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**J. H. Videman, S. A. Nazarov, A. Sequeira**

# ASYMPTOTIC MODELLING OF A PISTON WITH A COMPLETELY WETTED SURFACE

ABSTRACT. A quasi-stationary model is constructed for a lubricated piston by coupling a two-dimensional Reynolds equation with three-dimensional Navier-Stokes equations. This hybrid problem is shown to be well-posed in weighted Sobolev spaces with attached asymptotics

Dedicated to the 70th birthday of
Professor Vsevolod Solonnikov

## 1. Introduction

The problem of modelling the motion of a viscous liquid within a container that surrounds a moving piston or a rotating shaft has attracted the attention of scientists since the late $19th$ century, cf. [1]. The mathematical formulations of the corresponding problems have, however, been non-exhaustive from a physical point of view since they do not allow for a thin flow to take place between the piston (or the shaft) and the surrounding walls, cf. [2, 3] and, hence, model rather an injection than a lubrication problem. Besides, the assumption that the lateral walls of the piston stay dry causes a jump in the Dirichlet data for the fluid velocity at the edge where the piston meets the container walls and, consequently, makes the square integral of the velocity gradient (the Dirichlet integral) divergent. In [2], V. A. Solonnikov gave the final answer to the injection problem by showing that the equations admit solutions in special weighted function spaces where the velocity field remains bounded. We also refer to [3] where the obstacles, i.e. strong singularities in the velocity gradient, were avoided by changing the Dirichlet boundary conditions to partial slip conditions.

Here, we consider the problem of a slow (nearly steady-state) motion of a piston in a fixed finite cylinder where the lateral walls of the piston

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and the container are assumed to be separated by a narrow slit filled with a thin liquid film. Since we suppose that the motion of the piston is quasi-stationary, we can fix a reference position for the piston and model the problem by considering the steady motion of a Navier-Stokes fluid around the (fixed) piston with prescribed non-zero fluid velocity on its surface. Note that the same formulation also covers the case of a percolation flow through the piston (the piston acting as a filter) or through the walls of the container and that if the piston is circular and rotating the problem is related to the classical journal bearing problem in lubrication theory, see Section 4.2.

Let us introduce our notation. Consider a fixed and bounded cylindrical domain

$$Q = \{x = (y, z) \,:\, y = (y_1, y_2) \in \Omega\,,\, z \in (L_-\,, L_+)\}\,,$$

where $\Omega \subset \mathbb{R}^2$ is a simply-connected smooth cross-section and $L_- < L_+$. Assume that inside $Q$ there is a smaller cylinder, $Q_0 = \omega \times (l_-, l_+)$, with $\omega \subset \Omega$ and $L_- < l_- < l_+ < L_+$, and that the vertical walls of the cylinders $Q$ and $Q_0$ are very close to each other. Denote the average width of the gap by $h > 0$, a small parameter, and assume that the domain $Q_h := Q \setminus \overline{Q_0}$ is filled with a viscous incompressible fluid. The cylinders walls can be non-parallel or even distorted although, to simplify the notation, we consider here only the case of parallel walls; see also Remark 3.2.

Denoting by $\partial Q$ and $\partial Q_0$ the boundaries of $Q$ and $Q_0$, respectively; thus $\partial Q_h = \partial Q \cup \partial Q_0$; the Navier-Stokes problem can be formulated as follows

$$\begin{cases} -\nu\Delta_x v(x) + (v(x)\cdot\nabla_x)v(x) + \nabla_x p(x) = f(x), & x \in Q_h, \\ -\nabla_x \cdot v(x) = 0, & x \in Q_h, \\ v(x) = 0, & x \in \partial Q, \\ v(x) = g(x), & x \in \partial Q_0 \end{cases} \tag{1.1}$$

where $v = (v_1, v_2, v_3)$ is the fluid velocity and $p$ the fluid pressure. Moreover, $f$ and $g$ are given vector functions standing for the exterior volume forces and for the velocity of the piston, respectively.

The domain $Q_h$ is naturally divided into three parts, namely into the cylindrical top and bottom parts

$$Q^+ = \Omega \times (l_+, L_+)\,, \qquad Q^- = \Omega \times (L_-, l_-)\,,$$

and into the thin annular gap around the piston

$$Q_h^0 = \Big\{ x = (n, z, s) \, : 0 < n < h\, H(s)\, , \, l_- < z < l_+ \, , s \in \gamma \Big\}. \qquad (1.2)$$

Here $(n, z, s)$ denote the natural curvilinear coordinates measured towards the piston from the surface of the outer cylinder, $z$ directed along the axes of the cylinders and $s$ standing for the arc length on the contour $\partial\Omega$. Moreover, $H$ is a smooth function defining the lateral wall $T_h^0$ of the piston, i.e.

$$T_h^0 = \Big\{ (n, z, s) \, : \, n = h\, H(s)\, , \, z \in (l_-, \, l_+)\, , \, s \in \gamma \Big\}. \qquad (1.3)$$

The objective of this paper is to show that the piston problem, as we have presented it, can be modeled by considering simultaneously certain limit problems in the regions $Q^+$, $Q^-$ and $T^0$, see the figure above, where $T^0$ is obtained from $Q_h^0$ when $h \to 0$.

Now, it is clear that if, for example, the piston is moving downwards, the flux coming out from the thin slit $Q_h^0$ to $Q_h^+$ is linked to the flux flowing into the gap $Q_h^0$ over the edge $\gamma^- = \gamma \times \{l_-\}$. Hence, the data for the different limit problems must be compatible and it turns out that the problems must be solved simultaneously or at least in a certain fixed order.

We emphasize that the small geometrical parameter $h$ is used here mainly for modelling the piston problem since we focus on the investigation of the problem in a hybrid domain $Q^+ \cup T^0 \cup Q^-$. In $Q^\pm$ the limit problems consist of three-dimensional Navier-Stokes equations in a domain with edge singularities. However, the usual weighted Sobolev or Hölder space approach is not adequate since it does not allow non-zero fluxes into the edges $\gamma^\pm$. We follow the idea, suggested in [4, 5], of enlarging the weighted spaces by asymptotic terms that take into account the extra singularities. The construction of the weighted Sobolev spaces with attached asymptotics and the analysis of the three-dimensional limit problems in $Q^\pm$ will be discussed in Section 2.

In the thin domain $Q_h^0$, the main limit problem to be solved is a two-dimensional Reynolds equation in $T^0$. Boundary conditions are found by matching the asymptotic structures of the solutions in $Q_h^0$ and $Q^\pm$ at the edges $\gamma^\pm$. In this matching procedure, one needs to take into account the boundary layer effect around $\gamma^\pm$. To shorten and clarify the presentation, we have, however, omitted the discussion of this additional limit problem which, in any case, appears only for technical reasons. For more details concerning the matching procedure, see [6, 7], among others.

The two-dimensional Reynolds equation is derived and studied in Section 3. Non-standard boundary conditions and transmission conditions joining together the three limit problems are introduced for solving the equation. The boundary conditions consist of non-homogeneous Dirichlet conditions at $z = l_\pm$, with unknown (arbitrary) constants $\mathfrak{p}^0_\pm$ as data, and of (non-local) integral Neumann conditions at the edges $\gamma^\pm$. Theorem 3.1 shows how to solve the coupled problem in the hybrid domain.

Finally, in Section 4 we present two examples. One concerns the flow in a vertical cylinder with a slowly moving piston and the other the flow between rotating eccentric cylinders.

## 2. The three-dimensional limit problems

Let us study the following Navier-Stokes problems in the disjoint domains $Q^\pm$ obtained formally from the original problem (1.1) by taking $h = 0$, which cuts out the end parts $Q^\pm$ from the container $Q_h$ by the surfaces $\sigma^\pm = \Omega \times \{l_\pm\}$

$$\begin{cases} -\nu\Delta_x v^\pm(x) + (v^\pm(x)\cdot\nabla_x)v^\pm(x) + \nabla_x p^\pm(x) = f^\pm(x), & x \in Q^\pm, \\ -\nabla_x \cdot v^\pm(x) = 0, & x \in Q^\pm, \\ v^\pm(x) = g^\pm(x), \qquad x \in \partial Q^\pm \setminus (\gamma^\pm \cup \Gamma^\pm)\,, \end{cases} \tag{2.1}$$

where $\gamma^\pm$ and $\Gamma^\pm = \gamma \times \{L_\pm\}$ stand for the edges of the boundary $\partial Q^\pm$. Let us also denote by $\sigma_h^\pm = \omega \times \{l_\pm\}$ the end surfaces of the piston. Since problems $(2.1)_+$ and $(2.1)_-$ are completely analogous we will omit the indices $\pm$ in notations.

Assuming that the data are small, we linearize (2.1) around the trivial solution and obtain the Stokes problem

$$\begin{cases} -\nu\Delta_x v(x) + \nabla_x p(x) = f(x), & x \in Q, \\ -\nabla_x \cdot v(x) = f_\nabla(x), & x \in Q, \\ v(x) = g(x), & x \in \partial Q \setminus (\gamma \cup \Gamma)\,, \end{cases} \tag{2.2}$$

where $f_\nabla$ appears only for technical reasons.

Let $s \in \mathbb{N}_0 := \{0, 1, \dots\}$ and $\beta \in \mathbb{R}$, and let $V^s_\beta(Q)$ denote the function space obtained by completing $C_0^\infty(\overline{Q}\setminus(\gamma\cup\Gamma))$ with respect to the weighted norm

$$\|w; V^s_\beta(Q)\| = \Big(\sum_{k=0}^{s} \int_Q d(x)^{2(\beta-s+k)} |\nabla_x^k w(x)|^2\, dx\Big)^{1/2} \tag{2.3}$$

where $\nabla_x^k w$ is a collection of all k-$th$ order derivatives of $w$ and $d$ a smooth positive function in $\overline{Q} \setminus (\gamma \cup \Gamma)$ coinciding with $\mathrm{dist}(x, \gamma \cup \Gamma)$ in the neighbourhood of the edges $\gamma$ and $\Gamma$. The Stokes problem (2.2) is associated with the continuous mapping

$$\begin{aligned} &\mathcal{D}_\beta^l V(Q) := V_\beta^{l+1}(Q)^3 \times V_\beta^l(Q) \longrightarrow \\ &\mathcal{R}_\beta^l V(Q) := V_\beta^{l-1}(Q)^3 \times V_\beta^l(Q) \times V_\beta^{l+1/2}(\partial Q)^3 . \end{aligned} \tag{2.4}$$

Here $l \in \mathbb{N} := \{1, 2, \dots\}$, $\beta \in \mathbb{R}$ and $V_\beta^{l+1/2}(\partial Q)$ is the trace space on $\partial Q \setminus (\gamma \cup \Gamma)$ for $V_\beta^{l+1}(Q)$ equipped with the natural norm $\|g; V_\beta^{l+1/2}(\partial Q)\| = \inf \|G; V_\beta^{l+1}(Q)\|$ where *infimum* is taken over all $G$ coinciding with $g$ on $\partial Q \setminus (\gamma \cup \Gamma)$.

It is known, cf. [8], that the Stokes problem operator (2.4) is Fredholm if and only if

$$|\beta - l| < 1 . \tag{2.5}$$

Moreover, if (2.5) holds true, the Fredholm operator (2.4) is of index zero, any non-trivial solution $(v, p) \in \mathcal{D}_\beta^l V(Q)$ to the homogeneous problem (2.2) is but the constant pressure $(0, 0, 0, \mathrm{const})$ and the compatibility condition for problem (2.2) turns into

$$\int_Q f_\nabla(x)\, dx + \int_{\partial Q} g_N \, ds_x = 0 , \tag{2.6}$$

where $g_N$ is the projection of the vector $g$ on the unit outward normal $N(x)$.

If $v \in V_\beta^{l+1}(\Omega)^3$ then, by virtue of (2.3), the integral

$$\int_Q d(x)^{2(\beta - l - 1)} |v(x)|^2 \, dx \tag{2.7}$$

converges. On the other hand, if the indices $\beta, l$ are such that condition (2.5) is valid then, for the integral (2.7) to be finite, $v$ must vanish at the edges $\gamma$ (and $\Gamma$). This, however, is not true for the solution of the original problem (1.1) since the piston moves and, hence, $g$ can be non-zero everywhere on the surface $\sigma^\pm$. Besides, when $\pm g_N^\pm > 0$ the compatibility condition (2.6) can not be satisfied since the fluid is incompressible and the container walls fixed. In order to overcome this mismatching, we enlarge the function spaces by introducing weighted spaces with separated asymptotics in the same way as was done in [5, 9, 10] for domains with conical (angular) outlets to infinity.

Let $\mathfrak{D}^l_\beta(Q)$ be the linear space consisting of vector functions $(v, p)$ that admit an asymptotic representation in the form

$$\begin{aligned} v(x) &= \chi(r)[r^{-1}\mathcal{V}^1(\varphi, s) + r^0\mathcal{V}^0(\varphi, s)] + \widetilde{v}(x)\,, \\ p(x) &= \chi(r)[r^{-2}\mathcal{P}^1(\varphi, s) + r^{-1}\mathcal{P}^0(\varphi, s)] + \widetilde{p}(x)\,, \end{aligned} \tag{2.8}$$

where $\chi \in C_0^\infty(\mathbb{R})$ is a cut-off function with a small support satisfying $\chi(r) = 1$ for $r \in [0, r_0]$ and $(r, \varphi)$ are local polar coordinates defined by $n = r\cos\varphi$ and $z = r\sin\varphi$ on the wedge $\{(n, z, s) \in \mathbb{R}^3 : n > 0\,, z > 0\,, s \in \mathbb{R}\}$. The angular parts $\mathcal{V}^i$ and $\mathcal{P}^i$ in (2.8) are arbitrary[1] functions verifying

$$\mathcal{V}^i \in H^{l+1+i}(\gamma \longrightarrow H^{l+1}(\Upsilon)^3)\,, \mathcal{P}^i \in H^{l+1+i}(\gamma \longrightarrow H^l(\Upsilon))\,, \quad i = 0, 1\,, \tag{2.9}$$

where $\Upsilon = (0, \pi/2)$, and the remainder terms $\widetilde{v}$ and $\widetilde{p}$ fulfil the inclusion

$$e(\widetilde{v}, \widetilde{p}) \in \mathcal{D}^l_\beta V(Q)\,. \tag{2.10}$$

**Remark 2.1.** The first attached terms in (2.8) are introduced in order to be able to handle the nonlinear term $(v \cdot \nabla)v$ in the Navier-Stokes equations (2.1) which does not belong to $\mathcal{R}^l_\beta V(Q)$ if $(v, 0) \in \mathcal{D}^l_\beta V(Q)$ and (2.5) holds. The term $r^{-1}\mathcal{V}^1$ drives also a non-zero flux from the vertex of the right-angle $\{(n, z) \in \mathbb{R}^2 : n > 0\,, z > 0\}$. The second attached terms take into account the jump in the Dirichlet boundary data at the edge $\gamma$.

**Remark 2.2.** Note that the attached terms have unbounded Dirichlet integrals although in the original problem formulated in $Q_h$ these type of singularities are not present. The singularities and, consequently, the attached terms, appear due to the modelling since we close the gap between the piston and the container walls while letting $h \to 0$. Let us also stress that the discontinuity in the Dirichlet data could be handled even without attaching as in [2].

To guarantee that the attached term $\chi\,(r^0\mathcal{V}^0, r^{-1}\mathcal{P}^0)$ does not belong to the space $\mathcal{D}^l_\beta V(Q)$, meant for the remainder, we strengthen restriction (2.5) by assuming that

$$\beta - l \in (-1, 0)\,. \tag{2.11}$$

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[1] We emphasize that in contrast to [4, 11] the angular parts in (2.8) are arbitrary not only in the variable $s$ on the edge but also in the angular variable $\varphi$. In other words, our spaces are enlarged by $H^s(\Gamma \longrightarrow H^\mu(\Upsilon))$ whilst in the cited works the attached spaces are simply $H^s(\Gamma)$.

The norm in $\mathfrak{D}^l_\beta V(Q)$ is defined as the sum of norms of functions $(\widetilde{v}, \widetilde{p})$ and $(\mathcal{V}^i, \mathcal{P}^i)$ introduced in (2.10), (2.4), (2.3) and (2.9),

$$\begin{aligned} &\|(\mathcal{V}^i, \mathcal{P}^i); H^{l+1+i}(\gamma \to H^{l+1}(\Upsilon)^3 \times H^l(\Upsilon))\| \\ := &\left\{ \sum_{m=0}^{l+1+i} \int_\gamma \|(\partial_s^m \mathcal{V}^i(s,\cdot), \partial_s^m \mathcal{P}^i(s,\cdot)); H^{l+1}(\Upsilon)^3 \times H^l(\Upsilon))\|^2 \, ds \right\}^{\frac{1}{2}} . \end{aligned} \tag{2.12}$$

Observe that the space $\mathfrak{D}^l_\beta V(Q)$ depends on the choice of the cut-off function $\chi$ neither algebraically nor topologically.

The continuity of mapping (2.4) implies that the Stokes operator is also continuous as

$$\mathfrak{D}^l_\beta(Q) \to \mathfrak{R}^l_\beta(Q)\,, \tag{2.13}$$

if we define $\mathfrak{R}^l_\beta(Q)$ to be the space of vector functions $(f, f_\nabla, g)$ such that

$$\begin{aligned} f(x) &= \chi(r)[r^{-3}\mathcal{F}^1(\varphi,s) + r^{-2}\mathcal{F}^0(\varphi,s)] + \widetilde{f}(x)\,, \\ f_\nabla(x) &= \chi(r)[r^{-2}\mathcal{F}^1_\nabla(\varphi,s) + r^{-1}\mathcal{F}^0_\nabla(\varphi,s)] + \widetilde{f}_\nabla(x)\,, \\ g(x) &= \chi(r)[r^{-1}\mathcal{G}^1(\varphi,s) + r^0\mathcal{G}^0(\varphi,s)] + \widetilde{g}(x)\,, \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} &\mathcal{F}^i \in H^{l+1+i}(\gamma \longrightarrow H^{l-1}(\Upsilon)^3)\,, \qquad \mathcal{F}^i_\nabla \in H^{l+1+i}(\gamma \longrightarrow H^l(\Upsilon))\,, \\ &\mathcal{G}^i(0,\cdot)\,,\ \mathcal{G}^i(\pi/2,\cdot) \in H^{l+1+i}(\gamma)^3\,, \qquad (\widetilde{f}, \widetilde{f}_\nabla, \widetilde{g}) \in \mathcal{R}^l_\beta V(Q)\,, \end{aligned} \tag{2.15}$$

and the norm $\|(f, f_\nabla, g); \mathfrak{R}^l_\beta(Q)\|$ is equal to the sum of the norms of functions listed in (2.15).

Next, let us adjust the Banach spaces $\mathfrak{D}^l_\beta(Q)$ and $\mathfrak{R}^l_\beta(Q)$ to the specific problem we are looking at. Since the fluid is incompressible and the container walls are not moving, it holds

$$f_\nabla = 0\,, \qquad g|_{\partial Q \setminus \bar{\sigma}} = 0\,, \tag{2.16}$$

which for $(v, p)$ implies that

$$\nabla \cdot v = 0\,, \qquad v|_{\partial Q \setminus \bar{\sigma}} = 0\,. \tag{2.17}$$

On the other hand, since the velocity of the piston is certainly bounded, we take

$$\mathcal{G}^1(0, s) = 0. \tag{2.18}$$

We can also assume that

$$\mathcal{F}^1_\varphi = 0\,, \qquad \mathcal{F}^1_s = 0\,. \tag{2.19}$$

From the Stokes equations written for $(\mathcal{V}^1, \mathcal{P}^1)$ it then follows that

$$\mathcal{V}^1_\varphi = \mathcal{V}^1_s = 0\,, \qquad \mathcal{V}^1_r(0,s) = 0\,, \tag{2.20}$$

We shall denote by $\overset{\circ}{\mathfrak{R}}{}^l_\beta(Q)$ the subspace of $\mathfrak{R}^l_\beta(Q)$ obtained by imposing restrictions (2.16), (2.18) and (2.19). Analogously, we define $\overset{\circ}{\mathfrak{D}}{}^l_\beta(Q)$ as the subspace of $\mathfrak{D}^l_\beta(Q)$ obtained by assuming conditions (2.17) and (2.20).

Let us emphasize that the Stokes problem operator, now defined as $\overset{\circ}{\mathfrak{D}}{}^l_\beta(Q) \longrightarrow \overset{\circ}{\mathfrak{R}}{}^l_\beta(Q)$ still remains continuous. However, it is clear that it is not a Fredholm operator since its kernel is infinite dimensional. An obvious way to get rid of the non-trivial kernel is to impose a condition on the flux distribution along the edges $\gamma^\pm$; we return here back to the problems in $Q^\pm$

$$\int_0^{\frac{\pi}{2}} \mathcal{V}^{1\pm}_r(\varphi, s)\, d\varphi = \Phi^\pm(s)\,, \quad s \in \gamma^\pm\,. \tag{2.21}$$

Note that in the next section we will have

$$\Phi^\pm(s) = \pm\Phi(s), \quad s \in \gamma^\pm. \tag{2.22}$$

Consequently, comparing (2.9), at $i = 1$, and (2.21) we consider $\Phi \in H^{l+2}(\gamma^\pm)$ as part of the data of the problem, define

$$\overset{\bullet}{\mathfrak{R}}{}^l_\beta(Q^\pm) = \overset{\circ}{\mathfrak{R}}{}^l_\beta(Q^\pm) \times H^{l+2}(\gamma^\pm)\,,$$

and regard the operator of problem (2.2), (2.21) as a mapping $\overset{\circ}{\mathfrak{D}}{}^l_\beta(Q^\pm) \longrightarrow \overset{\bullet}{\mathfrak{R}}{}^l_\beta(Q^\pm)\,.$

The solvability of the Stokes problem $(2.2)_\pm$, $(2.21)_\pm$ in $\overset{\circ}{\mathfrak{D}}{}^l_\beta(Q^\pm)$ leads to the following result.

**Theorem 2.1.** *Let $\beta, l$ satisfy (2.11) and assume that $(f^\pm, 0, g^\pm, \Phi^\pm) \in \overset{\bullet}{\mathfrak{R}}{}^l_\beta(Q^\pm)$ fulfills the compatibility condition*

$$\int_{\partial Q^\pm} g^\pm_N(x)\, d\mathfrak{s}_x = \overline{\Phi}^\pm := \int_{\gamma^\pm} \Phi^\pm(s)\, ds\,,$$

*with the norm* $\|(f^{\pm}, 0, g^{\pm}, \Phi^{\pm}); \overset{\bullet}{\mathfrak{R}}{}^l_\beta(Q^{\pm})\| =: \varepsilon$ . *There exists* $\varepsilon_0 > 0$ *such that if* $\varepsilon \le \varepsilon_0$ *then the Navier-Stokes problem* (2.1), *supplemented with the flux conditions* (2.21) *and*

$$\int\limits_{Q^{\pm}} p^{\pm}(x)\, dx = 0\,, \tag{2.23}$$

*admits a unique solution in the ball* $\{(v^{\pm}, p^{\pm}) \in \overset{\circ}{\mathfrak{D}}{}^l_\beta(Q^{\pm}) \ : \ \|(v^{\pm}, p^{\pm}); \overset{\circ}{\mathfrak{D}}{}^l_\beta(Q^{\pm})\| \le c\,\varepsilon\}$. *This solution satisfies the estimate*

$$\|(v^{\pm}, p^{\pm});\ \overset{\circ}{\mathfrak{D}}{}^l_\beta(Q^{\pm})\| \le c\,\|(f^{\pm}, 0, g^{\pm}, \Phi^{\pm});\, \overset{\bullet}{\mathfrak{R}}{}^l_\beta(Q^{\pm})\|\,.$$

*Moreover, without the orthogonality conditions* (2.23), *problem* (2.1), (2.21) *admits solutions* $(v^{\pm}, p^{\pm} + c^{\pm}) \in \overset{\circ}{\mathfrak{D}}{}^l_\beta(Q^{\pm})$ *where* $c^{\pm} \in \mathbb{R}$.

**Sketch of the Proof.** Inserting the asymptotic forms (2.8) into the Stokes problem (2.2) and separating variables, we arrive at two systems of ordinary differential equations for the angular parts $(\mathcal{V}^1, \mathcal{P}^1)$ and $(\mathcal{V}^0, \mathcal{P}^0)$, and at a Stokes problem for the remainder $(\widetilde{v}, \widetilde{p})$, with a right-hand side $(\widetilde{f}, \widetilde{f}_\nabla, \widetilde{g})$ depending on the attached terms $\chi[\dots]$ in (2.8) but belonging into the proper space $\mathcal{R}^l_\beta V(Q)$.

The system of differential equations for $(\mathcal{V}^1, \mathcal{P}^1)$, in which the $s$-variable appears as a parameter, is supplied with boundary conditions at $\varphi = 0$ and $\varphi = \pi/2$ and with the flux condition (2.21). As was shown in [5], the problem has a solution if a certain compatibility condition is satisfied which in our case is automatically valid due to restrictions (2.16), (2.18). The problem for $(\mathcal{V}^0, \mathcal{P}^0)$ is uniquely solvable with any data. Moreover, inclusions (2.9) hold readily true and estimates are obtained in the norms (2.12). Having the attached terms at hand, we can find a solution (2.10) to the remaining Stokes problem by using the information on operator (2.4). Note that a direct calculation transforms the compatibility condition (2.6) for the right-hand side $(\widetilde{f}, \widetilde{f}_\nabla, \widetilde{g})$ into (2.23).

The Navier-Stokes problem is considered as a perturbation of the Stokes problem. Since the convective term does not effect the compatibility condition in the problem for $(\mathcal{V}^1, \mathcal{P}^1)$ (cf. [5], the final assertion is derived from the following lemmata.

**Lemma 2.1.** *Let $\mathfrak{D}$ and $\mathfrak{R}$ be Banach spaces, $\mathfrak{S} : \mathfrak{D} \to \mathfrak{R}$ a bounded invertible linear operator, such that $C_S = \|\mathfrak{S}^{-1} : \mathfrak{R} \to \mathfrak{D}\|$, and $\mathfrak{B} : \mathfrak{D} \times \mathfrak{D} \to \mathfrak{R}$ a bilinear and bi-continuous operator with the constant of bi-continuity $C_B$. Given any $\mathfrak{f} \in \mathfrak{R}$ with $\|\,\mathfrak{f}\,;\mathfrak{R}\,\| < \frac{1}{2}\,C_s^{-2}\,C_B^{-1}$, the nonlinear equation $\mathfrak{S}\,\mathfrak{u} + \mathfrak{B}(\mathfrak{u},\mathfrak{u}) = \mathfrak{f}$ admits a unique solution $\mathfrak{u} \in \mathfrak{D}$ in the ball $\|\,\mathfrak{u}\,;\mathfrak{D}\,\| < \frac{1}{2}\,C_s^{-1}\,C_B^{-1}$. Moreover $\mathfrak{u}$ satisfies the estimate $\|\,\mathfrak{u}\,;\mathfrak{D}\,\| < 2\,C_s\,\|\,\mathfrak{f}\,;\mathfrak{R}\,\|$.*

**Lemma 2.2.** *The mapping $\mathring{\mathfrak{D}}^l_\beta(Q) \ni (v,0),\,(w,0) \longmapsto ((v\cdot\nabla_x)w,0,0) \in \mathring{\mathfrak{R}}^l_\beta(Q)$ is bi-continuous, that is, there exists a positive constant $c$ such that*

$$\|((v\cdot\nabla_x)w,0,0);\mathring{\mathfrak{R}}^l_\beta(Q)\| \le c\,\|(v,0);\mathring{\mathfrak{D}}^l_\beta(Q)\|\,\|(w,0);\mathring{\mathfrak{D}}^l_\beta(Q)\|\,.$$

Note that Lemma 2.1 is just a variant of the Banach's fixed point principle. Lemma 2.2 can be verified by repeating word by word the proof of Lemma 6.1 in [10].

## 3. The Reynolds equation coupled with the Navier-Stokes problems

As $h \to 0$, the thin gap $Q^0_h$ shrinks to the surface $T^0 := (l_-, l_+) \times \gamma$, cf. (1.2). To describe the behavior of the solution in the gap, we make the following ansätze (cf. [12, 13, 14] and others).

$$\begin{aligned} v(h,x) &\sim h^{-2}\,0 + h^{-1}\mathfrak{v}^1(\eta,z,s) + h^0\mathfrak{v}^2(\eta,z,s) + \dots\,, \\ p(h,x) &\sim h^{-3}\mathfrak{p}^0(z,s) + h^{-2}\mathfrak{p}^1(\eta,z,s) + h^{-1}\mathfrak{p}^2(\eta,z,s) + \dots \end{aligned} \tag{3.1}$$

where $\eta = h^{-1}n$ is the stretched normal coordinate and $\mathfrak{v}^i$, $\mathfrak{p}^i$ are functions to be determined.

Writing the Navier-Stokes equations in coordinates $(n,s,z)$, making the change of variable $n \mapsto \eta$, using the ansätze (3.1) and collecting the terms of the highest order in $h$ (namely, $O(h^{-3})$ in the momentum equations, $O(h^{-2})$ in the divergence equation and $O(h^{-1})$ in the boundary conditions), one obtains, as the first limit problem, a homogeneous system for $(\mathfrak{v}^1,\mathfrak{p}^1)$; the system is homogeneous since the data are assumed to be bounded as $h \to 0$. This system can easily be solved resulting to

$$\begin{aligned} &\mathfrak{v}_n^1 = 0\,, \qquad \mathfrak{p}^1 = \mathfrak{p}^1(z,s)\,, \\ &\mathfrak{v}_z^1(\eta,z,s) = \frac{1}{2\nu}\,\eta(\eta - H(s))\partial_z \mathfrak{p}^0(z,s)\,, \\ &\mathfrak{v}_s^1(\eta,z,s) = \frac{1}{2\nu}\,\eta(\eta - H(s))\partial_s \mathfrak{p}^0(z,s) \end{aligned} \tag{3.2}$$

Collecting terms at the next orders, leads to a one-dimensional Stokes problem, for $(\mathfrak{v}_n^2, \mathfrak{p}^2)$, and to two scalar Dirichlet problems for the Poisson equation, for $\mathfrak{v}_z^2$ and $\mathfrak{v}_s^2$. These equations can be solved in terms of $\mathfrak{p}^0$ and $\mathfrak{p}^1$ provided the one-dimensional Stokes system satisfies the standard compatibility condition. As usual (cf. [12, 14]), this compatibility condition turns into the Reynolds equation

$$-\frac{1}{12\,\nu}\,\nabla_{(z,s)} \cdot H(s)^3\,\nabla_{(z,s)}\mathfrak{p}^0(z,s) = 0\,, \qquad (z,s) \in T^0\,. \tag{3.3}$$

**Remark 3.1.** Ansätze (3.1) allow the data on the lateral surface of the piston $T_h^0$ to be much larger than what was assumed in deriving (3.2). In fact, as was shown in [14] in a different geometrical situation, the data can be chosen as

$$g_n = h^0\,\mathfrak{g}_n(z,s) + \ldots\,, \quad g_z = h^{-1}\,\mathfrak{g}_z(z,s) + \ldots\,, \quad g_s = h^{-1}\,\mathfrak{g}_s(z,s) + \ldots \tag{3.4}$$

Now, however, that in this case the ansatz data has to satisfy the compatibility condition $\mathfrak{g}_z(l_\pm, s) = \mathfrak{g}_s(l_\pm, s) = 0$, $s \in \gamma$. For (3.4), the solution to the first limit problem can be written as $(3.2)_1$ and

$$\begin{aligned} \mathfrak{v}_z^1 &= \frac{1}{2\nu}\,\eta(\eta - H(s))\partial_z \mathfrak{p}^0 + H(s)^{-1}\eta\,\mathfrak{g}_z(z,s), \\ \mathfrak{v}_s^1 &= \frac{1}{2\nu}\,\eta(\eta - H(s))\partial_s \mathfrak{p}^0 + H(s)^{-1}\eta\,\mathfrak{g}_s(z,s)\,. \end{aligned}$$

The Reynolds equation takes the (non-homogeneous) form

$$-\frac{1}{12\,\nu}\,\nabla_{(z,s)} \cdot H(s)^3\,\nabla_{(z,s)}\mathfrak{p}^0(z,s) = -\mathfrak{g}^0(z,s)\,, \qquad (z,s) \in T^0\,, \tag{3.5}$$

where

$$\begin{aligned} \mathfrak{g}^0(z,s) &= \mathfrak{g}_n(z,s) - \nabla_{(z,s)}H(s) \cdot \mathfrak{g}'(z,s) + \frac{1}{2}\,\nabla_{(z,s)} \cdot H(s)\mathfrak{g}'(z,s) \\ \mathfrak{g}' &= (\mathfrak{g}_z, \mathfrak{g}_s)\,. \end{aligned}$$

Noting that the normal vector $N(h,x)$ to the surface $T_h^0$ is of the form $(1\,,\,0\,,-hH'(s)\,)+O(h)$, it is easy to see that the difference $\mathfrak{g}_n-\nabla_{(z,s)}H\cdot\mathfrak{g}'$ appearing in (3.5) is but the projection of the vector $(\mathfrak{g}_n, h^{-1}\mathfrak{g}_z, h^{-1}\mathfrak{g}_s)$ on the normal vector $N(h,x)$.

**Remark 3.2.** The above calculations are also valid in the case when the piston walls are not parallel to the walls of the container, i.e. $T_h^0 = \{(n,z,s)\,:\, n = h\,H(z,s),\ z\in(l_-,l_+)\,,\ s\in\gamma\,\}$, (cf. (1.3), where $H$ is a smooth positive function on $T^0$.

Before discussing the solvability of the Reynolds equation, let us consider the original problem (1.1) and see how to match the solutions of the different limit problems. The main issue in matching is to overcome the arbitrariness and overdetermination in the data of the limit problems. This will also define the order in which these problems need to be solved.

In the three-dimensional limit problems (2.1), the flux densities $\Phi^{\pm}\in H^{l+1}(\gamma^{\pm})$ must be prescribed in the flux condition (2.21), see Theorem 2.1. The total fluxes

$$\overline{\Phi}^{\,\pm}=\int\limits_{\gamma^{\pm}}\Phi^{\pm}(s)\,ds \tag{3.6}$$

are related to the data $g^{\pm}|_{\sigma^{\pm}}$ through the compatibility condition (2.23) and the fluxes are driven by the angular solutions $(\mathcal{V}^{1\pm},\mathcal{P}^{1\pm})$. These angular flux drivers, first transformed into Poiseuille flow solutions by the two-dimensional boundary layer equations (see, *e.g.*, [7, 10]), are matched with the asymptotic ansätze (3.1). This yields the relation

$$\pm\,\frac{1}{12\,\nu}\,H(s)^3\,\partial_z\mathfrak{p}^0(l_{\pm},s)=\Phi^{\pm}(s)\,, \tag{3.7}$$

normally taken as Neumann boundary condition for the Reynolds equation (3.3), cf. [12, 14]. Here, however, only the total fluxes but not the flux density distributions $\Phi^{\pm}$ are known. In fact, from (3.6) and (3.7) one obtains the integral conditions

$$\pm\frac{1}{12\,\nu}\int\limits_{\gamma^{\pm}}H(s)^3\,\partial_z\mathfrak{p}^0(l_{\pm},s)\,ds=\overline{\Phi}^{\,\pm}\,. \tag{3.8}$$

The conditions (3.8) do not, however, suffice to close the problem for $\mathfrak{p}^0$. Observing that the three-dimensional limit problems are linear with respect to the pressure so that one can add *any constants* to the pressure components in the asymptotic ansätze, we take

$$h^{-3}\mathfrak{p}^0_{\pm}+p^{\pm}(x) \tag{3.9}$$

as approximations of the pressure $p(h,x)$ inside $Q^{\pm}$. Matching the pressure components leads to the relations

$$\mathfrak{p}^0(l_{\pm},s)=\mathfrak{p}^0_{\pm}\,,\quad s\in\gamma^{\pm}\,, \tag{3.10}$$

which can be taken as Dirichlet boundary conditions for the Reynolds equation. Let us emphasize that $\mathfrak{p}^0_{\pm}$ in (3.10) are unknown constants to be determined after solving the whole problem.

We will now show that the Reynolds equation (3.3) is well-posed while complemented with the boundary conditions (3.10) and (3.8). Let us begin by giving a weak formulation to the problem adapting the general scheme of [15].

Let $\mathcal{H}(T^0)$ be a subspace of the usual Sobolev space $H^1(T^0)$ consisting of scalar functions that, at the sides $\gamma^{\pm}$ of $T^0$, take (any) constant value. By a weak solution of the non-homogeneous Reynolds equation (3.4), with the boundary conditions (3.10) and (3.7), we understand an element $\mathfrak{p}^0\in\mathcal{H}(T^0)$ such that

$$\begin{gathered}\frac{1}{12\,\nu}\,(H^3\,\nabla_{(z,s)}\mathfrak{p}^0,\nabla_{(z,s)}\mathfrak{q})_{T^0}=-(\mathfrak{g}^0,\mathfrak{q})_{T^0}+\overline{\Phi}^{+}\mathfrak{q}_{+}+\overline{\Phi}^{-}\mathfrak{q}_{-}\\ \forall\mathfrak{q}\in\mathcal{H}(T^0)\,,\end{gathered} \tag{3.11}$$

where $\mathfrak{q}_{\pm}=\mathfrak{q}|_{\gamma^{\pm}}\in\mathbb{R}$ and $(\,\cdot\,,\,\cdot\,)_{T^0}$ denotes the scalar product in $L^2(T^0)$. Note that the integral identity (3.11) can be easily derived from (3.4) taking the scalar product in $L^2(T^0)$ with $\mathfrak{q}$, integrating by parts and using (3.8).

**Proposition 3.1.** *Assume that* $\{\mathfrak{g}^0,\overline{\Phi}^{\pm}\}\in L^2(T^0)\times\mathbb{R}\times\mathbb{R}$. *There exists a solution to problem* (3.11) *if and only if the compatibility condition*

$$\int_{T^0}\mathfrak{g}^0(z,s)\,dz\,ds=\overline{\Phi}^{+}+\overline{\Phi}^{-} \tag{3.12}$$

*is valid. This solution is defined up to an additive constant but if subject to the orthogonality condition*

$$\int_{T^0}\mathfrak{p}^0(z,s)\,dz\,ds=0 \tag{3.13}$$

*it becomes unique and satisfies the estimate*

$$\|\mathfrak{p}^0;H^1(T^0)\|\,\le\,c\left(\|\mathfrak{g}^0;L^2(T^0)\|+|\overline{\Phi}^{+}|+|\overline{\Phi}^{-}|\right). \tag{3.14}$$

*Moreover, if* $\mathfrak{g}^0 \in H^{\varkappa}(T^0)$ *with* $\varkappa \in \mathbb{R}_+$*, then* $\mathfrak{p}^0 \in H^{\varkappa+2}(T^0)$ *and under* (3.13) *it holds*

$$\|\mathfrak{p}^0; H^{\varkappa+2}(T^0)\| \leq c\left(\|\mathfrak{g}^0; H^{\varkappa}(T^0)\| + |\overline{\Phi}^+| + |\overline{\Phi}^-|\right).$$

**Proof.** Let us add the term $\mu(\mathfrak{p}^0, \mathfrak{q})_{T^0}$, where $\mu \in \mathbb{R}$, to the left-hand side of (3.11) and denote the resulting integral identity by $(3.12)_\mu$. For $\mu > 0$ the left-hand side of $(3.11)_\mu$ defines a scalar product in $H^1(T^0)$ and by the Riesz representation theorem there exists a unique solution $\mathfrak{p}^0_\mu$ to problem $(3.11)_\mu$ satisfying the estimate (3.14). On the other hand, the term $\mu(\mathfrak{p}^0, \mathfrak{q})_{T^0}$ gives rise to a symmetric compact perturbation of the operator of problem $(3.11)_\mu$ and, consequently, problem $(3.11)=(3.11)_0$ satisfies the Fredholm alternative. Hence, the existence of a unique solution to problem (3.11), (3,13) becomes evident after observing that the only solutions to the homogeneous problem (3.11) are constants. Since this solution defines $\mathfrak{p}^0_\pm$, estimate (3.15) in Sobolev–Slobodetskii spaces follows from (3.11) and from the trace inequality while considering $\mathfrak{p}^0$ as a solution to the Dirichlet problem (3.14), (3.10). □

**Remark 3.3.** For ansatz (3.2) it holds $\mathfrak{g}^0 = 0$. Hence, (3.13) implies for the total fluxes (3.7) that $\overline{\Phi}^+ = \overline{\Phi}^-$. This same relation holds also for ansatz (3.4) if $\mathfrak{g}^0_N = 0$ since from (3.6) it follows that $\int_{T^0} \mathfrak{g}^0 dz ds = 0$. Note that in case of percolation it may happen that $\mathfrak{g}^0_N \neq 0$ and, consequently, the integral in (3.13) does not necessarily vanish.

Let us make final comments on the order in which the limit problems ought to be considered. First, for the total fluxes $\overline{\Phi}^\pm$, found from the compatibility conditions $(2.23)_\pm$, one solves the Reynolds equation (3.3), with the boundary conditions (3.10), (3.8), and calculates with the help of (3.2) the pressure $h^{-3}\mathfrak{p}^0$ and the velocity $h^{-1}\mathfrak{v}^1$ in the thin gap. In particular, the constant pressures $\mathfrak{p}^0_+$ and $\mathfrak{p}^0_-$ are then known and one can calculate the derivative $\partial_z\, \mathfrak{p}^0(l_\pm, s)$ from which, in view of (3.7), one obtains the flux densities $\Phi^\pm$. With the flux densities and the constant pressures in hand, one finally solves the three-dimensional limit problems in $Q^\pm$.

We can now join the results of Theorem 2.1 and Proposition 3.1.

**Theorem 3.1.** *Let* $f^\pm \in V^l_\beta(Q^\pm)^3$, $g^\pm \in V^{l+1/2}_\beta(\partial Q^\pm)^3$ *and* $\mathfrak{g}^0 \in H^{l+3/2}(T^0)$*, where* $l, \beta$ *satisfy* (2.11)*. Define*

$$\varepsilon := \sum_\pm \|(f^\pm, g^\pm); V^l_\beta(Q^\pm)^3 \times V^{l+1/2}_\beta(\partial Q^\pm)^3\| + \|\mathfrak{g}^0; H^{l+3/2}(T^0)\|,$$

*and assume that the compatibility condition*

$$\sum_{\pm} \int_{\partial Q^{\pm}} g_N^{\pm} \, d\mathfrak{s}_x = \int_{T^0} \mathfrak{g}^0(s,z) \, ds dz$$

*is valid. There exists* $\varepsilon_0 > 0$ *such that if* $\varepsilon \le \varepsilon_0$, *then the hybrid problem obtained by coupling the limit problems*

(1) *the Navier-Stokes problems* $(2.1)_{\pm}$ *supplied with the augmented conditions* $(2.21)_{\pm}$ *where the fluxes* $\Phi^{\pm}$ *are found from* (3.7), *and the orthogonality condition* (2.24);

(2) *the Reynolds equation* (3.4) *complemented with the boundary conditions* (3.10), *the flux conditions* (3.8), (2.23) *and with the orthogonality condition* (3.13),

*admits a unique solution* $(v^{\pm}, p^{\pm} + h^{-3}\mathfrak{p}^0_{\pm}, \mathfrak{p}^0)$ *in the ball*

$$\Big\{ (v^{\pm}, p^{\pm}, \mathfrak{p}^0_{\pm}, \mathfrak{p}^0) \in \mathring{\mathfrak{D}}^l_{\beta}(Q^{\pm}) \times \mathbb{R}^2 \times H^{l+7/2}(T^0)$$
$$: \sum_{\pm} \Big( \|(v^{\pm}, p^{\pm}); \mathring{\mathfrak{D}}^l_{\beta}(Q^{\pm})\| + |\mathfrak{p}^0_{\pm}| \Big) + \|\mathfrak{p}^0; H^{l+7/2}(T^0)\| \le c\,\varepsilon \Big\}.$$

## 4. Examples

In this section we present two examples to show how the asymptotic models could be used in engineering problems. In the first example, the main issue is to create the data (the piston velocity and the flux distribution) for the limit problems in $Q^{\pm}$ from the existing knowledge (the weight and the shape of the piston). The second example is related to the classical problem of a hydrodynamic journal bearing.

### 4.1. Slow motion of a piston.

Consider straight cylinders $Q$ and $Q_0$ with cross-sections $\Omega$ and $\omega$ and lengths $L = L_+ - L_-$ and $l = l_+ - l_-$, respectively. Denote the areas of $\Omega$ and $\omega$ by $|\Omega|$ and $|\omega|$ and assume that the clearance distance $h$ (film thickness) between the cylinders is constant and small in comparison with the cross-section diameters. Suppose that the outer cylinder $Q$ is held fixed and that $Q_0$ (the piston) is sliding slowly, without rotation, due to some external force $-We_z$, *e.g.* due to its weight. Since the film thickness is constant, i.e. $H(s) = 1$, wee see that the Reynolds equation (3.3) has a solution of the form $\mathfrak{p}^0(z) = az + p_0$, with $a, p_0 \in \mathbb{R}$. The boundary condition (3.10) implies that $\mathfrak{p}^0_+ - \mathfrak{p}^0_- = al$, where $l = l_+ - l_-$, and from the orthogonality condition (3.13) it follows that $p_0 = -a(l_+ + l_-)/2$.

In view of (3.9), the resultant pressure force acting on the piston

$$F = \Big( \int\limits_{\sigma_h^-} p(y, l_-)\, dy - \int\limits_{\sigma_h^+} p(y, l_+)\, dy \Big)\, e_z \simeq -\, |\omega|\, (\mathfrak{p}^0_+ - \mathfrak{p}^0_-) h^{-3} e_z\, .$$

must be balanced by the external load, thus $a = -\, h^3 (l|\omega|)^{-1} W$. Note that $l|\omega|$ is the volume of the piston and $(l|\omega|)^{-1} W$ its mean density. From relation (3.7), one calculates the (constant) flux density distribution

$$\Phi = -\, \frac{h^3}{12\,\nu}\, \frac{W}{l\,|\omega|}\ .$$

Since the piston moves only vertically, $g_N^{\pm} = \mp g_z^{\pm}$ is constant. Hence, the compatibility condition (2.23) implies that

$$g_z^{\pm} = \frac{|\gamma|}{|\omega|}\, \Phi = -\, \frac{h^3}{12\,\nu}\, \frac{W}{l\,|\omega|}\, \frac{|\gamma|}{|\omega|}\,, \tag{4.1}$$

where $|\gamma|$ denotes the length of the contour $\gamma = \partial\omega$. Knowing $g^{\pm} = (0, 0, g_z^{\pm})$ and $\Phi^{\pm} = \pm\Phi$ one can solve the three-dimensional limit problems (2.1) in $Q^{\pm}$ and obtain the velocity and pressure fields $(v^{\pm}, p^{\pm})$. The final pressure distributions in $Q^{\pm}$ are given by $p^{\pm} \mp \frac{1}{2}\, |\omega|^{-1} W$. Comparing pistons of different shape but with same volume and weight, it follows from (4.1) that the fall of the piston is slowest if it is circular.

**4.2 Flow between rotating eccentric cylinders.**

Assume that the cylinders $Q$ and $Q_0$ are circular with $R$ and $R_0$ denoting the radii of $Q$ and $Q_0$, respectively. Suppose that the cylinders are laying down horizontally and that the inner cylinder $Q_0$ is rotating at an angular velocity $\varpi$ within the fixed hollow outer cylinder $Q$ which is filled with a viscous incompressible fluid, *e.g.* motor oil. This example is related to the classical lubrication flow problems first introduced by Reynolds in 1886, cf. [1]. However, in contrast to the journal bearing problems where the rotation of the inner cylinder (journal) creates a thin film flow only between the journal and its bearing (outer cylinder), cf. [16], we take the bearing to be longer than the journal and study also the effect of the journal rotation on the flow in the end parts of the bearing.

Due to the rotation and vertical load, say gravity, the journal is shifting to the right (left) if the rotation is counter-clockwise (clockwise) and, hence, the cylinders become eccentric. Denoting by $a$ the displacement of the journal center from the bearing center and assuming that $\delta := a/R_0 \ll 1$, it is a simple geometrical exercise, cf. [16] p. 90, to show that, with an error $O(\delta^2)$, the film thickness $\mathbf{H}$, measuring the width of the gap between the cylinders, is approximated by

$$\mathbf{H}(s) = R - R_0 - a\cos\varphi\,, \qquad \varphi = R^{-1}\,s \tag{4.2}$$

where $\varphi$ is the angular coordinate measured from the radial line of the bearing that crosses the journal center. Usually, expression (4.2) is written as

$$\mathbf{H}(s) = \gamma(1 - \epsilon\cos\varphi)\,,$$

with $\gamma = R - R_0$ denoting the radial clearance and $\epsilon = a/\gamma$ the eccentricity ratio. Taking $h = \gamma$ and recalling that $\mathbf{H} = hH$, we finally arrive at the following approximation (up to an error $O(\epsilon^2)$) for the non-dimensional film thickness $H$

$$H(s) = 1 - \epsilon\cos\varphi\,.$$

The Reynolds equation (3.4) takes the form

$$-\partial_s\,H(s)^3\,\partial_s\,\mathfrak{p}^{\,0}(z,s) - \partial_z\,H(s)^3\,\partial_z\,\mathfrak{p}^{\,0}(z,s) = -6\varpi\nu R\,\partial_s\,H(s)\,, \tag{4.3}$$

where we have taken into account that $\mathfrak{g}_n = \mathfrak{g}_z = 0$ and $\mathfrak{g}_s = \varpi R\,e_s$. Assuming that the journal is light so that $\epsilon \ll 1$, we can look for the solution to (4.3) in the form

$$\mathfrak{p}^0(z,s) = \mathfrak{p}^0_0 + \epsilon\,\sin\varphi\,\mathfrak{p}^0_1(z)\,.$$

From the compatibility condition (2.23) and the boundary condition (3.10) it follows that $\mathfrak{p}^0_0 = \mathfrak{p}^0_+ = \mathfrak{p}^0_-$ and $\mathfrak{p}^0_1(l_\pm) = 0$, since the total fluxes $\overline{\Phi}^\pm$ must vanish.

At the first order in $\epsilon$, we get from (4.3)

$$-\partial_z^2\,\mathfrak{p}^0_1(z) + R^{-2}\mathfrak{p}^0_1(z) = -6\,\varpi\nu\,, \quad z \in (l_-, l_+)\,, \qquad \mathfrak{p}^0_1(l_\pm) = 0\,.$$

Hence, setting $\lambda_\pm = l_\pm R^{-1}$ and $\lambda = \lambda_+ = \lambda_-$, we conclude that

$$\mathfrak{p}^0_1(z) = \frac{6\,\varpi\nu}{e^{l/R} - e^{-l/R}}$$
$$\Big((e^{-\lambda_+} - e^{-\lambda_-})\,e^{-z/R} - (e^{\lambda_+} - e^{\lambda_-})\,e^{z/R}\Big) - 6\varpi\nu R\,.$$

Finally, the flux distributions at $\gamma^\pm$ can be calculated from (3.7) yielding

$$\Phi^\pm(s) = \pm\,\epsilon\,\varpi R\,\tanh\frac{l}{2R}\,\sin\varphi\,(1-\cos\varphi)^3\,, \qquad \varphi = R^{-1}\,s.$$

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Institute of Mechanical Engineering Problems

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Instituto Superior Técnico, Departamento de Matemática
Av. Rovisco Pais, 1, 1049–001 Lisboa, Portugal