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SPECIAL FUNCTION RELATED TO THE SCATTERING OF THE WHISPERING GALLERY MODE AT THE POINT OF LOCAL STRAIGHTENING

ABSTRACT. Scattering of the whispering gallery mode on the point of local straightening of the boundary is under consideration. It is constructed and examined a special function describing the main features of the wave field in the vicinity of the point of local straightening.

1. Introduction

The theory of whispering gallery modes (WGM) is a well-known part of the diffraction theory. These modes are solutions of the corresponding boundary problems which are localized near the boundary. The corresponding asymptotic construction was derived about 4 decades ago (Babich, Buldyrev, Lazutkin, Molotkov,...) and is presented in books [1, 2]. But some generic problems connected with WGM are open up to now. Among these is the scattering of WGM at the point of inflection or the point of local straightening of the boundary. At such points the main existence condition of such modes (positivity of effective radius of curvature) fails. These problems were discussed in some papers (the point of inflection in [3–7], the point of local straightening in [8–10]), but they are not completely solved up to now. Particularly, the description of the wave field in the vicinity of singular points is absent.

Note, that the asymptotic theory of the WGM is based on the corresponding Airy integrals [1]. These special functions defined as contour integrals describe the main features of the WGM, in particular, they decrease exponentially as their argument tends to $+\infty$ and decrease as power with oscillations as argument tends to $-\infty$. The asymptotic behavior of wave field is determined by the order of saddle points in Airy integrals. This fact leads to the asymptotic growth of the solution near the critical line (caustic line), where two simple saddle points coalesce into one double saddle point.

In this paper we consider the analogue of the Airy integral for the situation, when WGM scatters on the point of local straightening of the boundary. We discuss this problem in the parabolic equation framework [1].

2. Special function for the boundary problem with point of local straightening

We consider the next model boundary problem: we search for a function $\Psi(z,t)$, which is a solution of the equation

$$i\omega\Psi_t + \Psi_{zz} - 3\omega^2 t^2 z\Psi = 0, \tag{1}$$

satisfying the following conditions:

a) $\Psi(0,t) = 0$,

b) $\Psi(z,t)$ exponentially decreases at $z \to +\infty$,

c) $\Psi(z,t)$ reproduces scaled and deformed Airy function at $t \to -\infty$.
The last condition corresponds to the incoming WGM [9, 10].

We consider a special function defined by the contour integral

$$\Psi(z,t) = \int\limits_L dv \exp\{i\omega M(z,t,v)\}, \tag{2}$$

with

$$M(z,t,v) = z(v^3 - t^3) - t(v^3 - t^3)^2 - \frac{3}{2}t^4(v^3 - t^3) - \frac{9}{14}t^7 + \frac{9}{14}v^7. \tag{3}$$

Here the contour L begins in the sector $(6\pi/7, \pi)$, and ends in the sector $(4\pi/7, 5\pi/7)$ at infinity (Fig. 1). Note, that the function $M(t,z,v)$ is a polynomial, which degree equals to 7.

2.1. Saddle points.

Integral (2) has generally 6 saddle points, which are moving when values of t, z are changing. Their positions define the asymptotic behavior of our integral in different domains of variables t, z. Note, that contribution of the saddle point depends on its order [ ]11.

The equation for the saddle points is:

$$\frac{dM}{dv} = \frac{9v^6}{2} + \frac{3t^4v^2}{2} + 3zv^2 - 6tv^5 = 0. \tag{4}$$

This equation splits in two parts: $v^2 = 0$ and

$$9v^4 + 3t^4 + 6z - 12tv^3 = 0. \tag{5}$$

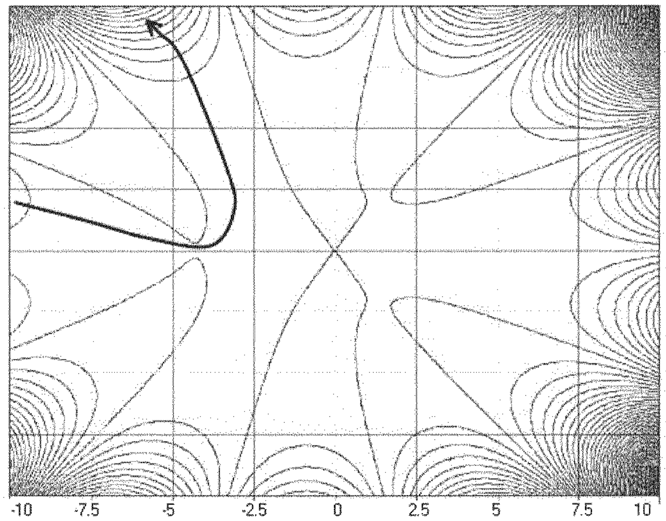

Fig. 1

Hence the point $v_0 = 0$ is a double saddle point in the general case and we have another 4 saddle points, solutions of equation (5). Considering simultaneously equations $dM/dv = 0$, $d^2M/dv^2 = 0$, we conclude, that the saddle point $v = t$ is a double one at $z = 0$. If $z = -t^4/2$ the saddle point $v = 0$ has the fifth order. The saddle point $v = 0$ is of the sixth order at $z = t = 0$.

Some computer calculations of the topology of level lines of the function $N(t, z, v)$ at different z, t were carried out in order to describe the saddle points, which are contributory for our integral and determine its asymptotic behavior. On the basis of the obtained information the plane of arguments z, t can be divided on 4 parts, in which integral (2) has different asymptotic behavior (Fig. 2).

1. If the point (z, t), as $t < 0$, is situated above the line $z = 0$, or as $t > 0$ is situated above line $z + t^4/2$ (region A), our integral has only one simple saddle point v_c. This point belongs to the upper half-plane and $\mathrm{Re}(iM(z, t, v_c)) < 0$.

2. If the point (z, t) is situated between the lines $z = 0, t < 0$ and $z = -t^4/2, t < 0$ (region B), our integral has two simple real saddle points $v_{1,2}$, and $\mathrm{Re}(iM(z, t, v_{1,2})) = 0$. When $-t >> 1$ our function depends on the variable z like Airy function, it exponentially decreases as z tends to

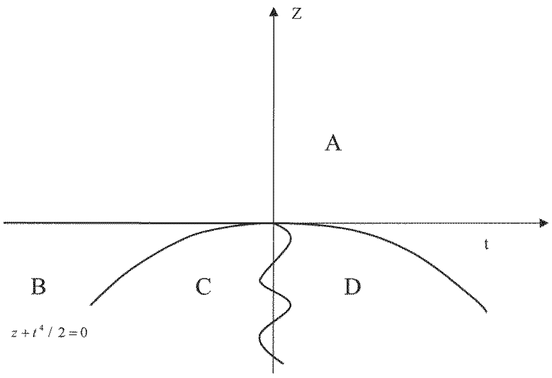

Fig. 2

$+\infty$ and decreases as power with oscillations as z tends to ∞.

3. Below line $z = -t^4/3$ there is the region C, where our integral gets 3 saddle points – double saddle point $v_0 = 0$, real simple saddle point and complex simple saddle point with positive imaginary part. For the both real saddle points $\mathrm{Re}(iM(z,t,v_c)) = 0$, for the complex one $\mathrm{Re}(iM(z,t,v_c)) < 0$. So, contribution of the last saddle point is exponentially small.

4. Below the line $z = -t^4/3$ it is situated the region D, where our integral has two real saddle points, one of which is double point $v_0 = 0$. In these points $\mathrm{Re}(iM(z,t,v_c)) = 0$.

2.2. Behavior in the vicinity of the critical lines.

These results are valid outside the asymptotically small vicinities of the critical lines and points, in which the order the saddle points is changed. Here we consider the behavior of special function (2) in these vicinities.

1. At first we consider the line $z = 0, t < 0$. As was mentioned above, on this line the saddle point of our integral $v = t$ is a double one. Substituting $v = t + s\omega^{-1/3}$, $z = \zeta\omega^{-2/3}$, we obtain:

$$\omega M(z,t,v) = 3t^2\zeta s + 3t^4 s^3 + O(\omega^{-1/3}),$$

so in the leading asymptotic order with respect to the large parameter ω we have:

$$\Psi(\zeta\omega^{-2/3}, t) = \omega^{-1/3} R_1(\zeta, t),$$

where

$$R_1(\zeta, t) = \int_{L_1} \exp(3it^2(\zeta s - t^2 s^3))ds$$

is scaled Airy integral and the contour L_1 corresponds here to the contour for Airy integral. The factor $\omega^{-1/3}$ agrees with the double saddle point for the line $z = 0$. The function $R_1(\zeta, t)$ describes our special function in the $\omega^{-2/3}$-vicinity of critical line in the leading asymptotical order respect the/ large parameter ω.

2. In the vicinity of the critical line $z + t^4/2 = 0$ we substitute $\theta = z + t^4/2$. Then

$$M(z, t, v) = 9v^7/(14) - tv^6 + \theta v^3 + q(z, t),$$

$$q(z, t) = -t^7/7 - zt^3.$$

So, after scaling $v = \omega^{-1/6}s, \theta = \omega^{-1/2}\delta$, one can get:

$$\Psi(z, t) = \omega^{-1/6} \exp(i\omega q(z, t))R_2(\delta, t),$$

$$R_2(\delta, t) = \int_{L_3} \exp(i(-ts^6 + \delta s^3))ds,$$

where contour L_3 starts and ends at infinity, in sectors $(5\pi/6, \pi)$ and $(\pi/2, 2\pi/3)$ respectively (here we suppose, that $t > 0$). The function $R_2(\delta, t)$ describes the behavior of special function (2) in the $\omega^{-1/6}$-vicinity of this critical line. The factor $\omega^{-1/6}$ corresponds to the presence of the contributory saddle point of 5 order. As it follows from results of [12], the function $R_2(\delta, t)$ can be expressed in terms of parabolic cylinder function.

3. We get in the asymptotically small vicinity of the critical point $z = t = 0$ after scaling $v = \omega^{-1/7}s$, $t = \omega^{-1/7}\tau$, $\theta = \omega^{-1/7}x$:

$$\Psi(z, t) = \omega^{-1/7} \exp(i\omega q(z, t))\Xi(x, \tau),$$

$$\Xi(x, \tau) = \int_{C_2} \exp(i(9s^7/(14) + xs^3 - \tau s^6))ds,$$

where contour C_2 is similar to the initial contour L. Here factor $\omega^{-1/7}$ corresponds to the order of the contributory saddle point $v = 0$.

2.3. Asymptotic behavior of special function.

Information about contributory saddle points obtained above gives us the possibility to calculate in explicit form the asymptotic of our special function at different values of variables z, t. It follows from the fact, that equation (5) has fourth order and its solutions can be written in radicals. We can determine the roots of this equation which are contributory for our integral. But we discuss here only general behavior of our special function in different regions.

In the region A special function (2) decreases exponentially as z tends to $+\infty$. In the region B it oscillates, decreasing as power when $\mid z \mid$ is increasing. In regions C, D it oscillates too, but the amplitude of oscillations has the more high asymptotical order due to presence of the double contributory saddle point v_0.

2.4. Summary.

So, let us summarize the main properties of special function (2).

1. It is a solution of equation (1) at any z, t.

2. It decreases exponentially as $z \to +\infty$.

3. The critical point $z = t = 0$ (the point of local straightening) is point of focusing of wave field.

4. Let consider the behavior of integral (2) as $t \to -\infty$. It follows from equation (4), that we have in this situation 2 simple contributory saddle points, which tends to the value t. After substitution $v = t + (9t^4)^{-1/3}s$ polynomial (3) can be rewritten as

$$M(z,t,v) \sim z3^{1/3}t^{2/3}s + s^3/3 + O(t^{-5/3}),$$

and we conclude, that $M(z,t,v)$ coincides up to scaling and minor corrections with standard polynomial in the exponent for the Airy function (and contour of integration coincides with the contour for the Airy function). So, integral (2) reproduces at $t \to -\infty$ scaled and deformed Airy function, in other words, the known asymptotic of solution of the model boundary problem – the incoming WGM, [9].

These facts gives the possibility to treat integral (2) as natural analogue of the Airy function for our model boundary problem. Note, that the boundary condition at $z = 0$ has small significance in the asymptotic description of the WGM defining the subordinate terms of its asymptotic expansion. We omit here the discussion of the boundary condition at $z = 0$ for our special function.

Similar results are obtained for the model problem with point of inflection of the boundary. In this case model equation reads:

$$i\omega\Psi_t + \Psi_{zz} + 2\omega^2 tz\Psi = 0.$$

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