$$

follows and we can extract a subsequence $\{\xi_\delta\}$ such that for any $t_0 > 0$

$$\xi_\delta \to \xi_* \quad \text{in} \quad L^2(t_0,T), \qquad \delta\xi_\delta \rightharpoonup 0 \quad \text{in} \quad W_2^1(t_0,T),$$

$$\xi_\delta \to \xi_* \quad \text{a.e. in} \quad (0,T).$$

From the estimates of Proposition 5.1 and (5.5), it also follows that $\{\nabla v_\delta\}$ is precompact in $L^2(Q_T)$ and we may extract a subsequence $\{v_\delta\}$ such that

$$v_\delta \to v \quad \text{in} \quad W_2^{1,0}(Q_T), \quad g_\delta \to \|D(v)\|^2_{2,\mathbb{T}} \quad \text{a.e.} \quad t\in(0,T).$$

Passing to the limit in the identity (5.3), we get

$$\xi_*(t) = \mu(\xi_*(t))\|D(v(t))\|^2_{2,\mathbb{T}} \quad \text{for a.e.} \quad t\in(0,T).$$

Resolving this equation with respect to ξ_* if α denotes the inverse function of $\beta(\xi) = \xi/\mu(\xi)$, we get the relation

$$\xi_*(t) = \alpha\Big(\|D(v(t))\|^2_{2,\mathbb{T}}\Big) \quad \text{for a.e.} \quad t \in (0, T).$$

Passing to the limit in the equation (2.8), we derive the (2.11) for (v, p). Theorem 2.3 is proved for $\mu \in C^1$, $\mu' \leqslant 0$. If $\mu \in C^0$ and only monotone nonincreasing, we regularize μ in order to obtain (5.7) and (5.8), and the proof follows as in the previous case.

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