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EXPONENTIALLY SMALL SPLITTING OF SEPARATRICES FOR WHISKERED TORI IN HAMILTONIAN SYSTEMS

Abstract. We study the existence of transverse homoclinic orbits in a singular or weakly hyperbolic Hamiltonian, with 3 degrees of freedom, as a model for the behaviour of a nearly-integrable Hamiltonian near a simple resonance. The example considered consists of an integrable Hamiltonian possessing a 2-dimensional hyperbolic invariant torus with fast frequencies $\omega/\sqrt{\varepsilon}$ and coincident whiskers or separatrices, plus a perturbation of order $\mu = \varepsilon^p$, giving rise to an exponentially small splitting of separatrices. We show that asymptotic estimates for the transversality of the intersections can be obtained if ω satisfies certain arithmetic properties. More precisely, we assume that ω is a quadratic vector (i.e. the frequency ratio is a quadratic irrational number), and generalize the good arithmetic properties of the golden vector. We provide a sufficient condition on the quadratic vector ω ensuring that the Poincaré–Melnikov method (used for the golden vector in a previous work) can be applied to establish the existence of transverse homoclinic orbits and, in a more restrictive case, their continuation for all values of $\varepsilon \to 0$.

1. Introduction and main results

The detection of *transverse homoclinic orbits* to an invariant object is one of the main tools to prove the existence of chaotic motion in a dynamical system. Such a detection becomes complicated in the case of a Hamiltonian system ε-close to a completely integrable one. Between the KAM tori, there appear generically whiskered tori which carry on non-coincident whiskers, giving rise to the phenomenon called *splitting of separatrices*, which is *exponentially small* with respect to ε.

In the case of a one-dimensional whiskered torus (periodic orbit) of a Hamiltonian with 2 degrees of freedom, V. F. Lazutkin introduced in a seminar paper [13] *complex parameterizations* for the invariant manifolds, obtaining in this way an analytic periodic function for the splitting, with zero mean. The width of the strip of analyticity of this function appears explicitly in the exponent of the splitting and it turns out that only one (the first) harmonic of the perturbation is relevant for the size of the splitting (see [10, 11]).

However, when the dimension of the whiskered tori is greater than one, the expression of the quasiperiodic splitting function becomes more intricate, since it depends on the arithmetic properties of the frequencies of the whiskered tori. Indeed, the effect of the small divisors is present in the most important part of the splitting: the exponent. This was first noticed by Chirikov [2], and later on by Lochak [15, 16] and Simó [21], and was first proven by Delshams et al. [7] for the pendulum under a fast quasiperiodic forcing (see also [1]). Later on, the splitting of separatrices for a 2-dimensional whiskered torus in a Hamiltonian system with 3 degrees of freedom was dealt by Sauzin and co-workers [20, 14], Rudnev and Wiggins [19], Pronin and Treschev [18], and also Simó and Valls [22], who also considered the homoclinic bifurcations that can take place. It is important to say that the main tool that has been used to establish the splitting for whiskered tori with two or more frequencies is the validation of the expression provided by a direct application of the *Poincaré–Melnikov method.*

In fact, in the Hamiltonian setting, it turns out [9, 3] that the splitting vector distance and the Melnikov vector function are the gradient of scalar functions, called respectively splitting potential and Melnikov potential. This implies that transverse homoclinic orbits to whiskered tori correspond to non-degenerate critical points of the splitting potential.

The arithmetic properties of the frequencies of the whiskered torus are very important. As a matter of fact, all the rigorous expressions found up to now involve only two frequencies and some famous *quadratic* numbers, like the golden number. The theory of continued fractions is essentially used to separate between primary resonances (in the case of the golden number, the ones associated to Fibonacci numbers) and other weaker resonances.

In this context, the existence of transverse homoclinic orbits to a 2-dimensional whiskered torus, with frequency the golden vector, of a Hamiltonian with 3 degrees of freedom was proved in [20, 14], but not for all values of $\varepsilon \to 0$, since at some sequence of values of ε the dominant harmonics of the splitting function change, and homoclinic bifurcations could take place. Some examples of such bifurcations have been described in [22].

This result was improved in the same situation in [6] with the help of a careful analysis of the Melnikov function and its dominant harmonics, and applying also accurate bounds for the size of the error term provided (from the use of flow-box coordinates) in [8]. Indeed, it was shown in [6] that the dominant harmonics of the splitting function correspond to the dominant harmonics in the Melnikov approximation, providing asymptotic estimates for the splitting. With such estimates, it is possible to show the existence of exactly 4 transverse homoclinic orbits, and their *continuation* for all values of the perturbation parameter $\varepsilon \longrightarrow 0$ (with no bifurcations).

We consider in this work some concrete perturbations with 3 degrees of freedom with an infinite number of harmonics, and study how far the results quoted above can be generalized to any quadratic frequency vector (i.e. a quadratic number as the frequency ratio). Using a generalization of the arithmetic properties of the golden vector to other quadratic vectors, it is possible to carry out a suitable analysis of the Melnikov function and its dominant harmonics, as well as the size of the remaining harmonics. Under a suitable condition on the quadratic vector, we obtain asymptotic estimates for the splitting function, which allow us to establish the existence of a certain number of transverse homoclinic orbits, although bifurcations of some of such orbits may occur for ε close to some critical values (like in [22]).

In the best case (the golden vector and other noble frequency vectors) we can ensure the continuation of transverse homoclinic orbits for all $\varepsilon \longrightarrow 0$ like in [6]. For some other quadratic vectors, at least we can ensure the existence of transverse homoclinic orbits, although bifurcations of them may occur for some critical values of ε.

Next we give a more precise description of the setting and the background, and the new results obtained in the present work.

1.1. Setup: A singular Hamiltonian with 3 degrees of freedom.

We consider a Hamiltonian system, with 3 degrees of freedom, depending on two perturbation parameters ε and μ. In canonical coordinates $(x, y, \varphi, I) \in \mathbb{T} \times \mathbb{R} \times \mathbb{T}^2 \times \mathbb{R}^2$, with the symplectic form $\mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}\varphi \wedge \mathrm{d}I$, our Hamiltonian is defined by

$$H(x, y, \varphi, I) = H_0(x, y, I) + \mu H_1(x, \varphi), \tag{1}$$

$$H_0(x, y, I) = \langle \omega_\varepsilon I \rangle + \frac{1}{2}\langle \Lambda I, I \rangle + \frac{y^2}{2} + \cos x - 1, \tag{2}$$

$$H_1(x, \varphi) = h(x) f(\varphi). \tag{3}$$

We assume $\varepsilon > 0$, and also $\mu > 0$ with no loss of generality. The vector

ω_ε of *fast frequencies* considered in (2) will be given by a *quadratic vector*

$$\omega = (1, \Omega), \qquad \omega_\varepsilon = \frac{\omega}{\sqrt{\varepsilon}}. \tag{4}$$

In other words, the frequency ratio Ω is a quadratic irrational number. We also consider in (2) a symmetric (2×2)-matrix Λ, such that H_0 satisfies the condition of *isoenergetic nondegeneracy*:

$$\det \begin{pmatrix} \Lambda & \omega \\ \omega^\top & 0 \end{pmatrix} \neq 0. \tag{5}$$

For the perturbation (3), we deal with the following concrete analytic periodic functions:

$$h(x) = \cos x - \nu, \quad \text{with} \quad \nu = 0 \quad \text{or} \quad \nu = 1, \tag{6}$$

$$f(\varphi) = \sum_{k \in \mathcal{Z}} f_k \cos(\langle k, \varphi \rangle - \sigma_k), \text{ with } f_k = e^{-\rho|k|} \text{ and } \sigma_k \in \mathbb{T}, \tag{7}$$

$$\mathcal{Z} = \{k = (k_1, k_2) \in \mathbb{Z}^2 : k_2 > 0 \text{ or } (k_2 = 0, k_1 \geqslant 0)\} \tag{8}$$

(the set $\mathcal{Z}$ is introduced to avoid repetitions in the Fourier expansion of $f(\varphi)$).

The Hamiltonian H_0 (that corresponds to $\mu = 0$) has a 2-parameter family of 2-dimensional whiskered tori given by the equations $I = \text{const}$, $x = y = 0$. The stable and unstable whiskers of each torus coincide, forming in this way a unique homoclinic whisker. We shall focus our attention on a concrete *whiskered torus*, located at $I = 0$, whose inner flow has ω_ε as the frequency vector. We denote $\mathcal{W}_0$ the homoclinic whisker associated to this torus, and consider for it the parameterization

$$\mathcal{W}_0 : (x_0(s), y_0(s), \theta, 0), \quad s \in \mathbb{R}, \ \theta \in \mathbb{T}^n, \tag{9}$$

$$x_0(s) = 4 \arctan e^s, \quad y_0(s) = \frac{2}{\cosh s}. \tag{10}$$

The inner flow on $\mathcal{W}_0$ is given by $\dot{s} = 1$, $\dot{\theta} = \omega\varepsilon$.

The two parameters ε and μ will not be independent. On the contrary, they will be linked by a relation of the type $\mu = \varepsilon^p$ with a suitable $p > 0$ (the smaller p the better), i.e. we consider a *singular* problem for $\varepsilon \to 0$ (also called *weakly hyperbolic*, or *a priori stable*). The main motivation for this singular setting is that it can be considered as a model for the behaviour of a nearly-integrable Hamiltonian near a simple resonance (see for instance [8, 5]).

Our choice in (4) of a quadratic frequency vector is motivated by the arithmetic properties of such vectors. An important and well-known property is that quadratic vectors satisfy a *Diophantine condition*:

$$|\langle k, \omega \rangle| \geqslant \frac{\gamma}{|k|^\tau} \qquad \forall k \in \mathbb{Z}^2 \setminus \{0\}, \tag{11}$$

with $\tau = 1$ and some $\gamma > 0$ (concerning the value of γ, see remark 1 at the end of Section 2). Other important properties of quadratic vectors to be used are discussed in Section 2.

Under conditions (5) and (11), the *hyperbolic KAM theorem* implies that, for μ small enough, the whiskered torus persists, as well as its local whiskers. We point out that the difference between the two values of ν in (6) is that in the case $\nu = 0$ the whiskered torus persists with some shift and deformation, whereas in the case $\nu = 1$ it remains fixed under the perturbation, though the whiskers do suffer some deformation. The Lyapunov exponent of the torus, which initially is 1, becomes a close amount b. Besides, in the isoenergetic case considered here, the frequency vector ω_ε of the torus becomes perturbed to a close and proportional vector:

$$\tilde{\omega}_\varepsilon = b' \omega_\varepsilon = \frac{b' \omega}{\sqrt{\varepsilon}} \tag{12}$$

The amounts b and b' tend to 1 as $\mu \to 0$, and $b' = 1$ in the case $\nu = 1$ (see [8, Th. 1] for a precise statement).

Concerning the Fourier expansion (7), the constant $\rho > 0$ gives the complex width of analiticity of $f(\varphi)$. In principle, the phases σ_k can be chosen arbitrarily, although some quite general condition on these phases will have to be fulfilled for the validity of our results (see Section 4).

1.2. Background: The splitting function and the Poincaré–Melnikov method.

When the local whiskers are extended to global ones, one can expect in general the existence of splitting between the stable and unstable whiskers (denoted $\mathcal{W}^+$ and $\mathcal{W}^-$), since they will no longer coincide. To study this splitting, symplectic *flow-box coordinates* (S, E, φ, I) are introduced in [8], in some neighbourhood containing a piece of both whiskers (and excluding the torus, where such coordinates are not valid). In the flow-box coordinates, the Hamiltonian equations become very simple: $\dot{S} = b$, $\dot{E} = 0$, $\dot{\varphi} = \tilde{\omega}_\varepsilon$, $\dot{I} = 0$ (recall that b and $\tilde{\omega}_\varepsilon$ are the perturbed Lyapunov exponent and frequencies). Besides, those coordinates can be constructed in such a way that the stable whisker is given by a coordinate plane,

$$\mathcal{W}^+ : \quad (s, 0, \theta, 0), \quad |s| \leqslant s^*, \ \theta \in \mathbb{T}^n, \tag{13}$$

where the parameters (s, θ) are inherited from (9–10), with some translation in s. Then, the unstable whisker can be parameterized, in the same neighbourhood, as

$$\mathcal{W}^- : \quad (s, \mathcal{E}(s,\theta), \theta, \mathcal{M}(s,\theta)), \quad |s| \leqslant s^*, \ \theta \in \mathbb{T}^n, \tag{14}$$

and the inner flow on both whiskers is given by $\dot{s} = b$, $\dot{\theta} = \tilde{\omega}_\varepsilon$. To study the splitting, it is enough to consider the vector function $\mathcal{M}$, called the *splitting function* (the function $\mathcal{E}$ is directly related to $\mathcal{M}$ by the energy conservation).

The use of flow-box coordinates implies the quasiperiodicity of the splitting function $\mathcal{M}$, an important property related with its exponential smallness. More precisely, the function $\mathcal{M}$ is $\hat{\omega}_\varepsilon$*-quasiperiodic*:

$$\mathcal{M}(s,\theta) = \mathcal{M}(0, \theta - \hat{\omega}_\varepsilon s), \quad \text{where} \quad \hat{\omega}_\varepsilon := \frac{\tilde{\omega}_\varepsilon}{b} = \frac{b'\omega}{b\sqrt{\varepsilon}}. \tag{15}$$

Another important property of $\mathcal{M}$, related to the Lagrangian properties of the whiskers, is that it is the gradient of a scalar function $\mathcal{L}$, called the *splitting potential* (see also [3]):

$$\mathcal{M}(s,\theta) = \partial_\theta \mathcal{L}(s,\theta)$$

(and hence $\mathcal{M}$ has zero average with respect to θ). Then, the transverse homoclinic orbits can be studied on $s = 0$ (or any other section $s = \text{const}$), as nondegenerate critical points of $\mathcal{L}(0,\theta)$.

Applying the Poincaré–Melnikov method, it is possible to give a first order approximation in μ for the splitting, in terms of the *Melnikov potential* and the *Melnikov function*, defined in [8] (see also [3]) in terms of an absolutely convergent integral:

$$L(s,\theta) = -\int_{-\infty}^{\infty} [h(x_0(s+bt)) - h(0)] \cdot f(\theta + \tilde{\omega}_\varepsilon t)\mathrm{d}t + \text{const},$$

$$M(s,\theta) = \partial_\theta L(s,\theta). \tag{16}$$

These functions are also $\hat{\omega}_\varepsilon$-quasiperiodic, since they are defined in terms of the perturbed Lyapunov exponent b and the perturbed frequencies $\tilde{\omega}_\varepsilon$ introduced in (12). As a consequence, the *error term* defined as

$$\mathcal{R}(s,\theta) = \mathcal{M}(s,\theta) - \mu M(s + s^{(0)}, \theta) \tag{17}$$

is also $\hat{\omega}_\varepsilon$-quasiperiodic. The amount s^0, not very relevant, compensates the translation of the parameterizations (13–14) with respect to the initial parameterization (9).

In order to validate the Poincaré–Melnikov method in the singular case $\mu = \varepsilon^p$, the main difficulty is that the first order approximation given by the Melnikov function is exponentially small in ε, as shown in Section (3). In principle, it turns out that the Poincaré–Melnikov method can be applied only if μ is exponentially small in ε (see for instance [3]), but not in our case $\mu = \varepsilon^p$. Nevertheless, exponentially small upper bounds for the error term (17) can also be obtained, and the method holds in the singular case if p is large enough. The key point in order to obtain such exponentially small estimates is to carry out the bounds on *complex domains* of the parameters (s, θ), and use the quasiperiodic properties of the splitting.

Note that the initial homoclinic whisker $\mathcal{W}_0$ can be defined in the complex domain for $|\mathrm{Im} s| < \pi/2$, $|\mathrm{Im}\theta| < \rho$. These restrictions are due to the singularity of (10) at $s = \pm i\pi/2$, and to the expression (7) involving ρ as the width of analyticity. This domain is reduced along the successive steps leading to define the splitting function and potential. One of the main achievements of [8] is to construct the flow-box coordinates in such a way that the loss of complex domain is controlled by a free small parameter δ, with $\delta \ll \pi/2$ and $\delta \ll \rho$. Then, choosing $\delta = \varepsilon^a$ for some $a > 0$ and using that the involved functions are analytic, quasiperiodic and with zero average, it is possible to obtain exponentially small estimates (see [8, 6] for more details).

With all these ingredients, estimates for the splitting function $\mathcal{M}(s, \theta)$ can be obtained in the singular case, under some restriction $p > p^*$. In the paper [6], where the frequency ω considered is the golden vector, we proved the existence of exactly 4 transverse homoclinic orbits, and their continuation for all values of $\varepsilon \to 0$. In fact, some improvement of the exponents can be given for the case of a fixed torus. Because of this, the exponent p^* depends on the value of ν in (6):

$$p^* = 2 \quad \text{if} \quad \nu = 1, \tag{18}$$

$$p^* = 3 \quad \text{if} \quad \nu = 0. \tag{19}$$

1.3. Description of the results.

Our goal is to study the existence of transverse homoclinic orbits for the Hamiltonian (1–7), assuming in (4) a quadratic frequency vector.

Our aim is to study how far it is possible to generalize the results for the golden vector, obtained in [6], to other quadratic frequencies.

Since we deal with the singular case $\mu = \varepsilon^p$, we need to show that, in the Poincaré–Melnikov approximation (17) for the whole splitting function $\mathcal{M}(s, \theta)$, the term $\mu M(s + s^{(0)}, \theta)$ (exponentially small in ε) dominates, in some sense, the error term $\mathcal{R}(s, \theta)$. A natural approach to this is to provide asymptotic estimates (or at least lower bounds) of the dominant harmonics of the Melnikov potential L. As we will show, such dominant harmonics are closely related to the small divisors of the frequency vector ω. In a subsequent step, we have to see that the estimates obtained for the dominant harmonics of L are big enough in order to be valid also for the dominant harmonics of the splitting potential $\mathcal{L}$ (recall that $\mathcal{M} = \partial_\theta \mathcal{L}$), showing that they overcome the part coming from $\mathcal{R}$.

Note that the quasiperiodicity (16) of the splitting function $\mathcal{M}(s, \theta)$ allows us to restrict to the section $s = 0$, and the (simple) zeros of $\mathcal{M}(0, \theta)$ give rise to (transverse) homoclinic orbits. These (simple) zeros are given by (nondegenerate) critical points of the splitting potential $\mathcal{L}(0, \theta)$. In our main result (Theorem 6), we give conditions for the existence of simple zeros of $\mathcal{M}(0, \theta)$, with asymptotic estimates of the associated eigenvalues of $\partial_\theta \mathcal{M}$.

Let us give a short summary of the results presented. First, in Section 2 we study the *arithmetic properties* of quadratic frequencies, carrying out a complete analysis of the associated *resonances* (to be strict, we should call them *quasi-resonances*), which originate the small divisors appearing in the coefficients of the Melnikov potential. Such an analysis is possible thanks to the *arithmetic properties* of quadratic vectors, and is carried out as a direct generalization of the analysis done in [6] for the golden vector. The main idea used is that a quadratic vector is always an eigenvector of some unimodular matrix [12]. This leads to a classification of such resonances of ω into "primary" and "secondary" ones.

In Section 3 we provide estimates for the Fourier coefficients of the splitting potential $\mathcal{L}$, showing what the *dominant harmonics* are, among the ones associated to primary resonances, and giving upper bounds for the remaining primary ones, and also for all the secondary ones. To prove this result we proceed as in [6], first obtaining estimates for the Fourier coefficients of the Melnikov potential, and then applying the upper bounds given in [8] for the error term.

Since we look for nondegenerate critical points on $\mathbb{T}^2$ of the splitting potential $\mathcal{L}$, we need at least the 2 most dominant harmonics. We show that, when ε goes across the critical values ε_n defined in (40), some changes in the dominance occur. In fact, for ε close to ε_n, we have to consider the 3 most dominant harmonics because the second and third ones are of the same magnitude.

In Section 4 we study the nondegenerate critical points of $\mathcal{L}$ (which correspond to simple zeros of $\mathcal{M}$) and obtain our *main result* (Theorem 6), concerning the existence of a certain number of *transverse homoclinic orbits*. More precisely, we give an asymptotic estimate for the minimum eigenvalue (in modulus) of the splitting matrix $\partial_\theta \mathcal{M}(0, \theta_*)$, for each zero θ_* of the function $\mathcal{M}(0, \cdot)$. This eigenvalue provides a measure of the transversality of the homoclinic orbits. In order to prove the continuation of the transverse homoclinic orbits for the example (1–7), we assume a quite general condition, described in (59), on the phases of the Fourier expansion of the function $f(\varphi)$ in (7).

In the best case, this result is valid in both the cases of 2 or 3 dominant harmonics, and ensures the *continuation* (without bifurcations) of the corresponding homoclinic orbits for all values of $\varepsilon \longrightarrow 0$. Nevertheless, this requires a condition on the quadratic frequency vector $\omega = (1, \Omega)$, that we call the *strong separation condition* (60), ensuring that the influence of secondary resonances can be neglected with respect to primary resonances, and the required dominant harmonics can always be found among the primary resonances. Unfortunately, it seems that such a condition is fulfilled only by the golden vector, given by $\Omega = (\sqrt{5-1}/2)$, and also (consequently) by the "noble" vectors (the ones that can be reduced to the golden vector by a unimodular transformation).

Nevertheless, in other cases one can check a *weak separation condition* (61) that can be used in the case of 2 dominant harmonics, and ensures the existence of transverse homoclinic orbits, at least for ε not very close to the critical values ε_n, but not the continuation of these orbits for all $\varepsilon \longrightarrow 0$. If this weaker condition is not fulfilled, the study of transverse homoclinic orbits becomes more involved because both primary and secondary resonances should be taken into account.

To end this introduction, we describe some notations used in this work. To express the bounds of functions we write $|f| \preceq g$ if we can bound $|f| \leqslant cg$, with some positive constant c not depending on any of the parameters that will be relevant to us, ε and μ. In this way, we do not describe the (usually complicated) dependence on amounts like $\rho, \Omega, \ldots$ and include this dependence in the 'constants'. We use the notation $f \sim g$ if we can bound $g \preceq f \preceq g$. Finally, the notation $f \simeq g$ simply means that they are nearly equal, in the sense that their difference can be neglected.

2. Quadratic frequencies

The analysis of the small divisors becomes relatively simple in the case of a *quadratic frequency vector*: $\omega = (\omega_1, \omega_2)$ such that ω_2/ω_1 is a quadratic irrational number. We assume with no restriction that ω is of the form

$$\omega = (1, \Omega), \qquad 0 < \Omega < 1,$$

where Ω is a quadratic irrational number. Our aim is to take advantage of the nice properties of quadratic irrationals, generalizing the results given in [6] for the case of the golden number, $\Omega = (\sqrt{5} - 1)/2$.

The important feature to be applied is that quadratic vectors are eigenvectors of suitable integer (2×2)-matrices. More precisely, applying a result established in [12], there exists a *unimodular* matrix T (i.e. a square matrix with integer entries and determinant ± 1) having a (unique) eigenvalue λ with $|\lambda| > 1$, whose associated eigenvector is ω. Denoting $\delta = \det T = \pm 1$, the other eigenvalue of T is δ/λ. (In fact, a generalization of the matrix T to higher dimensions is considered in [12]. In our 2-dimensional case, the matrix T can be constructed from the continued fraction of the number Ω; see [17] as a related reference).

It will be a consequence of Theorem 2 below that, for the quadratic vector ω, the small divisors $\langle k, \omega \rangle$ satisfy the Diophantine condition (11), with $\tau = 1$ and some $\gamma > 0$. With this fact in mind, like in [6] we define, for every $k \in \mathbb{Z}^2 \setminus \{0\}$, its associated *"numerator"* as

$$\gamma_k = \gamma_k(\omega) := |\langle k, \omega \rangle| \cdot |k| \tag{20}$$

(for integer vectors, we use the notation $|k| = |k|_1 = |k_1| + |k_2|$), and note that always $\gamma_k \geqslant \gamma$. We are going to provide a simple classification of the (quasi-)resonances associated to ω according to the size of their numerators γ_k.

We say that $k \in \mathbb{Z}^2 \setminus \{0\}$ is *admissible* (or ω-admissible) if $|\langle k, \omega \rangle| < 1/2$, and denote $\mathcal{A}$ the set of admissible integer vectors. The analysis of the resonances can be restricted to the set $\mathcal{A}$, since for any $k \notin \mathcal{A}$ one has $|\langle k, \omega \rangle| > 1/2$ and hence $\gamma_k > |k|/2$.

We now consider the matrix

$$U = (T^\top)^{-1},$$

whose eigenvalues are $1/\lambda$ and $\delta\lambda$, and we denote u, v their associated eigenvectors, respectively. One readily sees that $\langle v, \omega \rangle = 0$.

We stress the following fundamental equality:

$$\langle Uk, \omega\rangle = \langle k, U^{\top}\omega\rangle = \frac{1}{\lambda}\langle k, \omega\rangle. \tag{21}$$

This implies that if $k \in \mathcal{A}$, then also $Uk \in \mathcal{A}$. We say that k is *primitive* if $k \in \mathcal{A}$ but $U^{-1}k \notin \mathcal{A}$. We deduce from (21) that the primitive vectors are exactly the ones satisfying

$$\frac{1}{2|\lambda|} < |\langle k, \omega\rangle| < \frac{1}{2}. \tag{22}$$

It is clear that the admissible vectors are those of the form $k^0(j) = (-\mathrm{rint}(j\Omega), j)$, where $j \neq 0$ is an integer and $\mathrm{rint}(j\Omega)$ denotes the closest integer to $j\Omega$. Then, we have $\langle k^0(j), \omega\rangle = j\Omega - \mathrm{rint}(j\Omega)$. When $k^0(j)$ is primitive, we also say that j is primitive, and denote $\mathcal{P}$ the set of primitive integers j.

For any given $j \in \mathcal{P}$, we define the *resonant sequence* generated by j as the following sequence of (admissible) integer vectors:

$$s(j,n) := U^{n-1}k^0(j), \qquad n \geqslant 1. \tag{23}$$

The following simple result says that such resonant sequences cover the whole set of admissible vectors.

Lemma 1. *For any $k \in \mathcal{A}$, there exist $j \in \mathcal{P}$ and an integer $n \geqslant 1$, both unique, such that $k = s(j,n)$.*

Proof. If $k \in \mathcal{A}$, one finds a unique primitive vector in the sequence $U^{-n}k$, $n \geqslant 0$. Indeed, using (21) one has the equality $|\langle U^{-n}k, \omega\rangle| = |\lambda|^n |\langle k\omega\rangle|$, and hence only one of the vectors $U^{-n}k$ satisfies (22). •

The motivation for defining the sequences $s(j,\cdot)$ is that they provide a classification of the resonances, because the numerators $\gamma_{s(j,n)}$ become nearly constant when $n \to \infty$. In fact the numerators $\gamma_{s(j,n)}$ oscillate around a *"limit numerator"*, which we denote γ_j^*. In the next result we establish the existence of this limit and provide an explicit formula to compute it.

Theorem 2. *For any $j \in \mathcal{P}$, there exists the limit numerator*

$$\gamma_j^* := \lim_{n\to\infty} \gamma_{s(j,n)} = |\langle k^0(j), \omega\rangle| \cdot K(j), \qquad K(j) := \left| k^0(j) - \frac{\langle k^0(j), \omega\rangle}{\langle u, \omega\rangle u} \right|,$$

and one has:

(a) $\gamma_{s(j,n)} = \gamma_j^* + \mathcal{O}(\lambda^{-2n})$, $n \geqslant 1$.

(b) $|s(j,n)| = K(j)|\lambda|^{n-1} + \mathcal{O}(|\lambda|^{-n})$, $n \geqslant 1$.

(c) $\dfrac{(1+\Omega)|j|-a}{2|\lambda|} < \gamma_j^* < \dfrac{(1+\Omega)|j|+a}{2}$, $a = \frac{1}{2}\Big(1 + \frac{|u|}{|\langle u,\omega\rangle|}\Big)$.

Proof. We start by writing $k^0(j)$ as the following linear combination of the eigenvectors of U:

$$k^0(j) = c_1 u + c_2 v, \qquad c_1 = \frac{\langle k^0(j),\omega\rangle}{\langle u,\omega\rangle}, \tag{24}$$

where the value of c_1 has been obtained by taking a scalar product with ω in the linear combination. Then, we see that

$$|c_2 v| = |k^0(j) - c_1 u| = K(j). \tag{25}$$

We deduce from (24) and definition (23) that

$$s(j,n) = \frac{c_1}{\lambda^{n-1}} u + c_2 (\delta\lambda)^{n-1} v.$$

Then,

$$|s(j,n)| = |\lambda|^{n-1}|c_2 v| + \mathcal{O}(|\lambda|^{-n}),$$
$$|\langle s(j,n),\omega\rangle| = \frac{|\langle c_1 u,\omega\rangle|}{|\lambda|^{n-1}} = \frac{|\langle k^0(j),\omega\rangle|}{|\lambda|^{n-1}},$$

and we obtain

$$\gamma_{s(j,n)} = |\langle k^0(j),\omega\rangle| \cdot |c_2 v| + \mathcal{O}(\lambda^{-2n}),$$

whose limit for $n \to \infty$ is

$$\gamma_j^* = |\langle k^0(j),\omega\rangle| \cdot |c_2 v|.$$

The expressions obtained for $|s(j,n)|$, $\gamma_{s(j,n)}$ and γ_j^*, with (25), imply most of the statements.

Finally, we have to find upper and lower bounds for the limit γ_j^*. Since j is primitive we can use (22) to give bounds for $|\langle k^0(j),\omega\rangle|$. Besides, we can give bounds for $K(j)$ using that

$$\begin{aligned}|K(j) - (1+\Omega)|j|| &\leqslant |K(j) - |k^0(j)|| + ||k^0(j)| - (1+\Omega)|j|| \leqslant |c_1 u| + \frac{1}{2} \\ &\leqslant \frac{|u|}{2|\langle u,\omega\rangle|} + \frac{1}{2} = a,\end{aligned}$$

where we have bounded $|c_1|$ from (22), and have used the equality $|k^0(j)| = |j| + |\mathrm{rint}(j\Omega)|$ with the fact that $\mathrm{rint}(j\Omega)$ is the closest integer to $j\Omega$. Then, the upper and lower bounds for $|\langle k^0(j), \omega\rangle|$ and $K(j)$ imply those for γ_j^*. •

We shall always assume that $j > 0$ with no restriction.

The lower bound in (c) shows that, although we cannot expect the limits γ_j^* to be increasing in j, they tend to infinity as $j \to \infty$. This says that the main resonances associated to ω can be found among the sequences (23) generated by the first few primitives. We denote

$$\gamma^* = \liminf_{|k|\to\infty} \gamma_k = \min_{j\in\mathcal{P}} \gamma_j^* = \gamma_{j_0}^*. \tag{26}$$

In this way, the "most resonant" integer vectors are those belonging to the resonant sequence generated by j_0. We call them *primary resonances*, and use for them the notation

$$s_0(n) = s(j_0, n).$$

We then call *secondary resonances* the integer vectors belonging to any of the remaining sequences $s(j,\cdot)$, $j \neq j_0$. We also define the "normalized" limit numerators $\tilde\gamma_j^*$ in such a way that the minimum of them is $\tilde\gamma_{j_0}^* = 1$, and introduce a further parameter $\tilde\gamma^{**} \geqslant 1$, measuring the *separation* between primary and secondary resonances:

$$\tilde\gamma_j^* = \frac{\gamma_j^*}{\gamma^*}, \qquad \tilde\gamma^{**} = \min_{j\in\mathcal{P}\setminus\{j_0\}} \tilde\gamma_j^*. \tag{28}$$

We point out that γ^* and $\tilde\gamma^{**}$ will be important constants for us: γ^* appears in the constant C_0 defined in (35), directly related with the exponentially small estimates for the splitting, and $\tilde\gamma^{**}$ tells us whether it is enough to consider primary resonances in order to study the splitting and its transversality, or secondary resonances are also significative.

To end, we illustrate the results of this section for several examples of quadratic vectors $\omega = (1, \Omega)$. For each number Ω given, we provide the matrices T and U, the eigenvalue λ (which allows us to decide whether a given integer j is primitive or not), the minimum γ^* of the limit numerators, the separation $\tilde\gamma^{**}$, and the first few primitives $k^0(j)$ with their associated normalized limits $\tilde\gamma_j^*$, as well as a lower bound for the remaining ones. The first example is the golden number studied in [6].

Example 1

$\Omega = (\sqrt{5}-1)/2 = 0.618034$

$$T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

$\lambda = 1.618034$

$\gamma^* = 0.723607$

$\tilde{\gamma}^{**} = 4$

$k^0(j)$	$\tilde{\gamma}_j^*$
$(-1,1)$	1
$(-2,4)$	4
$(-4,7)$	5
$(-6,9)$	9
$(-7,12)$	11
$(-9,14)$	11
$(-11,17)$	19
$j \geqslant 18$	> 11.974169

Example 2

$\Omega = \sqrt{2}-1 = 0.414214$

$$T = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$$

$\lambda = 2.414214$

$\gamma^* = 0.5$

$\tilde{\gamma}^{**} = 2$

$k^0(j)$	$\tilde{\gamma}_j^*$
$(0,1)$	1
$(-1,3)$	2
$(-2,4)$	4
$(-2,6)$	8
$(-3,8)$	7
$(-4,9)$	7
$(-5,11)$	14
$j \geqslant 12$	> 6.572330

Example 3

$\Omega = (\sqrt{21}-3)/2 = 0.791288$

$$T = \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix}$$

$\lambda = 4.791288$

$\gamma^* = 0.390891$

$\tilde{\gamma}^{**} = 3$

$k^0(j)$	$\tilde{\gamma}_j^*$
$(-1,1)$	1
$(-2,2)$	4
$(-2,3)$	5
$(-3,4)$	3
$(-5,6)$	7
$(-6,7)$	15
$(-6,8)$	12
$j \geqslant 9$	> 4.030927

Example 4

$\Omega = (\sqrt{15}-3)/2 = 0.436492$

$$T = \begin{pmatrix} 7 & 2 \\ 3 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & -3 \\ -2 & 7 \end{pmatrix}$$

$\lambda = 7.872983$

$\gamma^* = 0.370901$

$\tilde{\gamma}^{**} = 1.5$

$k^0(j)$	$\tilde{\gamma}_j^*$
$(0,1)$	1.5
$(-1,2)$	1
$(-1,3)$	3.5
$(-2,4)$	4
$(-2,5)$	3.5
$(-3,6)$	9
$(-3,8)$	15
$j \geqslant 9$	> 2.030016

Remarks.

(1) It is an obvious consequence of Theorem 2 that the exponent in the Diophantine condition (11) is $\tau = 1$. Besides, the constant γ can be taken as the minimum of all the numerators γ_k, $k \neq 0$. Nevertheless, it is more significant to replace γ by the asymptotic

value γ^* defined in (26).

(2) We shall implicitly assume the hypothesis that the primitive j_0 giving the minimum in (26) is unique, and hence $\tilde{\gamma}^{**} > 1$. In fact, this happens for all the cases we have explored, provided we choose the matrix T suitably.

(3) We see in the examples given above that the limit numerators for the different resonant sequences are integer multiples of a certain number. This fact can be proved rigorously (see [6] for the case of the golden number), and implies a wide separation among the different limit numerators, except for some of them whose limits may coincide.

3. Dominant harmonics of the splitting potential

To show that the splitting potential $\mathcal{L}$ has nondegenerate critical points, we have to consider at least the 2 most dominant harmonics in its Fourier expansion. As we see below, which the dominant harmonics are depends on ε. Nevertheless, for some values of ε we will have to consider the 3 most dominant harmonics because the second and the third ones can be of the same magnitude.

Taking into account that $\mathcal{L}$ is $\hat{\omega}_\varepsilon$-quasiperiodic, we can consider for its Fourier expansion the following expressions:

$$\mathcal{L}(s,\theta) = \sum_{k\in\mathbb{Z}^2} \mathcal{L}_k^* e^{i\langle k,\theta-\hat{\omega}_\varepsilon s\rangle} = \sum_{k\in\mathcal{Z}} \mathcal{L}_k \cos(\langle k,\theta-\hat{\omega}_\varepsilon s\rangle - \tilde{\sigma}_k), \tag{29}$$

where $\mathcal{L}_k$, $\tilde{\sigma}_k$ are real, $\mathcal{L}_k \geqslant 0$ (recall that $\mathcal{Z}$ is defined in (8)). For every $k\in\mathcal{Z}$, the coefficients of the exponential form and the trigonometric form are related by $\mathcal{L}_k^* = \frac{1}{2}\mathcal{L}_k e^{-i\tilde{\sigma}_k}$, $\mathcal{L}_{-k}^* = \overline{\mathcal{L}_k^*} = \frac{1}{2}\mathcal{L}_k e^{i\tilde{\sigma}_k}$.

If some condition on the quadratic vector ω is satisfied, all the involved dominant harmonics of $\mathcal{L}$ will be found among the primary resonances: $k = s_0(n)$. We give below in Lemma 4 an estimate for the dominant harmonics among the primary ones, as well as bounds for both the remaining primary harmonics, and all the secondary harmonics.

We recalled in Section 1.2 that a first order approximation in μ for the splitting potential $\mathcal{L}$ is given by the Melnikov potential L, defined in (16). Thus, we can study the Fourier coefficients of L in order to find the dominant ones, analogously to the approach followed in [6] for the golden vector. Applying the bounds on the error term (17) given in [8], it is possible to show that this dominance persists in the whole splitting potential $\mathcal{L}$.

Let us compute the Fourier coefficients of the Melnikov potential (16):

$$L(s,\theta) = -\sum_{k\in\mathcal{Z}\setminus\{0\}} f_k \int_{-\infty}^{\infty} (\cos x_0(s+bt)-1)\cos(\langle k,\theta+\tilde\omega_\varepsilon t\rangle - \sigma_k)\mathrm{d}t$$

$$= \sum_{k\in\mathcal{Z}\setminus\{0\}} L_k \cos(\langle k,\theta-\hat\omega_\varepsilon s\rangle - \sigma_k),$$

$$L_k = 2f_k \int_{-\infty}^{\infty} \frac{\cos\langle k,\tilde\omega_\varepsilon t\rangle}{\cosh^2 bt}\mathrm{d}t = \frac{2\pi\langle k,\hat\omega_\varepsilon\rangle f_k}{b\sinh(\frac{\pi}{2}\langle k,\hat\omega_\varepsilon\rangle)} = \frac{2\pi|\langle k,\hat\omega_\varepsilon\rangle| e^{-\rho|k|}}{b\sinh|\frac{\pi}{2}\langle k,\hat\omega_\varepsilon\rangle|} \qquad (30)$$

(we take $L_0 = 0$ to have zero average). The integral has been computed by residues, and we have also used the formula $\cos x_0(bt)-1 = -2/\cosh^2 bt$. Notice that the value of ν in (6) does not influence the Melnikov potential, and that the phases σ_k in the Fourier expansion of $L(0,\theta)$ are the same as in the function $f(\varphi)$ given in (7). According to (17), we can expect $\mathcal{L}_k$, $\tilde\sigma_k$ in (29) to be a perturbation of L_k, σ_k.

In the analysis of the coefficients, we first proceed in a rough way in order to motivate the definitions of C_0, ε_n and $h_i(\varepsilon)$ given below. To estimate the size of the coefficients L_k in (30), we use as in [6] the arithmetic properties of ω established in Section 2. Taking into account the definition of γ_k in (20), and the fact that b and b' are μ-close to 1, we have

$$|\langle k,\hat\omega_\varepsilon\rangle| = \left|\langle k, \frac{b'\omega}{b\sqrt{\varepsilon}}\rangle\right| \simeq \frac{\gamma_k}{|k|\sqrt{\varepsilon}}. \qquad (31)$$

Then, we can give from (30) the following approximation for the coefficients:

$$L_k \simeq \alpha_k e^{-\beta_k}, \qquad (32)$$

where

$$\alpha_k = \frac{4\pi\gamma_k}{|k|\sqrt{\varepsilon}[1-\exp\{-\frac{\pi\gamma_k}{|k|\sqrt{\varepsilon}}\}]}, \quad \beta_k = \rho|k| + \frac{\pi\gamma_k}{2|k|\sqrt{\varepsilon}}. \qquad (33)$$

The largest coefficients L_k will be given by the smallest exponents β_k.

A more suitable expression for those exponents is:

$$\beta_k = \frac{C_0\sqrt{\gamma_k}}{2\varepsilon^{1/4}}\Big(\frac{2\rho|k|\varepsilon^{1/4}}{C_0\sqrt{\gamma_k}} + \frac{C_0\sqrt{\gamma_k}}{2\rho|k|\varepsilon^{1/4}}\Big), \qquad (34)$$

where we denote $\tilde{\gamma}_k = \gamma_k/\gamma^*$ analogously to (28), and we consider the important constant

$$C_0 := \sqrt{2\pi\rho\gamma^*}. \tag{35}$$

We deduce from (34) the lower bound

$$\beta_k \geqslant \frac{C_0\sqrt{\tilde{\gamma}_k}}{\varepsilon^{1/4}}, \tag{36}$$

which suggests that the size of the exponent β_k is strongly related (if k is admissible) to the sequence $s(j,\cdot)$, defined in (23), to which k belongs, due to the fact that the numerators tend to a constant for each sequence. Indeed, we know from Theorem 2 that, for k belonging to a given sequence $s(j,\cdot)$, the limit of the $\tilde{\gamma}_k$ is the number $\tilde{\gamma}_j^*$ defined in (28). This says that the smallest exponents β_k can be found among the primary resonances $s_0(\cdot)$, defined in (27).

Let us study which primary resonances give the smallest exponents. Recall from Theorem 2 the approximations

$$\tilde{\gamma}_{s_0(n)} = 1 + \mathcal{O}(\lambda^{-2n}), \tag{37}$$

$$|s_0(n)| = K(j_0)|\lambda|^{n-1} + \mathcal{O}(|\lambda|^{-n}). \tag{38}$$

Then, taking $k = s_0(n)$ in (34) we get

$$\beta_{s_0(n)} \simeq \frac{C_0}{2\varepsilon^{1/4}}\Big(\frac{2\rho K(j_0)|\lambda|^{n-1}\varepsilon^{1/4}}{C_0} + \frac{C_0}{2\rho K(j_0)|\lambda|^{n-1}\varepsilon^{1/4}}\Big) = \frac{C_0 g_n(\varepsilon)}{\varepsilon^{1/4}}, \tag{39}$$

where we have considered the decreasing sequences

$$\varepsilon_n := \Big(\frac{C_0}{2\rho K(j_0)|\lambda|^{n-1}}\Big)^4 = \frac{\varepsilon_0}{\lambda^{4n}}, \quad \varepsilon_n' := \sqrt{\varepsilon_n\varepsilon_{n-1}} = \frac{\varepsilon_0}{\lambda^{4n-2}}, \tag{40}$$

and the functions

$$g_n(\varepsilon) := \frac{1}{2}\Big[\Big(\frac{\varepsilon}{\varepsilon_n}\Big)^{1/4} + \Big(\frac{\varepsilon_n}{\varepsilon}\Big)^{1/4}\Big] = \cosh\Big(\frac{\ln\varepsilon - \ln\varepsilon_n}{4}\Big) = g_0(\lambda^{4n}\varepsilon),$$

which contain the main information on the size of $\beta_{s_0(n)}$. It is clear that each g_n has its minimum at $\varepsilon = \varepsilon_n$. Notice that, as a function of $\ln\varepsilon$, the graph of g_n is simply the graph of g_0 translated a distance $4n\ln|\lambda|$. This is illustrated in Figure 1, using logarithmic scale for ε for the sake of clarity. Note that for n large, the neglected terms in (37–38) become smaller, and the approximation obtained through the function $g_n(\varepsilon)$ becomes better.

We now consider for $n \geqslant 1$ the intervals $\mathcal{I}_n = [\varepsilon'_{n+1}, \varepsilon'_n]$, $\mathcal{I}'_n = [\varepsilon'_{n+1}, \varepsilon_n,$ $\mathcal{I}''_n = [\varepsilon_n, \varepsilon'_n]$, and define the following functions:

$$\begin{aligned} &h_1(\varepsilon) = g_n(\varepsilon), && \text{for } \varepsilon \in \mathcal{I}_n, \\ &h_2(\varepsilon) = g_{n+1}(\varepsilon),\ h_3(\varepsilon) = g_{n-1}(\varepsilon),\ h_4(\varepsilon) = g_{n+2}(\varepsilon), && \text{for } \varepsilon \in \mathcal{I}'_n, \\ &h_2(\varepsilon) = g_{n-1}(\varepsilon),\ h_3(\varepsilon) = g_{n+1}(\varepsilon),\ h_4(\varepsilon) = g_{n-2}(\varepsilon), && \text{for } \varepsilon \in \mathcal{I}''_n. \end{aligned} \tag{41}$$

By connecting the successive intervals $\mathcal{I}_n$, we get that these functions are continuous for all $0 < \varepsilon < \varepsilon'_1$, and satisfy the equality

$$h_i(\lambda^4 \varepsilon) = h_i(\varepsilon) \tag{42}$$

for any ε. In other words, the functions h_i are $4 \ln |\lambda|$-periodic in $\ln \varepsilon$. See Figure 1 for an illustration of the functions $h_i(\varepsilon)$.

An equivalent way to introduce these functions is to define

$$h_1(\varepsilon) = \frac{1}{2}\Big[\Big(\frac{\varepsilon}{\varepsilon_1}\Big)^{1/4} + \Big(\frac{\varepsilon_1}{\varepsilon}\Big)^{1/4}\Big], \qquad \varepsilon \in \mathcal{I}_1,$$

$$h_2(\varepsilon) = \begin{cases} \frac{1}{2}\Big[\Big(\frac{\varepsilon}{\varepsilon_2}\Big)^{1/4} + \Big(\frac{\varepsilon_2}{\varepsilon}\Big)^{1/4}\Big], & \varepsilon \in \mathcal{I}'_1, \\ \frac{1}{2}\Big[\Big(\frac{\varepsilon}{\varepsilon_0}\Big)^{1/4} + \Big(\frac{\varepsilon_0}{\varepsilon}\Big)^{1/4}\Big], & \varepsilon \in \mathcal{I}''_1, \end{cases}$$

and similarly for $h_3(\varepsilon)$ and $h_4(\varepsilon)$, and extend them according to (42).

Defining the constants

$$A_i = \frac{1}{2}(|\lambda|^{i/2} + |\lambda|^{-i/2}), \tag{43}$$

we can easily check the following bounds for the functions $h_i(\varepsilon)$:

$$1 \leqslant h_1(\varepsilon) \leqslant A_1 \leqslant h_2(\varepsilon) \leqslant A_2 \leqslant h_3(\varepsilon) \leqslant A_3 \leqslant h_4(\varepsilon) \leqslant A_4,$$

where equalities can take place only for $\varepsilon = \varepsilon_n, \varepsilon'_n$. More precisely, for $\varepsilon = \varepsilon_n$ we have $h_1 < h_2 = h_3 < h_4$, and for $\varepsilon = \varepsilon'_n$ we have $h_1 = h_2 < h_3 = h_4$ (see Figure 1 again).

For any given $\varepsilon < \varepsilon_1$, we define $N_i = N_i(\varepsilon)$, $i = 1, 2, 3, 4$, as the 4 integers $n \geqslant 1$ minimizing $g_n(\varepsilon)$. This means that

$$g_{N_1}(\varepsilon) \leqslant g_{N_2}(\varepsilon) \leqslant g_{N_3}(\varepsilon) \leqslant g_{N_4}(\varepsilon) \leqslant g_n(\varepsilon) \quad \forall n \neq N_1, N_2, N_3, N_4. \tag{44}$$

For ε belonging to a concrete interval $\mathcal{I}_n$, the first minimum is given by $N_1 = n$. The second, third and fourth minima are $N_2 = n \pm 1$, $N_3 = n \mp 1$

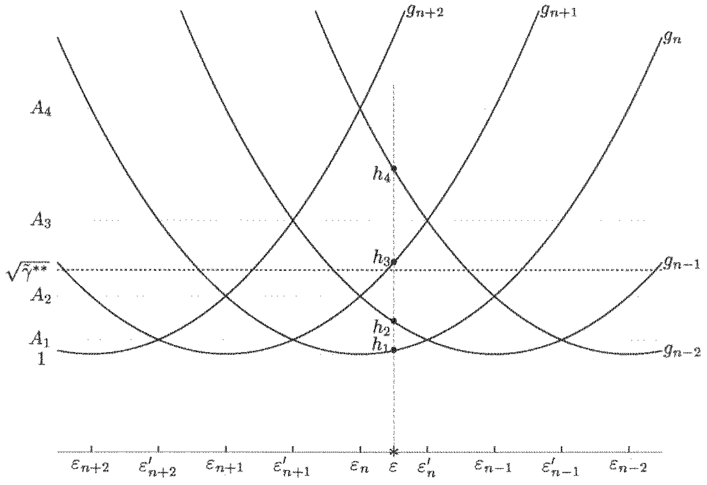

Fig. 1. The functions $h_i(\varepsilon)$ (using logarithmic scale for ε).

and $N_4 = n \pm 2$ respectively, and the signs depend on the subinterval to which ε belongs: $\mathcal{I}'_n$ or $\mathcal{I}''_n$. In this way, the integers N_i are *consecutive* (but not ordered). The main fact to be used is that the values of the 4 minima are given by the functions h_i defined in (41). Indeed, one easily checks that

$$g_{N_i}(\varepsilon) = h_i(\varepsilon), \quad i = 1, 2, 3, 4. \tag{45}$$

Notice that there is some ambiguity in the definition of $N_i(\varepsilon)$ at the endpoints of the intervals, but the important fact is that they are critical values at which some of the $N_i(\varepsilon)$ giving the minima change when ε goes across them.

For the sake of shortness, we also denote

$$S_i = S_i(\varepsilon) := s_0(N_i(\varepsilon)), \quad i = 1, 2, 3, 4,$$

the primary resonances indexed by the minimizing integers. As a consequence of Theorem 2 and the definition of N_i, one easily deduces the following estimate, to be used later:

$$|S_i| \sim |\lambda|^{N_i} \sim \varepsilon^{-1/4}, \quad i = 1, 2, 3, 4. \tag{46}$$

The next lemma implies that the 3 most dominant harmonics of the splitting potential, among the primary ones, are the (consecutive) ones corresponding to S_1, S_2, S_3, giving an asymptotic estimate for the coefficients $\mathcal{L}_{S_i}$, as well as a bound comparing the phases $\tilde{\sigma}_k$ with the original phases σ_k in (7) (this is affected by the translation $s^{(0)}$ that appears in (17)). Besides, the lemma provides an estimate for the sum of all the coefficients $\mathcal{L}_k$ (recall that $\mathcal{L}_k \geqslant 0$) associated to primary resonances, except the l dominant ones ($0 \leqslant l \leqslant 3$), in terms of the first neglected harmonic among the primary ones, $\mathcal{L}_{S_{l+1}}$, as well as an upper bound for the sum of all the coefficients $\mathcal{L}_k$ associated to secondary resonances. The sum of the two bounds can be considered as a bound of the difference between the splitting potential and the main part of it, given by the dominant harmonics. In fact, since we are interested in some derivative of the Melnikov potential, we consider the sum of (positive) amounts of the type $|k|^m \mathcal{L}_k$. The constant C_0 in the exponentials has been defined in (35).

We recall that the notations "$\preceq$" and "$\sim$" were introduced at the end of Section 1.3.

Lemma 3. *Assume that $\varepsilon \preceq 1$ and $\mu = \varepsilon^p$, $p > p^*$, with p^* as defined in* (18–19). *Then, one has:*

(a) $\mathcal{L}_{S_i} \sim \frac{\mu}{\varepsilon^{1/4}} \exp\left\{ -\frac{C_0 h_i(\varepsilon)}{\varepsilon^{1/4}} \right\}$,
$|\tilde{\sigma}_{S_i} - \sigma_{S_i} - s^{(0)}\langle S_i, \hat{\omega}_\varepsilon\rangle| \preceq \frac{\mu}{\varepsilon^{p^*}}$, $i = 1, 2, 3, 4$.

(b) $\displaystyle\sum_{\substack{k \in s_0(\cdot) \\ k \neq S_1, \dots, S_l}} |k|^m \mathcal{L}_k \sim \frac{1}{\varepsilon^{m/4}} \mathcal{L}_{S_{l+1}}$, $0 \leqslant l \leqslant 3$, $m \geqslant 0$.

(c) $\displaystyle\sum_{k \notin s_0(\cdot)} |k|^m \mathcal{L}_k \preceq \frac{\mu}{\varepsilon^{(m+1)/4}} \exp\left\{ -\frac{C_0 \sqrt{\gamma^{**}}}{\varepsilon^{1/4}} \right\}$, $m \geqslant 0$.

Proof. The proof follows essentially as in [6], and we only give here a sketch of the proof. The main idea is to deduce the results for the coefficients $\mathcal{L}_k$ of the splitting potential comparing them with the coefficients L_k of the Melnikov potential, with the help of the bound for the error term (17) provided in [8]. In the notation (29), the Fourier coefficients (in the exponential form) are $\mathcal{M}_k^* = ik\mathcal{L}_k^*$ and $M_k^* = ikL_k^*$ respectively. Then, we see from (17) that the Fourier coefficients of the error term $\mathcal{R}(s, \theta)$ are $\mathcal{R}_k^* = ik(\mathcal{L}_k^* - \mu L_k^* e^{-is^{(0)}\langle k\hat{\omega}_\varepsilon\rangle})$, $k \neq 0$, and taking modulus and argument we get

$$|\mathcal{L}_k - \mu L_k| \preceq \frac{|\mathcal{R}_k^*|}{|k|}, \qquad |\tilde{\sigma}_k - \sigma_k - s^{(0)}\langle k\hat{\omega}_\varepsilon\rangle| \preceq \frac{|\mathcal{R}_k^*|}{|k|\mu L_k}. \tag{47}$$

It is given in [8, Th. 10] (see also [6,Th. 0]) a bound for the error term on a complex domain: $|\mathrm{Im} s| \leqslant \pi/2 - \delta$, $|\mathrm{Im}\rho| \leqslant \rho| - \delta$, where $\delta > 0$ is a small reduction. Choosing $\delta = \varepsilon^{1/4}$, the bound on such a domain can be written as

$$|\mathcal{R}| \preceq \frac{\mu^2}{\varepsilon^q},$$

with $q = 5/2$ if $\nu = 1$, and $q = 7/2$ if $\nu = 0$ (recall that there is some improvement in the case of a fixed torus). Since $\mathcal{R}$ is $\hat{\omega}_\varepsilon$-quasiperiodic, applying to it a standard result (see, for instance, [8, Lemma 11]) we get the following bound for its Fourier coefficients:

$$|\mathcal{R}^*_k| \preceq \frac{\mu^2}{\varepsilon^q} e^{-\tilde{\beta}_k} \simeq \frac{\mu^2}{\varepsilon^q} e^{-\beta_k},$$
$$\tilde{\beta}_k = (\rho - \varepsilon^{1/4})|k| + (\frac{\pi}{2} - \varepsilon^{1/4})|\langle k, \hat{\omega}_\varepsilon \rangle| \simeq \beta_k \tag{48}$$

where, as in [6], the perturbation terms with $\varepsilon^{1/4}$ in the exponent $\tilde{\beta}_k$ can be neglected thanks to the denominator $\varepsilon^{1/4}$ in (34), and the μ-small terms in $\hat{\omega}_\varepsilon$ can be neglected as in (31).

To establish (a) and (b) we only need to consider primary resonances. For the coefficients $\mathcal{L}_{s_0(n)}$ of the splitting potential, in a first step we consider the approximation given by the coefficients $L_{s_0(n)}$ of the Melnikov potential and look for the dominant ones. Then, in a second step we show that, if $\mu = \varepsilon^p$ with $p > p^*$, such dominance remains unchanged when the error term (17) is added.

Thus, we first look for the largest coefficients $L_{s_0(n)}$, i.e. the smallest exponents $\beta_{s_0(n)}$. We see from (39) and (44) that the 4 smallest exponents are the ones obtained for $n = N_i$. Since the functions $h_i(\varepsilon)$ have been defined in such a way that (45) holds, we deduce that

$$\beta_{S_i} \simeq \frac{C_0 h_i(\varepsilon)}{\varepsilon^{1/4}}, \qquad i = 1, 2, 3, 4.$$

We see from (46) that $\alpha_{S_i} \sim \varepsilon^{-1/4}$ in (32–33), and hence

$$L_{S_i} \sim \frac{1}{\varepsilon^{1/4}} \exp\Big\{ - \frac{C_0 h_i(\varepsilon)}{\varepsilon^{1/4}} \Big\}, \qquad i = 1, 2, 3, 4. \tag{49}$$

We have to recall that we are dealing with approximations, and we actually have a perturbation of the situations described, due to the terms neglected in (31) and (37–38). As thoroughly explained in [6], if we take into account the size of the terms neglected, we can see that, under our choice of μ, the asymptotic estimates for the dominant coefficients remain the same.

Now, to estimate the size of the coefficients $\mathcal{L}_{S_i}$ of the splitting potential, we use (fitark1–fitark2) together with (domprimary), and we get

$$|\mathcal{L}_{S_i} - \mu L_{S_i}| \preceq \frac{\mu^2}{\varepsilon^{q-1/4}} \exp\Big\{ -\frac{C_0 h_i(\varepsilon)}{\varepsilon^{1/4}} \Big\}, \qquad i = 1, 2, 3, 4.$$

This upper bound is dominated by the term $|\mu L_{S_i}|$, estimated in (aproxmelnidom), provided $\mu \preceq \varepsilon^p$ with $p > q - 1/2 = p^*$. Taking p^* as defined in (18–19), we obtain the first statement of (a), and the second one is proved in a similar way.

The proof of (b) works as in [6], bounding the sum of the coefficients $|s_0(n)|^m \mathcal{L}_{s_0(n)}$, excluding some (consecutive) dominant ones ($n \neq N_1, \ldots, N_l$), by a geometric series whose main term is the next dominant harmonic ($n = N_{l+1}$). It can also be shown that the term $|s_0(n)|^m$ does not affect such dominance.

Finally, we can prove (c) in a similar way, bounding the sum of the secondary coefficients $|k|^m \mathcal{L}_k$ with $k \in s(j, \cdot)$, $j \neq j_0$, by a geometric series. Now, an upper bound for the main term of this series can be given from the lower bound (36) for the exponent β_k, and using that the normalized numerators $\tilde{\gamma}_k$, for $k \in s(j, \cdot)$, tend to $\tilde{\gamma}_j^* \geqslant \tilde{\gamma}^{**}$. In fact, the sum also includes the coefficients associated to non-admissible k, i.e. not belonging to any sequence $s(j, \cdot)$ (see Section 2). Such coefficients are clearly dominated by the admissible ones, since we have $\gamma_k > |k|/2$ and we always find in (33) that $\beta_k \succeq 1/\sqrt{\varepsilon}$ for the non-admissible case. •

Remarks.

(1) We have shown that the harmonics S_i are the most dominant among the primary ones, but it is not excluded that some secondary harmonic can be more dominant than some of the S_i. This depends on the relation between the separation $\tilde{\gamma}^{**}$ and the constants A_i introduced in (43). Thus, if $A_i \leqslant \sqrt{\tilde{\gamma}^{**}} < A_{i+1}$, we can ensure that the i most dominant harmonics are primary (see Figure 1 again).

(2) To give a more refined bound in (c), we could define some functions (periodic in $\ln \varepsilon$) for the secondary resonances, analogous to the functions $h_i(\varepsilon)$ introduced in (41). Then, the number of (primary or secondary) dominant harmonics for which asymptotic estimates can be given could be bigger than in the previous

remark. It is not hard to carry out this approach for concrete examples of quadratic frequencies, but it seems more involved to give a general description of it.

(3) Assume that we consider a perturbation $f(\varphi)$ having only primary harmonics, instead of the "full" series considered in 7. Then, the Melnikov potential L has only primary harmonics. The splitting potential $\mathcal{L}$ can be "full", but its secondary harmonics would be μ^2-small (since they come from the error term). Then, the dominant harmonics would all be found among the primary ones, and this is not obstructed by the separation $\tilde{\gamma}^{**}$. An example of this type is given in [4], where the frequency vector is the golden one and the perturbation has only the harmonics associated to Fibonacci numbers.

4. Critical points of the splitting potential

We are going to use in this section the estimates given in Lemma 3, to show that the splitting potential $\mathcal{L}(0,\theta)$ has nondegenerate critical points (fixing $s = 0$). First, we will study the critical points for the approximations given by the 2 or 3 most dominant harmonics, among the primary ones:

$$\begin{aligned} \mathcal{L}^{(2)}(\theta) &= \sum_{i=1,2} \mathcal{L}_{S_i} \cos(\langle S_i, \theta\rangle - \tilde{\sigma}_{S_i}), \\ \mathcal{L}^{(3)}(\theta) &= \sum_{i=1,2,3} \mathcal{L}_{S_i} \cos(\langle S_i, \theta\rangle - \tilde{\sigma}_{S_i}). \end{aligned} \tag{50}$$

Afterwards, we discuss the persistence of these critical points in the whole function $\mathcal{L}(0,\theta)$. As the functions $h_i(\varepsilon)$ defined in (41) suggest, it seems natural to consider the 2 dominant harmonics for most values of ε, and 3 dominant harmonics for ε close to a critical value ε_n. Nevertheless, we stressed in remark 1 after Lemma 3 that some of the dominant harmonics may be secondary, depending on the separation $\tilde{\gamma}^{**}$. Then, the approximations defined in (50) would not be good enough.

Anyway, to begin we study the function $\mathcal{L}^{(2)}2$ for $\varepsilon \neq \varepsilon_n$, and the function $\mathcal{L}^{(3)}$ for $\varepsilon \neq \varepsilon'_n$. To fix ideas, we look at concrete intervals: we assume $\varepsilon \in (\varepsilon_n, \varepsilon_{n-1})$ in the first case, and $\varepsilon \in (\varepsilon'_{n+1}, \varepsilon'_n)$ in the second case. Recalling Figure 1, note that

$$\begin{aligned} &S_1 = s_0(n),\ S_2 = s_0(n+1),\ S_3 = s_0(n-1), && \text{for } \ \varepsilon \in (\varepsilon'_{n+1}, \varepsilon_7 n), \\ &S_1 = s_0(n),\ S_2 = s_0(n-1),\ S_3 = s_0(n+1), && \text{for } \ \varepsilon \in (\varepsilon_n, \varepsilon'_n), \\ &S_1 = s_0(n-1),\ S_2 = s_0(n), && \text{for } \ \varepsilon \in (\varepsilon'_n, \varepsilon_{n-1}). \end{aligned}$$

In order to have a simpler expression for the functions $\mathcal{L}^{(i)}(\theta)$, we carry out in both cases the linear change $(\theta_1, \theta_2) \mapsto (\psi_1, \psi_2)$ defined by

$$\psi_1 = \langle s_0(n-1), \theta \rangle - \tilde{\sigma}_{s_0(n-1)}, \quad \psi_2 = \langle s_0(n), \theta \rangle - \tilde{\sigma}_{s_0(n)}, \tag{51}$$

which can be written as

$$\psi = \mathcal{A}_n \theta - b_n, \quad \text{where } \ \mathcal{A}_n = \begin{pmatrix} s_0(n-1)^\top \\ s_0(n)^\top \end{pmatrix}, b_n = \begin{pmatrix} \tilde{\sigma}_{s_0(n-1)} \\ \tilde{\sigma}_{s_0(n)} \end{pmatrix}. \tag{52}$$

This change is not always one-to-one on $\mathbb{T}^2$. Indeed, calling

$$\delta = \det U = \pm 1, \qquad \tau = \operatorname{tr} U,$$

we have $U^2 = \tau U - \delta \mathrm{Id}$ and we deduce the following recurrence relation for the primary resonances:

$$s_0(n+1) = \tau s_0(n) - \delta s_0(n-1).$$

Using induction, we deduce from this relation that $|\det \mathcal{A}_n| = \kappa$ for all n, where we denote

$$\kappa := |\det \mathcal{A}_2| = |\det(k^0(j_0), U k^0(j_0))| \tag{53}$$

(a nonvanishing integer, since $k^0(j_0)$ is not an eigenvector of U). This says that the change (52) takes κ points (θ_1, θ_2) to 1 point (ψ_1, ψ_2). With the change (52), the functions $\mathcal{L}^{(2)}(\theta)$, $\mathcal{L}^{(3)}(\theta)$ move respectively to the following ones:

$$\begin{aligned} \mathcal{K}^{(2)}(\psi) &= A \cos \psi_1 + B \cos \psi_2, \\ \mathcal{K}^{(3)}(\psi) &= B\eta(1-Q) \cos \psi_1 + B \cos \psi_2 + B\eta Q \cos(\tau \psi_2 - \delta \psi_1 - \Delta\tilde{\sigma}), \end{aligned} \tag{54}$$

where we denote

$$\begin{aligned} &A = \mathcal{L}_{s_0(n-1)}, \quad B = \mathcal{L}_{s_0(n)}, \\ &\eta = \frac{\mathcal{L}_{s_0(n-1)} + \mathcal{L}_{s_0(n+1)}}{\mathcal{L}_{s_0(n)}}, \qquad Q = \frac{\mathcal{L}_{s_0(n+1)}}{\mathcal{L}_{s_0(n-1)} + \mathcal{L}_{s_0(n+1)}}, \\ &\Delta\tilde{\sigma} = \tilde{\sigma}_{s_0(n+1)} - \tau \tilde{\sigma}_{s_0(n)} + \delta \tilde{\sigma}_{s_0(n-1)} \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}. \end{aligned} \tag{55}$$

Note that A, B, η and Q are positive, because so are the coefficients $\mathcal{L}_k$ in (29). Looking at $\mathcal{K}^{(2)}$, we have $B = \mathcal{L}_{S_1}$, $A = \mathcal{L}_{S_2}$ for $\varepsilon \in (\varepsilon_n, \varepsilon'_n)$, and $A = \mathcal{L}_{S_1}$, $B = \mathcal{L}_{S_2}$ for $\varepsilon \in (\varepsilon'_n, \varepsilon_{n-1})$, i.e. the first and second dominant harmonics swap when ε goes across the value ε'_n.

Instead, when looking at $\mathcal{K}^{(3)}$ we have $B = \mathcal{L}_{S_1}$ for any $\varepsilon \in (\varepsilon'_{n+1}, \varepsilon'_n)$. So the first dominant harmonic of $\mathcal{K}^{(3)}$ is always $\cos\psi_2$, whereas the second and third ones swap when ε goes across ε_n. Note that η measures the size of the second and third harmonics with respect to the first one, and Q is an indicator of the relative weight of the second and third harmonics ($0 < Q < 1$). We study $\mathcal{K}^{(3)}$ in terms of η and Q, considering η as a perturbation parameter (note that $\eta \sim \mathcal{L}_{S_2}/\mathcal{L}_{S_1}$ is small except for ε close to the endpoints ε'_n, ε'_{n+1}). However, we have to point out that η and Q are not independent parameters, because they are both linked to ε.

In the next lemma, we show the existence of 4 critical points for η small enough and any Q, provided the difference of phases $\triangle\tilde{\sigma} \in \mathbb{T}$ is not very close to 0 or $\pi \pmod{2\pi}$. To measure this closeness, we denote

$$\tilde{\sigma}^* = \min(|\triangle\tilde{\sigma}|, |\triangle\tilde{\sigma} - \pi|).$$

Lemma 4.

(a) *The function $\mathcal{K}^{(2)}$ has exactly 4 critical points, all nondegenerate: $\psi^{(2)}_{(1)} = (0,0)$, $\psi^{(2)}_{(2)} = (0,\pi)$, $\psi^{(2)}_{(3)} = (\pi,0)$, $\psi^{(2)}_{(4)} = (\pi,\pi)$. At the critical points, $|\det \mathrm{D}^2\mathcal{K}^{(2)}(\psi^{(2)}_{(j)})| = AB$.*

(b) *Assume $\tilde{\sigma}^* > 0$ and define $E^{(\pm)}$, $\alpha^{(\pm)}$ by*

$$E^{(\pm)} = \sqrt{1 - 2Q(1-Q)(1-(\pm1)^{\tau}\cos\triangle\tilde{\sigma})},$$

$$\cos\alpha^{(\pm)} = \frac{(1-Q) + (\pm1)^{\tau} Q\cos\triangle\tilde{\sigma}}{E^{(\pm)}}, \quad \sin\alpha^{(\pm)} = -\frac{(\pm1)^{\tau}\delta Q\sin\triangle\tilde{\sigma}}{E^{(\pm)}}.$$

Then, for any $Q \in [0,1]$ and $0 < \eta \preceq \tilde{\sigma}^*$ the function $\mathcal{K}^{(3)}$ has exactly 4 critical points, all nondegenerate: $\psi^{(3)}_{(j)} = \psi^{(3)}_{(j),0} + \mathcal{O}(\eta)$, $j = 1,2,3,4$, where $\psi^{(3)}_{(1),0} = (\alpha^{(+)}, 0)$, $\psi^{(3)}_{(2),0} = (\alpha^{(-)}, \pi)$, $\psi^{(3)}_{(3),0} = (\alpha^{(-)} + \pi, 0)$, $\psi^{(3)}_{(4),0} = (\alpha^{(-)} + \pi, \pi)$. At the critical points,

$$|\det \mathrm{D}^2\mathcal{K}^{(3)}(\psi^{(3)}_{(1,3)})| = B^2(E^{(+)}\eta + \mathcal{O}(\eta^2)),$$

$$|\det \mathrm{D}^2\mathcal{K}^{(3)}(\psi^{(3)}_{(2,4)})| = B^2(E^{(-)}\eta + \mathcal{O}(\eta^2)).$$

Proof. We do not prove (a), because it is very simple. Instead, the proof of (b) requires some more work, and is carried out as in [6]. The critical points of $\mathcal{K}^{(3)}$ are the solutions of the following system of equations:

$$\sin\psi_2 = -\eta\delta\tau(1-Q)\sin\psi_1, \quad (1-Q)\sin\psi_1 - \delta Q\sin(\tau\psi_2-\delta\psi_1-\triangle\tilde{\sigma}) = 0. \tag{56}$$

It is clear that, for η small enough, the solutions of the first equation of (56) are two curves in $\mathbb{T}^2$. One of these curves is η-close to the line $\psi_2 = 0$, and the other one is η-close to the line $\psi_2 = \pi$. To get the solutions of (56) on the first curve, we replace $\psi_2 = \mathcal{O}(\eta)$ into the second equation, and obtain the equation $F_\eta^{(+)}(\psi_1) = 0$, with

$$\begin{aligned} F_\eta^{(+)}(\psi_1) &= (1-Q)\sin\psi_1 + Q\sin(\psi_1 + \delta\triangle\tilde{\sigma}) + \mathcal{O}(\eta) \\ &= E^{(+)}\sin(\psi_1 - \alpha^{(+)}) + \mathcal{O}(\eta). \end{aligned}$$

For $\eta = 0$, the solutions are clearly $\alpha^{(+)}$ and $\alpha^{(+)} + \pi$, except for the case that $E^{(+)} = 0$ (avoided with the condition $\tilde{\sigma}^* > 0$). Note that $E^{(+)} \geqslant \sqrt{(1+\cos\triangle\tilde{\sigma})/2} \succeq \tilde{\sigma}^*$ and, consequently, these solutions persist for $\eta \preceq \tilde{\sigma}^*$. The perturbed solutions obtained give rise to the critical points $\psi_{(1)}^{(3)}$, $\psi_{(3)}^{(3)}$.

Analogously, one can replace $\psi_2 = \pi + \mathcal{O}(\eta)$ into the second equation of (56), obtaining the equation $F_\eta^{(-)}(\psi_1) = 0$, with

$$\begin{aligned} F_\eta^{(-)}(\psi_1) &= (1-Q)\sin\psi_1 + (-1)^\tau Q\sin(\psi_1 + \delta\triangle\tilde{\sigma}) + \mathcal{O}(\eta) \\ &= E^{(-)}\sin(\psi_1 - \alpha^{(-)}) + \mathcal{O}(\eta), \end{aligned}$$

whose solutions are now η-perturbations of $\alpha^{(-)}$ and $\alpha^{(-)} + \pi$, except for the case that $E^{(-)} = 0$ (also avoided), leading to the critical points $\psi_{(2)}^{(3)}$, $\psi_{(4)}^{(3)}$.

The determinant is easily computed. We have

$$\begin{aligned} &\det \mathrm{D}^2\mathcal{K}^{(3)}(\psi) \\ &= B^2(\eta\cos\psi_2 \cdot ((1-Q)\cos\psi_1 + Q\cos(\tau\psi_2 - \delta\psi_1 - \triangle\tilde{\sigma})) + \mathcal{O}(\eta^2)) \end{aligned}$$

for any $\psi \in \mathbb{T}^2$. At the point $\psi_{(1)}^{(3)} = (\alpha^{(+)}, 0) + \mathcal{O}(\eta)$, we obtain

$$\det \mathrm{D}^2\mathcal{K}^{(3)}(\psi_{(1)}^{(3)}) = B^2(\eta(F_0^{(+)})'(\alpha^{(+)}) + \mathcal{O}(\eta^2)) = B^2(E^{(+)}\eta + \mathcal{O}(\eta^2)),$$

and similarly with $\psi_{(2)}^{(3)}$, $\psi_{(3)}^{(3)}$, $\psi_{(4)}^{(3)}$. $\bullet$

Remarks.

(1) The amount $\tilde{\sigma}^*$, which measures the distance from $\Delta\tilde{\sigma}$ to the "forbidden" values 0 and $\pi \pmod{2\pi}$, depends on n, as we see in (55). However, in Theorem 6 we will assume $\tilde{\sigma}^*$ greater than a concrete positive constant (independent of n) by imposing a simple condition on the phases $\sigma_{s_0(n)}$ of the initial perturbation (7).

(2) We can give a description of the *continuation* of the critical points of $\mathcal{K}^{(3)}$ as Q goes from 0 to 1 (recall that this corresponds to transfer the second dominance from the harmonic $\cos\psi_1$ to the harmonic $\cos(\tau\psi_2 - \delta\psi_1 - \Delta\tilde{\sigma})$). Assuming for instance that $0 < \Delta\tilde{\sigma} < \pi$, the point $\psi^{(3)}_{(1)}$ drifts on a line from $(0,0)$ to $(\Delta\tilde{\sigma}, 0)$ with the first component increasing, the point $\psi^{(3)}_{(2),0}$ drifts on a line from $(0,\pi)$ to $(\Delta\tilde{\sigma}+\pi, \pi)$ with the first component decreasing, etc. When one considers the perturbed points $\psi^{(3)}_{(j)} = \psi^{(3)}_{(j),0} + \mathcal{O}(\eta)$, such lines become close curves.

(3) If $\Delta\tilde{\sigma}$ is near to 0 or π, one of the determinants, given at first order by $E^{(\pm)}$, can be very small. Indeed, for $Q = 1/2$ one has $E^{(\pm)} = \sqrt{\frac{1}{2}(1 + (\pm 1)^\tau \cos\Delta\tilde{\sigma})}$. Then, studying more carefully the term $\mathcal{O}(\eta)$ neglected from the equations one could show that, near this value $Q = 1/2$, bifurcations of some of the 4 critical points can take place. Examples of such bifurcations have been shown in [22].

(4) Concerning the possibility of bifurcations, we stress here that two different situations may occur depending on whether $\tau = \mathrm{tr}U$ is odd or even. This can be seen from the expressions of $E^{(\pm)}$ given in the previous remark for $Q = 1/2$, studying when such expressions vanish. One gets that, if τ is odd, then for $\Delta\tilde{\sigma}$ close to 0 the critical points $\psi^{(3)}_{(2,4)}$ may bifurcate, whereas $\psi(3)_{(2,4)}$ continue, and for $\Delta\tilde{\sigma}$ close to π the situation is the opposite. Instead, if τ is even, then for $\Delta\tilde{\sigma}$ close to 0 the 4 critical points continue, and for $\Delta\tilde{\sigma}$ close to π the 4 critical points may bifurcate.

Next, we translate the results of Lemma 4 through the linear change (51). As said in (53), each critical point of the function $\mathcal{K}^{(i)}(\psi)$ gives rise to κ critical points of $\mathcal{L}^{(i)}(\theta)$. For each critical point $\theta^{(i)}_*$, we also find an estimate for (the modulus of) the minimum eigenvalue $m^{(i)}_*$ of the symmetric matrix $\mathrm{D}^2\mathcal{L}^{(i)}$ at this point. This eigenvalue is closely related with the transversality of the homoclinic orbit associated to the critical point.

Lemma 5.

(a) *The function* $\mathcal{L}^{(2)}$ *has exactly* 4κ *critical points* $\theta_*^{(2)}$*, all nondegenerate and satisfying*

$$m_*^{(2)} \sim \sqrt{\varepsilon}\mathcal{L}_{S_2}.$$

(b) *Assuming* $\tilde{\sigma}^* > 0$ *and* $\mathcal{L}_{S_2} \preceq \tilde{\sigma}^*\mathcal{L}_{S_1}$*, the function* $\mathcal{L}^{(3)}$ *has exactly* 4κ *critical points* $\theta_*^{(3)}$*, all nondegenerate and satisfying*

$$\tilde{\sigma}^*\sqrt{\varepsilon}\mathcal{L}_{S_2} \preceq m_*^{(3)} \preceq \sqrt{\varepsilon}\mathcal{L}_{S_2}.$$

Proof. For the minimum eigenvalue (in modulus) of $\mathrm{D}^2\mathcal{L}^{(2)}(\theta_*^{(2)})$, we use the following expression:

$$m_*^{(2)} = \frac{2|D|}{|T| + \sqrt{T^2 - 4D}}, \tag{57}$$

where we denote $D = \det \mathrm{D}^2\mathcal{L}^{(2)}(\theta_*^{(2)})$ and $T = \mathrm{tr}\,\mathrm{D}^2\mathcal{L}^{(2)}(\theta_*^{(2)})$. So we have to find estimates for D and T. It is clear that $\mathrm{D}^2\mathcal{L}^{(2)}(\theta_*^{(2)}) = \mathcal{A}_n^\top \mathrm{D}^2\mathcal{K}^{(2)}(\psi_*^{(2)})\mathcal{A}_n$ and, since $|\det \mathcal{A}_n| = \kappa$, we obtain directly from Lemma 4 that $|D| = \kappa^2 AB = \kappa^2\mathcal{L}_{S_1}\mathcal{L}_{S_2}$. On the other hand, writing $\mathrm{D}^2\mathcal{K}^{(2)}(\psi_*^{(2)}) = \begin{pmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{pmatrix}$ we see that

$$\begin{aligned} \mathrm{D}^2\mathcal{L}^{(2)}(\psi_*^{(2)}) &= k_{11}s_0(n-1)s_0(n-1)^\top \\ &\quad + k_{12}(s_0(n-1)s_0(n)^\top + s_0(n)s_0(n-1)^\top) \\ &\quad + k_{22}s_0(n)s_0(n)^\top \end{aligned}$$

and we obtain

$$T = k_{11}\langle s_0(n-1), s_0(n-1)\rangle + 2k_{12}\langle s_0(n-1), s_0(n)\rangle + k_{22}\langle s_0(n), s_0(n)\rangle. \tag{58}$$

Using that $|k_{11}| = A$, $|k_{22}| = B$ and $k_{12} = 0$, and also that $\mathcal{L}_{S_1} = \max(A, B)$ and estimate (46), we deduce that $|T| \sim \mathcal{L}_{S_1}/\sqrt{\varepsilon}$. Since $|D| \ll T^2$, we see from (57) that

$$m_*^{(2)} \sim \frac{|D|}{|T|} \sim \sqrt{\varepsilon}\mathcal{L}_{S_2}.$$

To estimate the minimum eigenvalue of $\mathrm{D}^2\mathcal{L}^{(3)}(\theta_*^{(3)})$, we can proceed analogously. Applying Lemma 4, we obtain

$$|D| = \kappa^2 B^2(E^{(\pm)}\eta + \mathcal{O}(\eta^2)) \sim \mathcal{L}_{S_1}^2 E^{(\pm)}\eta \sim E^{(\pm)}\mathcal{L}_{S_1}\mathcal{L}_{S_2} \succeq \tilde{\sigma}^*\mathcal{L}_{S_1}\mathcal{L}_{S_2},$$

under the following additional condition (required in Lemma 4):

$$\eta \sim \frac{\mathcal{L}_{S_2}}{\mathcal{L}_{S_1}} \preceq \tilde{\sigma}^*.$$

We can give an estimate for T using equality (58) again, but now with $|k_{22}| = B(1+\mathcal{O}(\eta)) \sim \mathcal{L}_{S_1}$ and $|k_{11}|, |k_{12} \preceq B\eta \sim \mathcal{L}_{S_2}$, obtaining the same estimate for $|T|$ as before and, consequently, the expected estimate for $m_*^{(3)}$. $\bullet$

After having studied the critical points of the approximations $\mathcal{L}^{(2)}$, $\mathcal{L}^{(3)}$, the last step is to study their persistence in the whole splitting potential $\mathcal{L}$. In order to apply the result on $\mathcal{L}^{(3)}$ in Lemma 5, and establish (for suitable quadratic frequencies) the existence and continuation of nondegenerate critical points for all $\varepsilon \to 0$, we will assume in (7) that the difference of phases

$$\triangle\sigma_n := \sigma_{s_0(n+1)} - \tau\sigma_{s_0(n)} + \delta\sigma_{s_0(n-1)}$$

keeps far away from 0 or $\pi \pmod{2\pi}$ for any n: for some fixed $\sigma^* > 0$,

$$\min(|\triangle\sigma_n|, |\triangle\sigma_n - \pi|) \geqslant \sigma^* \quad \forall n \geqslant 1. \tag{59}$$

As a concrete example such that condition (59) holds, we can consider in (7) a sequence of (primary) phases given by the recurrence $\sigma_{s_0(n+1)} = \tau\sigma_{s_0(n)} - \delta\sigma_{s_0(n-1)} + \pi/2$, $n \geqslant 1$, from any starting $\sigma_0(1)$, $\sigma_0(2)$. On the contrary, we stress that condition (59) does not hold in the case of a reversible perturbation, given by an even function $f(\varphi)$. In such a case, bifurcations of some of the homoclinic orbits, when ε goes across some critical values, have been described in [22].

The next theorem is formulated in terms of the splitting function, $\mathcal{M}(0,\theta) = \partial_\theta\mathcal{L}(0,\theta)$, which gives a measure of the distance between the whiskers. In this theorem, we establish under some conditions the existence of 4κ simple zeros of $\mathcal{M}(0,\theta)$, denoted θ_*, and we provide for these zeros an estimate for the minimum eigenvalue (in modulus) of the *splitting matrix* $\partial_\theta\mathcal{M}(0,\theta_*)$. As pointed out in [4], this minimum eigenvalue provides a lower bound for the transversality of the homoclinic orbit associated to the zero θ_*. Recall that the integer $\kappa \geqslant 1$ was introduced in (53).

To get the continuation for all $\varepsilon \to 0$ of the critical points, we have to assume that our quadratic frequency vector ω satisfies the *strong separation condition*:

$$\sqrt{\tilde{\gamma}^{**}} > 2A_2 - 1 \tag{60}$$

(recall that the constants A_i have been defined in (43)). Otherwise, we can get the persistence of all the critical points for ε not very close to the critical values ε_n (in other words, for ε close enough to the values ε'_n) provided ω satisfies the *weak separation condition*:

$$\sqrt{\tilde{\gamma}^{**}} > A_1. \tag{61}$$

We stress that the two separation conditions can be explicitly checked for concrete quadratic frequencies. For instance, in the four examples considered in Section 2, we checked that example 1 (the golden vector) satisfies (60), examples 2 and 3 satisfy (61), and example 4 satisfies none of them. Unfortunately, it seems from our numerical explorations that the only frequency vectors satisfying the strong condition (60) are the golden vector and any other noble vector (the ones that can be reduced to the golden vector by a unimodular transformation; the constants $\tilde{\gamma}^{**}$ and A_i are the same for all of them). Nevertheless, the result obtained may be relevant if we take into account that noble vectors are dense.

For the sake of completeness, we have also included a much simpler statement concerning the maximum size (in modulus) of the splitting function $\mathcal{M}(0,\theta)$, giving in this way an asymptotic estimate for the maximum splitting distance. Notice the difference in the exponents given by the functions $h_1(\varepsilon)$ and $h_2(\varepsilon)$ illustrated in Figure 1.

Theorem 6. *For the example introduced in* (1–7)*, assume that* $\varepsilon \preceq 1$ *and* $\mu = \varepsilon^p$*,* $p > p^*$*, with* p^* *as defined in* (18–19)*. Assume also that condition* (59) *on the phases is fulfilled. Then, one has:*

(a) *Under the weak separation condition* (61)*, the following estimate holds:*

$$\max_{\theta \in \mathbb{T}^2} |\mathcal{M}(0,\theta)| \sim \frac{\mu}{\sqrt{\varepsilon}} \exp\Big\{ -\frac{C_0 h_1(\varepsilon)}{\varepsilon^{1/4}} \Big\}.$$

(b) *Under the weak separation condition* (61)*, there exists* ζ*, with* $1 < \zeta < \lambda^2$*, such that if* ε *belongs to some interval* $(\zeta\varepsilon_n, \varepsilon_{n-1}/\zeta)$*, the function* $\mathcal{M}(0,\theta)$ *has exactly* 4κ *zeros* θ_**, all simple (with the integer* $\kappa \geqslant 1$ *defined in* (53)*).*

(c) *Under the strong separation condition* (6), *for any* $\varepsilon \preceq (\sigma^*)^{1/(p-p^*)}$, *the function* $\mathcal{M}(0,\theta)$ *has exactly* 4κ *zeros* θ_*, *all simple.*

In both cases (b) and (c), the minimum eigenvalue (in modulus) of $\partial_\theta \mathcal{M}(0,\cdot)$ *at each zero satisfies the estimate*

$$\sigma^* \mu \varepsilon^{1/4} \exp\left\{ - \frac{C_0 h_2(\varepsilon)}{\varepsilon^{1/4}} \right\} \preceq m_* \preceq \mu \varepsilon^{1/4} \exp\left\{ - \frac{C_0 h_2(\varepsilon)}{\varepsilon^{1/4}} \right\}.$$

Proof. We write $\mathcal{M} = \partial_\theta \mathcal{L}$. To prove (a), we consider as in (50) the approximation $\mathcal{L}^{(2)}$ given by the 2 most dominant harmonics. We can easily give an estimate for $\partial_\theta \mathcal{L}^{(2)}$ by writing it in the variables ψ as in (54), and applying Lemma 3 (we use the notation $|\cdot|$ for the supremum norm on $\mathbb{T}^2$):

$$|\partial_\theta \mathcal{L}^{(2)}| \sim \frac{1}{\varepsilon^{1/4}} \mathcal{L}_{S_1}, \qquad |\partial_\theta \mathcal{L}(0,\cdot) - \partial_\theta \mathcal{L}^{(2)}| \preceq \frac{1}{\varepsilon^{1/4}} (\mathcal{L}_{S_3} + \mathcal{S}),$$

where we denote, for the bound coming from secondary resonances,

$$\mathcal{S} = \frac{\mu}{\varepsilon^{1/4}} \exp\left\{ - \frac{C_0 \sqrt{\tilde{\gamma}^{**}}}{\varepsilon^{1/4}} \right\}.$$

Since we always have $h_1(\varepsilon) < h_3(\varepsilon)$, we obtain the expected asymptotic estimate (a) for $|\mathcal{M}(0,\cdot)|$.

To prove (b) and (c), we will show that $\mathcal{L}$ has nondegenerate critical points, choosing $\mathcal{L}^{(2)}$ or $\mathcal{L}^{((3)}$ as a suitable approximation, since we know from Lemma 5 that the critical points of these functions are all nondegenerate. The choice of the approximation depends on the closeness of ε to the values ε_n, ε'_n. More precisely, in some interval around ε'_n, we consider $\mathcal{L}^{(2)}$ and, near ε_n, we consider $\mathcal{L}^{(3)}$. We are going to study whether the two intervals where the approximations are valid intersect and the results are valid for all ε (small enough) or, on the contrary, some intervals of ε have to be excluded. This will depend on the separation $\tilde{\gamma}^{**}$.

First, for $\varepsilon \in (\varepsilon_n, \varepsilon_{n-1})$ we consider for the function $G(\theta) = \mathcal{M}(0,\theta) = \partial_\theta \mathcal{L}(0,\theta)$ the first approximation given by $G_0(\theta) = \partial_\theta \mathcal{L}^{(2)}(\theta)$. Recall that the zeros $\theta_*^{(2)}$ of G_0 are all simple, and an estimate for $m_*^{(2)}$, the minimum eigenvalue of $\mathrm{D}G_0(\theta_*^{(2)})$, has been given in Lemma 5(a). According to a quantitative version of the implicit function theorem (see for instance [6, Appendix]), this zero of G_0 persists as a (perturbed) zero θ_* of G provided inequalities of the following types are fulfilled:

$$|G - G_0| \preceq \frac{(m_*^{(2)})^2}{|\mathrm{D}^2 G_0|}, \qquad |\mathrm{D}G - \mathrm{D}G_0| \preceq m_*^{(2)}. \tag{62}$$

We have from Lemmas 3(b) and 5(a) the estimates

$$|G - G_0| \preceq \frac{1}{\varepsilon^{1/4}}(\mathcal{L}_{S_3} + \mathcal{S}), \qquad |\mathrm{D}G - \mathrm{D}G_0| \preceq \frac{1}{\varepsilon^{1/2}}(\mathcal{L}_{S_3} + \mathcal{S}),$$
$$m_*^{(2)} \sim \sqrt{\varepsilon}\mathcal{L}_{S_2}, \qquad |\mathrm{D}^2 G_0| \preceq \frac{1}{\varepsilon^{3/4}}\mathcal{L}_{S_1}.$$

Using these estimates, we see from (62) that we need

$$\mathcal{L}_{S_3} + \mathcal{S} \preceq \frac{\varepsilon^2 \mathcal{L}_{S_2}^2}{\mathcal{L}_{S_1}}. \tag{63}$$

Taking logarithms, we see that (63) can be written as the following inequality (with a suitable constant c):

$$2h_2(\varepsilon) - h_1(\varepsilon) \leqslant h_3^*(\varepsilon) + \frac{\varepsilon^{1/4}}{C_0}\ln(c\varepsilon^2), \quad h_3^*(\varepsilon) := \min(h_3(\varepsilon), \sqrt{\tilde{\gamma}^{**}}). \tag{64}$$

This inequality is going to be analysed later.

For $\varepsilon \in (\varepsilon'_{n+1}, \varepsilon'_n)$, we consider for $G(\theta)$ the first approximation given by $G_0(\theta) = \partial_\theta \mathcal{L}^{(3)}(\theta)$. In this case, to apply Lemma 5(b) to the zeros of $\theta_*^{(3)}$ of G_0, we have two check two additional conditions. The first one is that the difference of phases $\triangle\tilde{\sigma}_n = \tilde{\sigma}_{s_0(n+1)} - \tau\tilde{\sigma}_{s_0(n)} + \delta\tilde{\sigma}_{s_0(n-1)}$ is not very close to 0 or $\pi \pmod{2\pi}$. Indeed, we get from Lemma 3(a) that $|\triangle\tilde{\sigma}_n - \triangle\sigma_n| \preceq \mu\varepsilon^{-p^*}$. Recalling that $\mu = \varepsilon^p$, we see that $\mu\varepsilon^{-p^*} \preceq \sigma^*$, and deduce from from (59) the lower bound $\min(|\triangle\tilde{\sigma}_n|, |\triangle\tilde{\sigma}_n - \pi|) \succeq \sigma^*$. The second condition can be written as $\mathcal{L}_{S_2} \preceq \sigma^* \mathcal{L}_{S_1}$, which can be written as

$$h_1(\varepsilon) \leqslant h_2(\varepsilon) + \frac{\varepsilon^{1/4}}{C_0}\ln(c\sigma^*). \tag{65}$$

Then, we have given in Lemma 5(b) a lower bound for the minimum eigenvalue $m_*^{(3)}$. Applying the implicit function theorem as before, we see that the zero $\theta_*^{(3)}$ persists as a zero θ_* of G provided

$$\mathcal{L}_{S_4} + \mathcal{S} \preceq \frac{(\sigma^*\varepsilon)^2 \mathcal{L}_{S_2}^2}{\mathcal{L}_{S_1}},$$

which can be written as

$$2h_2(\varepsilon) - h_1(\varepsilon) \leqslant h_4^*(\varepsilon) + \frac{\varepsilon^{1/4}}{C_0}\ln(c(\sigma^*\varepsilon)^2), \quad h_4^*(\varepsilon) := \min(h_4(\varepsilon), \sqrt{\tilde{\gamma}^{**}}). \tag{66}$$

To get the continuation of the perturbed zeros we need that every ε satisfies (64) or (65–66). To study this, we can restrict ourselves to the interval $[\varepsilon_n, \varepsilon'_n]$ (the intersection of the two intervals considered above), and the results can easily be extended outside this interval using the symmetries of the functions $h_i(\varepsilon)$.

The behaviour of the involved functions on the interval $[\varepsilon_n, \varepsilon'_n]$ can be described as follows (see also Figure 1): the function h_1 increases from 1 to A_1, the function h_2 decreases from A_2 to A_1, the function h_3 increases from A_2 to A_3, the function h_4 decreases from A_4 to A_3, and the function $2h_2 - h_1$ decreases from $2A_2 - 1$ to A_1. Besides, the functions h_3 and $2h_2 - h_1$ intersect at some $\bar{\varepsilon}_n$ with a common value $\bar{A}$ (the values $\bar{\varepsilon}_n$ and $\bar{A}$ can be found explicitly if desired). Recall that the constants A_i, defined in (43), only depend on $|\lambda| > 1$. One can check that $A_2 < \bar{A} < 2A_2 - 1 < A_3$ (to check the last inequality, one may use that $|\lambda|^{3/2}(A_3 - 2A_2 + 1)$ is a polynomial in $|\lambda|^{1/2}$ with no real roots for $|\lambda| > 1$).

Now, to study the intervals where inequalities (64–66) are fulfilled, we have to replace h_3, h_4 by h_3^*, h_4^*. Hence we have to take into account the situation of $\sqrt{\tilde{\gamma}^{**}}$ with respect to the values A_i, $2A_2 - 1$, $\bar{A}$ considered above. Besides, there is a contribution coming from the term containing $\varepsilon^{1/4}$ and a logarithm in the three inequalities. This small term only gives rise to a small perturbation of the results, although it has to be seriously taken into account in inequality (65), which will be true for all $\varepsilon \in [\varepsilon_n, \varepsilon'_n)$ excluding a small neighbourhood close to ε'_n. On the other hand, note that (64) always implies (66). that the approximation given by the 2 dominant harmonics is valid, the one given by 3 dominant harmonics is valid as well, unless ε belongs to the small neighbourhood where inequality (65) does not hold.

One can check, taking those considerations into account, that if the strong separation condition (60) is fulfilled, then every ε in the whole interval considered satisfies (64) or (65–66), and hence the zeros persist. More precisely, we can consider 2 dominant harmonics only for $\varepsilon \in (\bar{\varepsilon}_n, \varepsilon'_n]$, and 3 dominant harmonics for $\varepsilon \in [\varepsilon_n, \bar{\varepsilon}_n]$.

If the strong condition does not work but instead the weak condition (61) is fulfilled, then there exists $1 < \zeta < \lambda^2$ such that every $\varepsilon \in (\zeta\varepsilon_n, \varepsilon'_n]$ satisfies (64) or (65–66). The value $\zeta\varepsilon_n$ is the solution of $2h_2(\varepsilon) - h_1(\varepsilon) = \sqrt{\tilde{\gamma}^{**}}$; it moves from ε_n to ε'_n as $\sqrt{\tilde{\gamma}^{**}}$ goes down from $2A_2 - 1$ to A_1 (at the end, the interval shrinks to ε'_n). If the weak condition (61) is not fulfilled, the interval is empty and both inequalities (64) and (66) are false. Note that the interval $(\zeta\varepsilon_n, \varepsilon'_n]$ becomes $(\zeta\varepsilon_n, \varepsilon_{n-1}/\zeta)$ if the results are extended using the symmetry of the functions h_i (considered as functions of $\ln\varepsilon$).

For the cases such that the persistence of the zeros has been established, the upper and lower bounds for the minimum eigenvalue at each zero come from Lemma 5. •

Remarks.

(1) In the weak condition case considered in (b), it can be seen from the proof that, for $\sqrt{\tilde{\gamma}^{**}} \leqslant \bar{A}$, condition (59) can be removed because we do not need to consider the 3 dominant harmonics case. We have ignored this in order to write a simpler statement of the theorem.

(2) As said in remark 3 after Lemma 3, if we consider in (7) a perturbation $f(\varphi)$ having only primary harmonics, then no obstruction comes from secondary harmonics, and the result of (c) would be valid independently of the separation $\tilde{\gamma}^{**}$. However, this would be a rather artificial example.

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References

1. G. Benettin, A. Carati, and G. Gallavotti, *A rigorous implementation of the Jeans–Landau–Teller approximation for adiabatic invariants.* — Nonlinearity, **10** (1997), 479–505.
2. B. V. Chirikov, *A universal instability of many-dimensional oscillator systems.* — Phys. Rep., **52**, No. 5 (1979), 263–379.
3. A. Delshams and P. Gutiérrez, *Splitting potential and the Poincaré–Melnikov method for whiskered tori in Hamiltonian systems.* — J. Nonlinear Sci., **10**, No. 4 (2000), 433–476.
4. A. Delshams and P. Gutiérrez. *Splitting and Melnikov potentials in Hamiltonian systems.* — J. Delgado, E. A. Lacomba, E. Pérez-Chavela, and J. Llibre, eds. *Hamiltonian Systems and Celestial Mechanics (HAMSYS-98). Proceedings of the III International Symposium*, Vol. 6 of World Scientific Monograph Series in Mathematics, pages 111–137. Held in Pátzcuaro, Michoacán, México, 7–11 December, 1998. World Scientific, Singapore (2000).
5. A. Delshams and P. Gutiérrez, *Homoclinic orbits to invariant tori in Hamiltonian systems.* — In: C.K.R.T. Jones and A. I. Khibnik, eds. Multiple-Time-Scale Dynamical Systems, Vol. 122 of IMA Vol. Math. Appl., pages 1–27. Held in Minneapolis, 27–31 October (1997). Springer-Verlag, New York (2001).

6. A. Delshams and P. Gutiérrez, *Exponentially small splitting for whiskered tori in Hamiltonian systems: Continuation of transverse homoclinic orbits.* — Preprint 03-112 (2002). mp-arc@math.untexas.edu.
7. A. Delshams, V. G. Gelfreich, À. Jorba, and T. M. Seara, *Exponentially small splitting of separatrices under fast quasiperiodic forcing.* — Comm. Math. Phys., **189** (1997), 35–71.
8. A. Delshams, P. Gutiérrez, and T. M. Seara, *Exponentially small splitting for whiskered tori in Hamiltonian systems: Flow-box coordinates and upper bounds.* — Preprint 03-134 (2002). mp-arc@math.utexas.edu.
9. L. H. Eliasson, *Biasymptotic solutions of perturbed integrable Hamiltonian systems.* — Bol. Soc. Brasil. Mat. (N.S.), **25**, No. 1 (1994), 57–76.
10. V. G. Gelfreich, *A proof of the exponentially small transversality of the separatrices for the standard map.* — Comm. Math. Phys., **201**, No. 1 (1999), 155–216.
11. V. G. Gelfreich and V. F. Lazutkin, *Splitting of separatrices: perturbation theory and exponential smallness.* — Russian Math. Surveys, **56**, No. 3 (2001), 499–558.
12. H. Koch, *A renormalization group for Hamiltonians, with applications to KAM theory.* — Ergodic Theory Dynam. System, **19**, No. 2 (1999), 475–521.
13. V. F.Lazutkin, *Splitting of separatrices for the Chirikov's standard map.* — (in Russian). Preprint VINITI (1984), 6372-84.
14. P. Lochak, J.-P. Marco, and D. Sauzin, *On the splitting of invariant manifolds in multidimensional near-integrable Hamiltonian systems.* — Preprint (1999).
15 . P. Lochak, *Effective speed of Arnold's diffusion and small denominators.* — Phys. Lett. A, **143**, No. 1–2 (1990), 39–42.
16 . P. Lochak, *Canonical perturbation theory via simultaneous approximation.* — Russian Math. Surveys, **47**, No. 6 (1992), 57–133.
17. J. Lopes Dias, *Renormalization scheme for vector fields on T^2 with a Diophantine frequency.* — Nonlinearity, **15**, No. 3 (2002), 665–679.
18. A. Pronin and D. Treschev, *Continuous averaging in multi-frequency slow-fast systems.* — Regul. Chaotic Dyn., **5**, No. 2 (2000), 157–170.
19. M. Rudnev and S. Wiggins, *On a homoclinic splitting problem.* — Regul. Chaotic Dyn., **5** No. 2 (2000), 227–242.
20. D. Sauzin, *A new method for measuring the splitting of invariant manifolds.* — Ann. Sci. École Norm. Sup. (4), **34** No. 2 (2001), 159–221.
21. C. Simó, *Averaging under fast quasiperiodic forcing.* — In: J. Seimenis, ed. Hamiltonian Mechanics: Integrability and Chaotic Behavior, Vol. 331 of NATO ASI Ser. B: Phys., pp. 13–34. Held in Toruń, Poland, 28 June–2 July 1993. Plenum, New York, 1994.
22. C. Simó and C. Valls, *A formal approximation of the splitting of separatrices in the classical Arnold's example of diffusion with two equal parameters.* — Nonlinearity, **14**, No. 6 (2001), 1707–1760.

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