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VIOLATION OF SYMMETRY IN THE SYSTEM OF THREE LATERALLY COUPLED QUANTUM WAVEGUIDES AND RESONANCE ASYMPTOTICS

Abstract. The system of three two-dimensional laterally coupled quantum waveguides with the Dirichlet boundary condition is considered. Violation of geometrical symmetry is studied. The behaviour of the resonance asymptotics and the transition "eigenvalue – resonance" are investigated.

Introduction

The spectral problem for the Dirichlet Laplacian in compound domains was intensively studied by many authors. From the point of view of physics this problem for coupled waveguides if especially interesting. It attracts a new wave of interest due to the development of nanoelectronics. Recently experimentalists began to produce the new class of objects: quantum dots and antidots, quantum wires (waveguides), etc. These systems are usually called mesoscopic because they are sufficiently large to be created experimentally, but sufficiently small to demonstrate quantum character of an electron. In these semiconductor devices the mean free path of the electron may be larger then the size of the system. In this situation physicists usually deal with a ballistic regime. The description of ballistic electron transport in many mesoscopic quantum systems reduces to the description of electron wave propagation in a system of waveguides or layers (see, for example, [1, 2, 3, 4, 5]). This paper deals with systems of two-dimensional waveguides coupled through small apertures. It has been proved in [6] that the Dirichlet Laplacian for the system of two waveguides coupled through small window has an eigenvalue λ_a close to the threshold. The authors used variational technique and obtained some estimates. Similar estimates were obtained for the case of n coupling windows [8]. The asymptotics of the eigenvalue in question was

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obtained in [9, 10]. Some further results are in [11]. Method of matching of the asymptotic expansions (in the semiwidth of the aperture a) for the corresponding solutions was used. The scheme of matching was a modification of that suggested in [12, 13]. The problem of resonance (quasi-eigenvalue) was considered in [14]. The asymptotics of resonances close to the Nth threshold is obtained for the problem of two coupled waveguides. It is very important for physical applications to investigate the behaviour of the resonance and possibility of its transformation in the eigenvalue (see, e.g., [15, 16]). The present paper is devoted to this problem.

We consider the spectral problem for the Dirichlet Laplacian $-\Delta$ in three two-dimensional strips Ω^1, Ω^2, Ω^3 of widths d_1, d_2 and d_3, correspondingly, coupled through apertures of widths $2a$ (Fig. 1a). If $d_1 = d_3$, the main space $\mathcal{H}$ can be represented as an orthogonal sum, $\mathcal{H} = H_a \oplus \mathcal{H}_s$, where $\mathcal{H}_a(\mathcal{H}_s)$ is a subspace of antisymmetric (symmetric) in respect to the centreline of Ω^2 functions. These subspaces are invariant subspaces of the Dirichlet Laplacian, consequently, one has the corresponding representation of the operator: $-\Delta = -\Delta_a \oplus \Delta_s$. Hence, the problem, actually splits into two boundary problems, each for two coupled strips (Fig. 1b) (with the Dirichlet or Neumann boundary condition on the former centreline). The spectra of the corresponding operators $(-\Delta_a, -\Delta_s)$ are shown on Fig. 1c. It is known (see above) that the operator $-\Delta_a$ has an eigenvalue close to the threshold. Consider breaking of symmetry, namely, let $d_1 \neq d_3$ $(d_3 < d_1 < d_2 < 2d_3)$. In this case there is no reduction of the problem, and the mentioned eigenvalue became a resonance (quasi eigenvalue). This effect is very important for the description of transport properties of the system. That is why it is interesting to look for this transition "resonance – eigenvalue". We shall deal with the asymptotics (in a) of the resonance. To look for the asymptotics of the resonance when $d_3 \to d_1$ we suppose that $d_1^2 = d_3^2 + \chi a^6$, $\chi > 0$. The main goal of the paper is to find the first terms of the asymptotic expansions of the resonance. We consider the Helmholtz equation $(\Delta + k^2)\varphi_a = 0$ with the Dirichlet boundary condition in a system of three coupled waveguides and seek for quasi eigenvalue $\lambda_a = k_a^2$ close to the threshold $\frac{\pi^2}{d_3^2}$. Method of mutching of asymptotic expansions of the solutions of boundary value problems is used.

Resonance asymptotics

Let us introduce the local coordinate system $x_1^{(12)}$, $x_2^{(12)}(x_1^{(23)}, x_2^{(23)})$

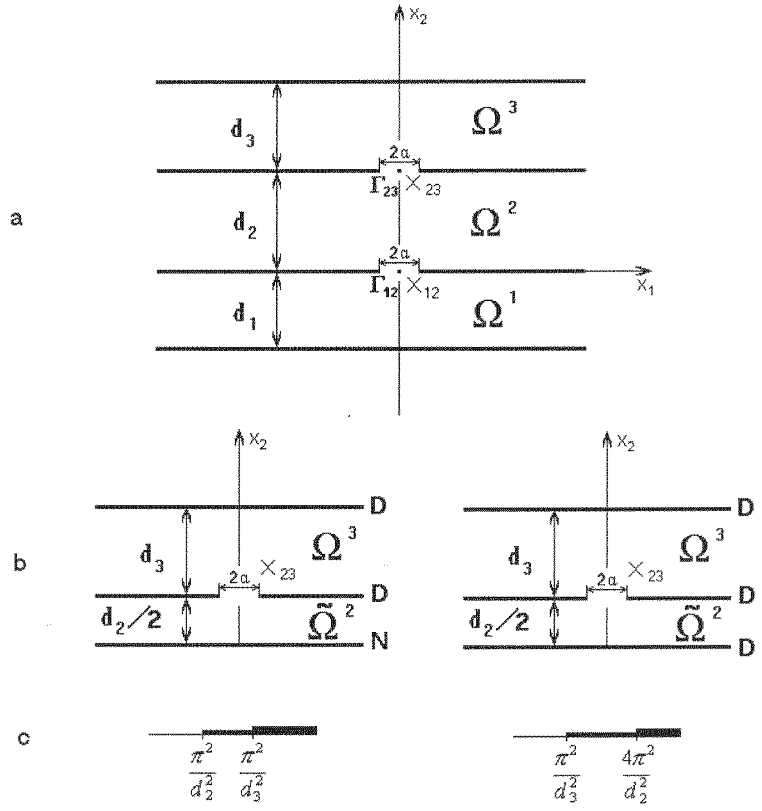

Fig. 1. a) Geometrical configuration of the system. $\Omega^1, \Omega^2, \Omega^3$ – waveguides, X_{12}, X_{23} – the centres of coupling windows Γ_{12}, Γ_{23} , correspondingly; b) Geometrical configuration for the reduced problems; letters $D(N)$ marks the type of boundary condition - Dirichlet (Neumann); c) Branches of the continuous spectra of the operators for the reduced problems..

with the origin at the centre of the opening Γ_{12} (Γ_{23}) in such a way that the axis $Ox_1^{(12)}$ ($Ox_1^{(23)}$) coincides with the waveguide wall and $x_2^{(12)} > 0$ ($x_2^{(23)} > 0$) in Ω^1 (Ω^2). We shall seek the asymptotic expansion of the quasi-eigenvalue in the following form:

$$\left(\frac{\pi^2}{d_3^2} - k_a^2\right)^{\frac{1}{2}} = \sum_{j=2}^{\infty} \sum_{i=0}^{[(j-1)/2]} k_{ji} a^j \left(\ln \frac{a}{a_0}\right)^i . \tag{1}$$

Here a_0 is some characteristic length unit. We shall find some first coefficients k_{ji} in (1). The expansions of the corresponding quasi-eigenfunction in different domains are as follows:

$$\varphi_a(x) = -\left(\frac{\pi^2}{d_3^2} - k_a^2\right)^{\frac{1}{2}} \sum_{j=0}^{\infty} a^j P_{j+1}^{(1)}\left(D_y, \ln\frac{a}{a_0}\right) G_1(x,y,k)\,|_{y=X_{12}},$$
$$x \in \Omega^1 \backslash S^{12}_{a_0\left(\frac{a}{a_0}\right)^{1/2}}, \tag{2}$$

$$\varphi_a(x) = \sum_{j=1}^{\infty} \sum_{i=0}^{[(j-1)/2]} v_{ji}^{(12)}\left(\frac{x}{a}\right) a^j \left(\ln\frac{a}{a_0}\right)^i, \; x \in S^{12}_{2a_0\left(\frac{a}{a_0}\right)^{1/2}}, \tag{3}$$

$$\varphi_a(x) = \left(\frac{\pi^2}{d_3^2} - k_a^2\right)^{\frac{1}{2}} \sum_{j=0}^{\infty} a^j (P_{j+1}^{(21)}\left(D_y, \ln\frac{a}{a_0}\right) G_2(x,y,k)\,|_{y=X_{12}}$$
$$+P_{j+1}^{(23)}\left(D_y, \ln\frac{a}{a_0}\right) G_2(x,y,k)\,|_{y=X_{23}}),$$
$$x \in \Omega^2 \backslash \left(S^{12}_{2a_0\left(\frac{a}{a_0}\right)^{1/2}} \cup S^{23}_{2a_0\left(\frac{a}{a_0}\right)^{1/2}}\right), \tag{4}$$

$$\varphi_a(x) = \sum_{j=1}^{\infty} \sum_{i=0}^{[(j-1)/2]} v_{ji}^{(23)}\left(\frac{x}{a}\right) a^j \left(\ln\frac{a}{a_0}\right)^i, \; x \in S^{23}_{2a_0\left(\frac{a}{a_0}\right)^{1/2}}, \tag{5}$$

$$\varphi_a(x) = -\left(\frac{\pi^2}{d_3^2} - k_a^2\right)^{\frac{1}{2}} \sum_{j=0}^{\infty} a^j P_{j+1}^{(3)}\left(D_y, \ln\frac{a}{a_0}\right) G_3(x,y,k)\,|_{y=X_{23}},$$
$$x \in \Omega^3 \backslash S^{23}_{2a_0\left(\frac{a}{a_0}\right)^{1/2}}. \tag{6}$$

Here X_{ij} is the centre of the opening Γ_{ij}, S_t^{ij} is the sphere of the radius t centered at X_{ij}, $v_{ji} \in W^1_{2,\mathrm{loc}}\left(\Omega^1 \cup \Omega^2 \cup \Omega^3\right)$, $P_m^{(rs)}$, $P_m^{(r)}$ are some polynomials on D_y(derivative operator) and $\ln\frac{a}{a_0}$, having the form

$$P_1^{(r)}(D_y, \ln\frac{a}{a_0}) = {}^r a_{10}^{(1)} \frac{\partial}{\partial n_y} P_m^{(r)}(D_y, \ln\frac{a}{a_0}) =$$
$$\sum_{j=1}^{m-1} \sum_{i=0}^{[(j-1)/2]} {}^r a_{ji}^{(m)} \left(\ln\frac{a}{a_0}\right)^i D_y^{m-j+i}, \quad m = 2,3,4,\ldots \tag{7}$$

$$P_1^{(rs)}(D_y, \ln\frac{a}{a_0}) = {}^{rs}a_{10}^{(1)}\frac{\partial}{\partial n_y}P_m^{(rs)}(D_y, \ln\frac{a}{a_0})$$

$$= \sum_{j=1}^{m-1}\sum_{i=0}^{[(j-1)/2]} {}^{rs}a_{ji}^{(m)}\left(\ln\frac{a}{a_0}\right)^i D_y^{m-j+i}, \quad m = 2, 3, 4, ..., \tag{8}$$

where ${}^{r}a_{ji}^{(m)}$, ${}^{rs}a_{ji}^{(m)}$ are some constants, $D_y^{2j+1} = \frac{\partial^{2j+1}}{\partial n_y^{2j+1}}$, $D_y^{2j+2} = \frac{\partial^{2j+2}}{\partial l_y \partial n_y^{2j+1}}$; $l = (1,0)$, $n = (0,-1)$, G_i is the Dirichlet Green function for the Helmholtz equation in the waveguide Ω^i:

$$G_s(x,y,k) = \sum_{n=1}^{\infty}\frac{1}{d_s}\left(\frac{n^2\pi^2}{d_s} - k^2\right)^{-1/2}$$

$$\sin\frac{n\pi x_2}{d_s}\sin\frac{n\pi y_2}{d_s}\exp\left(-\left(\frac{n^2\pi^2}{d_s} - k^2\right)^{1/2}|x_1 - y_1|\right). \tag{9}$$

The derivative of the Green function can be represented in a neighbourhood of the boundary point for spectral parameter close to $\frac{\pi^2}{d_3^2}$ in the form:

$$D_y^j G_s(x,y,k)\big|_{y=X_{pq}} = \frac{1}{d_s}\sin\frac{\pi x_2}{d_s}D_x^j\left(\sin\frac{\pi x_2}{d_s}\right)\Bigg|_{x=X_{pq}}\left(\frac{\pi^2}{d_s^2} - k_a^2\right)^{-\frac{1}{2}}$$

$$+\Phi_j(x,k)\ln\frac{r}{a_0} + g_j^{(s)}(x,k) + \sum_{i=0}^{[j/2]}\sum_{t=0}^{j-2i-1} b_{it}^{(j)}(k)r^{-j+2(i+t)}\sin(j-2i)\theta. \tag{10}$$

Here (r,θ) are polar coordinates. Functions $b_{it}^{(j)}(k)$, $\Phi_j(x,k)$, $g_j^{(l)}(x,k)$ are analytic in k in a neighbourhood of $\frac{\pi^2}{d_3}$; $\Phi_j(x,k) \in C^\infty(\Re^2)$, and are continuous in respect to x_2. It is known (see, e.g. [G]) that $g_j^{(l)}(x,k) \in C^\infty(\Omega^l)$ and

$$b_{00}^{(j)} = \frac{(-1)^{[(j+1)/2]}(j-1)!}{\pi}, \qquad b_{10}^{(3)} = \frac{k^2}{2\pi}, \tag{11}$$

$$\Phi_{1n}(0,k) = -\frac{k^2}{2\pi}. \tag{12}$$

Functions $v_{ji}^{(mn)}(x/a)$ are solutions of boundary value problems which are obtained by the following way. One substitutes series ((1), (3), (5) into the Helmholtz equation (for $k = k_a$). Than one replaces the variables in the local coordinates: $\xi = x/a$ and cancels equal terms of the same order of a and $\ln a$. Taking formal limit $a \to 0$, one gets the following boundary problem for $v_{ji}^{(mn)}\left(\frac{x}{a}\right)$:

$$\Delta_\xi v_{ji}^{(mn)} = -\sum_{p=0}^{j-3}\sum_{q=0}^{[p/2]-1} \Lambda_{pq} v_{j-p-2,i-q}^{(mn)}, \xi \in R^2 \backslash \gamma v_{ji}^{(mn)} = 0, \xi \in \bar{\gamma}, \quad (13)$$

where $\bar{\gamma} = \{\xi : \xi_2 = 0, \xi_1 \in (-\infty, -1] \cup [1, \infty)\}$, Λ_{pq} are the coefficients of the series:

$$k_a^2 = \sum_p \sum_q \Lambda_{pq} a^p \left(\ln \frac{a}{a_0}\right)^q.$$

Let us define the operator $M_{pq}^{(ij)}$ on sums $U(x,a)$ of type (2), (4), (6) in local coordinates with the centre X_{ij}. Expression (2) for $\varphi_a(x)$ can be transformed with use of (1) and (10). One expands coefficients of $U(x,a)$ in an asymptotic series for $r \to \infty$ and changes the variables $\xi = \frac{x}{a}$ ($\ln r$ is replaced by $\ln \rho + \ln a$). Denote the sum of all terms of the type $a^p (\ln a)^q \, \mathrm{H}(\xi)$ as M_{pq}. Let $M_p^{(ij)} = \sum_q M_{pq}^{(ij)}$.

To match asymptotic expansions (2) and (3), (3) and (4), (4) and (5), (5) and (6) one should make equal the coefficients in terms of the same order $a^p \left(\ln \frac{a}{a_0}\right)^q$.

After matching, we obtain the result. The first terms of the asymptotic expansion of the resonance are as follows:

$$\begin{aligned} k_a^2 = k_0^2 - k_{20}^2 a^4 - 2k_{20}\left(k_{40} + k_{41} \ln \frac{a}{a_0}\right) a^6 \\ - \left(k_{40}^2 + 2k_{40}k_{41} \ln \frac{a}{a_0} + k_{41}^2 \left(\ln \frac{a}{a_0}\right)^2\right) a^8 \\ + \text{higher order terms}, \end{aligned} \quad (14)$$

where

$$k_0^2 = \frac{\pi^2}{d_3^2}, \quad k_{20} = \frac{\pi^3}{4d_3^3},$$

$$k_{40} = -\frac{\pi^4}{32d_3^3}\left(g_x^{(1)} + g_x^{(2)}\right) + \frac{\pi^5}{16d_3^2 d_2^2\sqrt{d_3^2 - d_2^2}} + \frac{3\pi^5}{128d_3^5} - \frac{\chi}{\pi d_3}$$
$$+\frac{1}{2}(3\left(\frac{\pi^4}{16d_3^3}g_x^{(2)} - \frac{\pi^5}{16d_3^2 d_2^2\sqrt{d_3^2 - d_2^2}} - \frac{3\pi^5}{64d_3^5}\right)^2$$
$$+\left(\frac{\pi^4}{16d_3^3}\left(g_x^{(1)} + g_x^{(2)}\right) + \frac{2\chi}{\pi d_3}\right)^2 + \frac{\pi^8 f^2}{256 d_3^6})^{1/2}, \; k_{41} = -\frac{\pi^5}{16d_3^5}. \quad (15)$$

One can see that if $\chi = 0$ then k_{40} becomes real. It corresponds to the fact that in this case we have an eigenvalue instead of the resonance.

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